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Solving an Integro-Differential Equation by Legendre Polynomial and Block-Pulse Functions

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Abstract

In this paper, hybrid Legendre Block-Pulse functions are developed to find an approximate solution for an integro-differential equation. Hybrid Legendre Block-Pulse functions are developed by combining Block-Pulse functions on $[0, 1]$ and Legendre polynomials. By using this method integro-differential equations reduce to a system of linear equations.

Key words: Block-Pulse functions, Volterra and Fredholm integral equation, Integro-differential equation.

1 Introduction

In this paper, we will use a simple basis for solving an integro-differential equation. This basis is a combination of Block-Pulse functions on $[0, 1]$, and Legendre polynomials, that is called the hybrid Legendre Block-Pulse functions.

1.1 Definition

Consider the Legendre polynomials $p_m(t)$ on the interval $[-1, 1]$:

$$\begin{aligned} &p_0(t) = 1,\ p_1(t) = t, \dots, \\ &p_{m+1}(t) = \frac{2m+1}{m+1} + p_m(t) - \frac{m}{m+1} p_{m-1}(t), m = 1, 2, \dots \end{aligned} \tag{1.1}$$

The set $\{p_m(t);\ m = 0, 1, \dots\}$ in the Hilbert space $L^2[-1, 1]$ is a complete orthogonal set on $[1, 2]$.

1.2 Lemma

Let $x(t) \in H^k(-1,1)$ (a Sobolev space) and let $x_j(t) = \sum_{i=0}^{j} a_i L_i(t)$ be the best approximation polynomial of $x(t)$ in the L^2-norm, then

$$\|x(t) - x_j(t)\|_{L^2[-1,1]} \leq c_0 j^{-k} \|x(t)\|_{H^k(-1,1)}$$

where c_0 is a positive constant, which depends on the selected norm and is independent of $x(t)$, j (see [3]).

1.3 Definition

A set of Block-Pulse functions $b_i(\lambda), i = 1, 2, \ldots, m$, on the interval $[0, 1)$ are defined as follows:

$$b_i(\lambda) = \begin{cases} 1, & \frac{i-1}{m} \leq \lambda < \frac{i}{m}; \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

The Block-Pulse functions on $[0, 1)$ are disjoint, that is, for $i = 1, 2, \ldots, m$, they satisfy an orthogonality property on $[0, 1)$.

1.4 Definition

For $m = 0, 1, 2, \ldots, M - 1$ and $n = 1, 2, \ldots, N$ the hybrid Legendre Block-Pulse functions are defined as:

$$b(n, m, t) = \begin{cases} P_m(2Nt - 2n + 1), & \frac{n-1}{N} \leq t < \frac{n}{N}; \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

1.5 The operational matrix

If

$$B(t) = [b(1, 0, t), b(1, 1, t), \ldots, b(1, M - 1, t), b(2, 0, t), \ldots, b(N, M - 1, t)]^T$$

is a vector function of hybrid Legendre Block-Pulse functions on $[0, 1)$, the integration of the vector $B(t)$ can be obtained as:

$$\int_0^1 B(t')\, dt' \simeq PB(t) \tag{1.4}$$

where P is an $MN \times MN$ matrix, that is called the operation matrix for hybrid Legendre Block-Pulse functions. Then the operation matrix P has the following form [4, 5]

$$P = \begin{pmatrix} E & H & \dots & H \\ 0 & E & \dots & H \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & E \end{pmatrix} \tag{1.5}$$

where H is an $M \times M$ matrix and is defined as follows:

$$H = \frac{1}{N} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}; \tag{1.6}$$

also E is an $M \times M$ matrix on the interval $[0, \frac{1}{n})$ and is defined as follows [5, 6]:

$$E = \frac{1}{2N} \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & \dots & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & -\frac{1}{5} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\frac{1}{2M-3} & 0 & \frac{1}{2M-3} \\ 0 & 0 & 0 & 0 & \dots & 0 & -\frac{1}{2M-1} & 0 \end{pmatrix} \tag{1.7}$$

2 Function approximation

A function $x(t) \in L^2[0, 1]$ may be expanded as

$$x(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} X(n, m) b(n, m, t), \tag{2.1}$$

where

$$X(n, m) = \frac{(x(t), b(n, m, t))}{(b(n, m, t), b(n, m, t))} \tag{2.2}$$

where $(\cdot, \cdot)$ denotes the inner product. If the infinite series in (2.1) is truncated, then (2.1) can be written as

$$x(t) \simeq X_{NM}(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} X(n.m) b(n, m, t) = X^T B(t), \tag{2.3}$$

where $B(t)$ is a vector function and X is given by

$$X = [X(1,0), X(1,1), \dots, X(1, M-1), X(2,0), \dots, X(N, M-1)]^T.$$

We can also approximate the function $k(t,s) \in L^2([0,1]\times[0,1])$ as follows:

$$k(t,s) \simeq k_{NM}(t,s) = B^T(t) k B(s), \tag{2.4}$$

where k is an $MN \times MN$ matrix such that

$$k_{ij} = \frac{(B_i(t), (k(t,s), B_j(s)))}{(B_i(t), B_i(t))(B_j(s), B_j(s))}, \quad i,j = 1,2,\dots, MN. \tag{2.5}$$

We also define the matrix D as follows:

$$D = \int_0^1 B(t) B^T(t)\, dt. \tag{2.6}$$

For the hybrid Legendre Block-Pulse functions, D has the following form:

$$D = \begin{pmatrix} D_1 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_N \end{pmatrix}, \tag{2.7}$$

where D_i is defined as follows:

$$D_i = \frac{1}{N} \int T(t) T^T(t)\, dt.$$

3 Integro-differential equation

Consider the following integro-differential equation:

$$\begin{aligned} q(t)y'(t) &= \int_0^1 k(t,s)y(s)\, ds + r(t)y(t) + x(t), \\ y(0) &= y_0, \end{aligned} \tag{3.1}$$

where $x, q, r \in L^2[0,1)$, $k \in L^2([0,1]\times[0,1])$ and y is an unknown function [7]. If we approximate x, q, r, y' and k by (2.1)–(2.4) as follows:

$$x(t) \simeq X^T B(t),\ y(t) \simeq Y^T B(t),\ k(t,s) \simeq B^T(t) K B(s),$$

then

$$\begin{aligned} y(t) &= \int_0^t y'(t')\,dt' + y(0) \\ &\simeq \int_0^t Y'^T B(t')\,dt' + Y_0^T B(t) \\ &\simeq Y^T D B(t) + Y_0^T B(t) \\ &= (Y'^T D + Y_0^T) B(t). \end{aligned}$$

With substituting in (3.1) we have

$$Y^T = Y'^T D + Y_0^T \;\Rightarrow\; y(t) \simeq Y^T B(t).$$

4 Numerical experiments

Example 1. Consider the equation with exact solution $y(t) = e^t$:

$$\begin{aligned} y'(t) &= \int -0^1 e^{st} y(s)\,ds + y(t) + \frac{1 - e^{t+1}}{t+1}, \\ y(0) &= 1. \end{aligned}$$

The solution for $y(t)$ is obtained by the method of Section 3. Results are shown in Table 1.

Example 2. Consider the equation with exact solution $y(t) = \cos(2\pi t)$:

$$\begin{aligned} y'(t) &= \int_0^1 \sin(4\pi t + 2\pi s) y(s)\,ds + y(t) - \cos(2\pi t) - 2\pi \sin(2\pi t) - \frac{1}{2}\sin(4\pi t), \\ y(0) &= 1. \end{aligned}$$

The solution for $y(t)$ is obtained by the method of Section 3. Results are shown in Table 2.

Table 1: Results for Example 1.

N	M	$\|y - y_{NM}\|_2$
2	3	6.348×10^{-2}
4	3	5.319×10^{-4}
8	3	8.135×10^{-6}
16	3	5.187×10^{-6}
32	3	3.522×10^{-7}
64	3	2.334×10^{-8}

Table 2: Results for Example 2.

N	M	$\|y - y_{NM}\|_2$
2	3	1.274×10^{-1}
4	3	1.986×10^{-2}
8	3	3.674×10^{-3}
16	4	2.598×10^{-4}
32	4	9.907×10^{-5}
64	3	3.876×10^{-6}

5 Conclusion

If we solve the integro-differential equation using orthogonal continuous or piecewise constant functions, the accuracy of the method will be worse. Whereas, using hybrid Legendre and Block-Pulse functions the accuracy of system will improve using suitable M and N because the hybrid Legendre and Block-Pulse functions are orthogonal piecewise continuous functions and have high flexibility.

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