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Stability of Linear Systems with Discrete Feedback and Impulse Input*

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The well-known problem of stabilization for the feedback control system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ u(t) &= Cx(t)\end{aligned}$$

is to determine the matrix C of feedback coefficients such that the closed system

$$\dot{x}(t) = Ax(t) + BCx(t)$$

is exponentially stable.

We start our study under the basic assumption that this problem is solved succesfully. In the sequel, we use the label B instead of BC for the matrix describing the effect of the feedback. Namely, we consider the system

$$\dot{x}(t) = (A + B)\, x(t) \tag{1}$$

such that the estimate

$$|e^{(A+B)(t-s)}| \le Ke^{-\lambda(t-s)}, \quad t \ge s, \tag{2}$$

holds with positive K and λ.

1 Discrete feedback

Let us suppose now that the measurement of the state $x(t)$ of the system is available, by some technical reasons, only at the discrete moments of time cycling with the period ω. So we have, instead of the continuous feedback $Bx(t)$, the functional operator

$$(\mathbf{B}_\omega x)(t) = Bx(k\omega) \;\text{ for } t \in (k\omega,\, (k+1)\omega),$$

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and the system of functional differential equations

$$\dot{x}(t) = Ax(t) + (\mathbf{B}_\omega x)(t). \tag{3}$$

This system is a special case of the linear functional differential equation

$$\dot{x}(t) = \int_0^t d_s R(t,\, s)\, x(s) + f(t) \tag{4}$$

comprehensively studied nowadays (see the survey and recent results in [1, 2, 3]).

Namely, the function R for the system (3) is defined as follows:

$$R(t,\, s) = \begin{cases} 0, & s \le k\omega, \\ B, & k\omega < s < t, \\ A + B, & s = t, \end{cases} \qquad \text{if } t \in (k\omega,\, (k+1)\omega). \tag{5}$$

2 Linear functional differential equation with locally summable f

We review here the most interesting, for our special case, results on exponential stability of equation (4).

The function R is supposed to satisfy the standard condition, named condition (R) in [3]. This condition ensures that if z is a continuous function on $[0,\, \infty)$, then $(\mathbf{R}z)(t) \overset{\text{def}}{=} \int_0^t d_s R(t,\, s)\, z(s)$ is locally summable on $[0,\, \infty)$.

1. It is known that *this equation has a Cauchy function $C(t,\, s)$ such that its solutions admit the Cauchy formula*

$$x(t) = C(t,\, 0)\, x(0) + \int_0^t C(t,\, s) f(s)\, ds.$$

The functions $C(t, \cdot)$ have bounded variation on $[0,\, t]$ and may be discontinuous. The operator $(\mathbf{C}f)(t) \overset{\text{def}}{=} \int_0^t C(t,\, s) f(s)\, ds$ converts locally summable functions to continuous functions [1, 2, 3]).

We use the following notation. Let $\gamma \ge 0$. C_γ is the space of continuous functions $x : [0,\, \infty) \to \mathbb{R}^n$ with the bounded norm $\|x\| = \sup\limits_{t \in [0,\, \infty)} e^{\gamma t} |x(t)|$. M_γ is the space of measurable functions $f : [0,\, \infty) \to \mathbb{R}^n$ with the norm $\|f\| = \operatorname*{vrai\,sup}\limits_{t \in [0,\, \infty)} e^{\gamma t} |f(t)|$. We write C_0 and M_0 in the case of $\gamma = 0$.

2. *Let the Cauchy function of the system* (3) *satisfy the estimate*

$$|C(t,\, s)| \le N e^{-\gamma(t-s)}, \quad t \ge s, \tag{6}$$

with $\gamma > 0$, *and* $\beta \in [0, \gamma)$. *Then the operator* $\mathbf{C}$ *maps* M_β *into* C_β. *Hence for* $f \in M_\beta$ *all the solutions of equation* (4) *belong to* C_β.

Indeed, from (6) we get $e^{\beta t}|(\mathbf{C}f)(t)| \le \int_0^t Ne^{(\beta-\gamma)(t-s)}e^{\beta s}|f(s)|\,ds \le \dfrac{N}{\gamma-\beta}\|f\|$. □

The function R is said to satisfy the *δ-condition* [5] if there is $\delta > 0$ such that $R(t, s) = 0$ for $s < t - \delta$ [5, 3].

We say that R satisfies the *V-condition* if $\operatorname{vrai\,sup}_{t\ge 0} \operatorname{Var}_0^t R(t, \cdot) < \infty$.

Clearly our function (5) satisfies both δ- and V-condition (take $\delta = \omega$).

3. *Let* R *satisfy both* δ- *and* V-*conditions. Then, for any* $\gamma \ge 0$, *the operator* $\mathbf{R}$ *acts from* C_γ *to* M_γ.

To prove this, let $z \in C_\gamma$; then

$$\begin{aligned} |(\mathbf{R}z)(t)| &\le \operatorname{Var}_{t-\delta}^t R(t, \cdot) \cdot \max_{s\in[t-\delta,\, t]} |z(s)| \\ &\le \operatorname{Var}_0^t R(t, \cdot) \cdot \max_{s\in[t-\delta,\, t]} e^{-\gamma s}\|z\| \\ &\le e^{-\gamma t} \cdot \text{const} \cdot \|z\| \end{aligned}$$

for a.e. $t \ge 0$. □

Note that in the case $\gamma = 0$ the δ-condition is not needed.

4. *Suppose* R *satisfies both* δ- *and* V-*conditions.*

If the operator $\mathbf{C}$ *maps* M_0 *into* C_0 *then there exist positive* N *and* γ *such that the estimate* (6) *holds.*

If the operator $\mathbf{C}$ *maps* M_β *into* C_β, $\beta > 0$, *then for any* $\gamma \in (0, \beta)$ *there exists* $N > 0$ *such that* (6) *is valid.*

For the spaces M_0 and C_0, this is a direct consequence of [5, Theorem 2.4] (see also [3, Theorem 3.3.1]). For $\beta > 0$, see [3, Theorem 3.3.2]. □

3 Exponential stability for equation (3)

We prove that if ω is sufficiently small, then the new system (3) is still exponentially stable.

Theorem 1 *For* $\gamma \in (0, \lambda)$, *let the inequality*

$$|B|\Big(\frac{K}{\lambda-\gamma}|A+B| + 1\Big)\, e^{\gamma\omega}\omega < 1 \tag{7}$$

hold. Then the Cauchy function of the system (3) *admits the estimate* (6).

Proof. To get (6), we shall check the sufficient conditions presented in [3, Theorem 3.3.2].

1°. The operator $(\mathbf{R}x)(t) \overset{\text{def}}{=} Ax(t) + (\mathbf{B}_\omega x)(t)$ acts from C_γ to M_γ.

This is shown above (item 2.3).

2°. The operator $\mathbf{C}$ maps M_γ into C_γ.

To prove this, we use a general idea of W-method [1, 2, 3]. Namely, we define

$$(\mathbf{W}z)(t) = \int_0^t e^{(A+B)(t-s)} z(s)\, ds,$$

then $\mathbf{W} : M_\gamma \to C_\gamma$. Indeed, according to (2) we have

$$e^{\gamma t}|(\mathbf{W}z)(t)| \le \int_0^t K e^{(\gamma-\lambda)(t-s)} e^{\gamma s}|z(s)|\, ds \le \frac{K}{\lambda-\gamma}\|z\|.$$

The equality $x = \mathbf{C}f$ takes place if and only if x is a solution of the problem

$$\begin{aligned} &\dot{x}(t) = Ax(t) + (\mathbf{B}_\omega x)(t) + f(t), \\ &x(a) = 0. \end{aligned} \tag{8}$$

Let $f \in M_\gamma$. We shall seek the solution of the form $x = \mathbf{W}z$, where $z \in M_\gamma$; so $x \in C_\gamma$. Substituting $\mathbf{W}z$ for x in (8), we get the equation

$$z - (\mathbf{B}_\omega - B)\mathbf{W}z = f \tag{9}$$

in the space M_γ. For the solvability of equation (9), it is sufficient that

$$\|(\mathbf{B}_\omega - B)\mathbf{W}\| < 1. \tag{10}$$

We estimate this norm as follows.

Let $x = \mathbf{W}z$. We have got above the inequality $|x(t)| \le \dfrac{K}{\lambda-\gamma} e^{-\gamma t}\|z\|$. Since $\dot{x} = (A+B)x + z$,

$$|\dot{x}(t)| \le \Big(\frac{K}{\lambda-\gamma}|A+B| + 1\Big) e^{-\gamma t}\|z\|.$$

For $t \in (k\omega,\, (k+1)\omega)$ we have

$$\begin{aligned} \big|\big((\mathbf{B}_\omega - B)\mathbf{W}z\big)(t)\big| &= \big|B\big(x(k\omega) - x(t)\big)\big| \\ &\le |B| \sup_{s\in(k\omega,\, t)} |\dot{x}(s)|\,\omega \\ &\le |B|\Big(\frac{K}{\lambda-\gamma}|A+B| + 1\Big)\, e^{-\gamma k\omega}\,\omega\,\|z\| \\ &\le e^{-\gamma t}\,|B|\Big(\frac{K}{\lambda-\gamma}|A+B| + 1\Big)\, e^{\gamma\omega}\,\omega\,\|z\|. \end{aligned}$$

So the condition (7) implies (10). For $f \in M_\gamma$ the equation (9) has a solution $z \in M_\gamma$ and $x = \mathbf{C}f = \mathbf{W}z \in C_\gamma$. Thus $\mathbf{C} : M_\gamma \to C_\gamma$.

3°. The inequalities $\gamma < \lambda$ and (7) remain valid if γ is replaced by a bit greater value $\gamma + \Delta\gamma$. According to item 2.4, the Cauchy function admits the exponential estimate with any quantity in exponent less than $\gamma + \Delta\gamma$. This completes the proof. □

4 Perturbation by measures

Consider the perturbed system

$$\dot{x}(t) = Ax(t) + (\mathbf{B}_\omega x)(t) + F'(t), \tag{11}$$

where F' is a generalized derivative of the function F of locally bounded variation. Without loss of generality, we can assume that $F(t)$ is continuous at $t = 0$.

The concepts of solution for such a system are presented in the paper [7]. In particular, the solutions with memory are acceptable in the situation considered. In the case of system (11) this general concept may be described as follows. The function x is a *solution with memory* if the difference $x - F$ is locally absolutely continuous on $[0, \infty)$ and there exists a function $\bar{x}(t) \in [x(t-0),\, x(t+0)]$ such that

$$\frac{d}{dt}\big(x(t) - F(t)\big) = Ax(t) + (\mathbf{B}_\omega \bar{x})(t)$$

almost everywhere.

We see that, for a given initial value, such a solution is determined non-uniquely. In the paper [7], the basic facts are proved concerning the existence of solutions, compactness and convexity of the solution set, continuous dependence on the input F'. In the sequel, we mean each solution "to have memory".

The solutions with memory obey the Cauchy-type formula [7]

$$x(t) \in C(t,\, 0)\, x(0) + \int_0^t C(t,\, s)\, dF(s). \tag{12}$$

Since both functions $C(t, \cdot)$ and F may be discontinuous, we use here the multivalued Stieltjes integral introduced and studied in the paper [6] ([7] contains a brief survey of [6]).

For every $t > 0$, the integral $\int_0^t C(t, s)\, dF(s)$ is a nonempty convex closed subset of $\mathbb{R}^n$ and

$$\left\| \int_0^t C(t,\, s)\, dF(s) \right\| \le \int_0^t |C(t,\, s)|\, d\,\mathrm{Var}_0^s F \tag{13}$$

(we use the notation $\|A\| \overset{\text{def}}{=} \sup\limits_{a \in A} |a|$).

5 Accumulation of perturbations

Due to (13) and the exponential estimate (6), we are now in a position to study problems on accumulation of perturbations.

Theorem 2 *Let the estimate* (6) *be valid and* $U \subset \mathbb{R}^n$ *be a bounded subset. Denote by* $X(U)$ *the set of all solutions with memory* x *of system* (11) *such that* $x(0) \in U$.

i) Let $\beta \in [0, \gamma)$. *If* $V(F') \stackrel{\text{def}}{=} \sup_{t\in[0,\infty)} e^{\beta t} \operatorname{Var}_t^{t+1} F < \infty$, *then*

$$\sup_{t\in[0,\infty)} e^{\beta t}|x(t)| \le N\|U\| + \text{const} \cdot V(F')$$

for all $x \in X(U)$.

ii) If $v(F') \stackrel{\text{def}}{=} \varlimsup_{t\to\infty} e^{\beta t} \operatorname{Var}_t^{t+1} F < \infty$, *then*

$$\varlimsup_{t\to\infty} e^{\beta t}\,|x(t)| \le \text{const} \cdot v(F')$$

uniformly for all $x \in X(U)$.

Proof. The first term of the formula (12) satisfies the inequality

$$|C(t, 0)\, x(0)| \le N e^{-\gamma t}|x(0)|$$

and therefore is in accordance with each of the assertions i), ii) of the theorem.

We estimate now the integral term of (12). Let k be an integer such that $t \in [k, k+1)$. So we have

$$\begin{aligned}\left\| \int_0^t C(t, s)\, dF(s) \right\| &\le \int_0^t N e^{-\gamma(t-s)}\, d\operatorname{Var}^s F \\ &\le \sum_{i=1}^{k+1} \int_{i-1}^{i} N e^{-\gamma(t-s)}\, d\operatorname{Var}^s F \\ &\le \sum_{i=1}^{k+1} N e^{-\gamma(t-i)}\ \operatorname{Var}_{i-1}^{i} F.\end{aligned}$$

i) Since $\operatorname{Var}_{i-1}^{i} F \le e^{\beta - i\beta}\, V(F')$, summing the geometric progression we have

$$\left\| e^{\beta t} \int_0^t C(t, s)\, dF(s) \right\| \le N e^{\beta} \sum_{i=1}^{k+1} e^{(\gamma-\beta)(i-t)} V(F') \le \frac{N e^{2\gamma-\beta}}{e^{\gamma-\beta} - 1}\, V(F').$$

ii) Take $\varepsilon > 0$. There exists an integer k_ε such that $\sup\limits_{t \geq k_\varepsilon} e^{\beta t} \operatorname{Var}_{t-1}^{t} F < v(F') + \varepsilon$. Let $t > k_\varepsilon$. Hence

$$e^{\beta t} \sum_{i=k_\varepsilon}^{k+1} N e^{-\gamma(t-i)} \operatorname{Var}_{i-1}^{i} F \leq \frac{N e^{2\gamma-\beta}}{e^{\gamma-\beta}-1} (v(F') + \varepsilon).$$

Besides,

$$\begin{aligned} e^{\beta t} \sum_{i=1}^{k_\varepsilon - 1} N e^{-\gamma(t-i)} \operatorname{Var}_{i-1}^{i} F &\leq e^{\beta t} \sum_{i=1}^{k_\varepsilon - 1} N e^{-\gamma(t-i)} V(F') e^{-\beta(i-1)} \\ &\leq \frac{N\, V(F')\, e^{\beta}}{e^{\gamma-\beta}-1} e^{-(\gamma-\beta)(t-k_\varepsilon)}. \end{aligned}$$

So

$$\left\| e^{\beta t} \int_0^t C(t,\, s)\, dF(s) \right\| < \frac{N e^{2\gamma-\beta}}{e^{\gamma-\beta}-1} v(F') + \frac{N}{e^{\gamma-\beta}-1} \big(e^{2\gamma-\beta} + N\, V(F')\, e^{\beta}\big) \varepsilon$$

for sufficiently large t. □

Remark 1 In particular (take $\beta = 0$), if $\sup\limits_{t \geq 0} \operatorname{Var}_{t}^{t+1} F < \infty$, then all the solutions $x \in X(U)$ are uniformly bounded on $[0,\, \infty)$.

Conversely, if $\sup\limits_{t \geq 0} \operatorname{Var}_{t}^{t+1} F < \infty$ yields $\sup\limits_{t \geq 0} |x(t)| < \infty$, then the inequality (6) is true for the Cauchy function. This is proved by V. A. Tyshkevich [5, Theorem 2.2] who considered the system (4) having locally summable (ordinary) function f and continuous solutions. Clearly, this constraint is not too restrictive for the proof.

Other "inverse" results are presented in item 2.4 above.

Remark 2 The calculation of the integral $\int_0^t e^{-\gamma(t-s)}\, dv(s)$, where $\sup\limits_{t \geq 0} \operatorname{Var}_{t}^{t+1} v < \infty$, considered for the case of ordinary differential equations, was made by E. A. Barbashin [4, p. 197].

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