5–10 July 2004, Antalya, Turkey — Dynamical Systems and Applications,
Proceedings, pp. 614–623

On Asymptotic Behavior of Solutions to Linear Discrete Stochastic Equation

Alexandra Rodkina
Department of Mathematics and Computer Science,
University of the West Indies at Mona, Kingston, JAMAICA

Gregory Berkolaiko
Department of Mathematics, Texas A&M University,
TX 77843, U.S.A.

Abstract

We consider stochastic linear difference equations

$$X_{n+1} = X_n\Big(1 + \xi_{n+1}\Big) + S_n, \quad n = 1, 2, \dots, \quad X_0 = x_0,$$

where ξ_i are i.i.d. random variables with $\mathbf{E}\xi_i = 0$ and $\mathbf{Var} \ln|1 + \xi_i| < \infty$.

In the homogeneous case $S_n \equiv 0$ we obtain necessary and sufficient conditions for the fulfillment of the following:
a) $\lim_{n\to+\infty} X_n = 0$ holds a.s.;
b) $\lim_{n\to+\infty} |X_n| = \infty$ holds a.s.

In the non-homogeneous case we derive a sufficient condition which guarantees that $\lim_{n\to+\infty} X_n = 0$ holds a.s.

1 Introduction

The question of sufficient conditions which guarantee almost sure asymptotic stability of solutions of stochastic difference equations is crucial in diverse applications. Among such applications we can mention asset price evolution in discrete (B, S)-markets and population dynamics in mathematical biology.

In this paper we consider linear stochastic difference equation without drift

$$X_{n+1} = X_n\Big(1 + \xi_{n+1}\Big) + S_n, \quad n = 1, 2, \dots, \quad X_0 = x_0, \tag{1}$$

where ξ_i are independent and identically distributed (i.i.d.) random variables with $\mathbf{E}\xi_i = 0$ and $\mathbf{Var} \ln|1+\xi_i| < \infty$. We consider both homogeneous ($S_n \equiv 0$) and non-homogeneous cases. Our approach can be applied to more complicated stochastic difference equations without the condition on identical distribution of ξ_i and also in the presence of the drift term. However, in this paper we mostly restrict our attention to the simplest case (1) to demonstrate the idea of the method in simple terms.

Certain results dealing with almost sure asymptotic stability for homogeneous linear stochastic difference equation can be found in Neveu [7]. In Higham [2] and Schurz [13, 14] almost sure asymptotic stability was obtained for the trivial solution of the equation

$$X_{n+1} = X_n\Big(1 + \sigma_0 \xi_{n+1}\Big) + |c_0 \xi_{n+1}|(X_n - X_{n+1})$$

with zero drift and real parameters $|c_0| \geq |\sigma_0| > 0$. This equation was interpreted as a linear-implicit discretizations of Girsanov's stochastic differential equation $dX(t) = \sigma_0 X(t)\, dW(t)$ through balanced implicit methods with unbounded martingale-type of noise $\xi_{n+1} = W(t_{n+1}) - W(t_n) \in \mathcal{N}(0, t_{n+1} - t_n)$ along partitions $0 = t_0 < t_1 < \cdots < t_N = T$ driven by an underlying Wiener process W.

We note that to obtain criteria for almost sure asymptotic stability we do not apply Lyapunov functional's method as it was usual in many papers (see *e.g.* Kolmanovskii and Shaikhet [3, 4], Rodkina, Mao and Kolmanovskii [10], Rodkina [9], Rodkina and Schurz [11, 12]).

The idea of our method is very transparent: the modulus of the solution X_n to the homogeneous stochastic difference equations

$$X_{n+1} = X_n\Big(1 + \xi_{n+1}\Big), \quad n = 1, 2, \ldots, \quad X_0 = x_0, \tag{2}$$

is given by

$$|X_n| = |X_0| \prod_{i=0}^{n} \Big|1 + \xi_{n+1}\Big| = |X_0| \exp\left\{\sum_{i=0}^{n} \ln \Big|1 + \xi_{n+1}\Big|\right\}. \tag{3}$$

From the representation (3) we obtain that

$$\lim_{n\to\infty} X_n = 0 \quad \text{if and only if} \quad \sum_{i=0}^{\infty} \ln \Big|1 + \xi_{n+1}\Big| = -\infty \tag{4}$$

and

$$\lim_{n\to\infty} |X_n| = \infty \quad \text{if and only if} \quad \sum_{i=0}^{\infty} \ln \Big|1 + \xi_{n+1}\Big| = \infty. \tag{5}$$

In this paper we are going to derive conditions which insure fulfillment of one of the following

$$\sum_{i=0}^{\infty} \ln \Big|1 + \xi_{n+1}\Big| = -\infty \quad \text{or} \quad \sum_{i=0}^{\infty} \ln \Big|1 + \xi_{n+1}\Big| = \infty.$$

2 Auxiliary definitions and facts

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a complete filtered probability space. Let $\{\xi_i\}_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed (i.i.d.) random variables. We suppose that the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is naturally generated: $\mathcal{F}_{n+1} = \sigma\{\xi_{i+1} : i = 0, 1, ..., n\}$.

The notation $X_n = \{X_n(\omega)\}_{n\in\mathbb{N}}$ denotes a $\mathcal{F}_n$-measured stochastic process. We use the standard abbreviation "a.s." for the wordings "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbb{P}$ throughout the text.

For detailed definitions and facts from the theory of Stochastic Processes and Stochastic Differential Equations see, for example, Arnold [1], Neveu [7], Lipster and Shiryaev [5], Protter [8], Shiryaev [15] and Mao [6].

The folowing statements will be used in the proofs of the main results. The first is Kolmogorov Strong Law of Large Numbers which we cite from Shiryaev [15], Chapter IV, p. 389.

Theorem 1 (Kolmogorov). *Let $\{\xi_i\}_{i\in\mathbb{N}}$ be a sequence of independent random variables with $\theta_k^2 = \mathbf{Var}\xi_k < \infty$. Let $S_n = \xi_1 + \cdots + \xi_n$ and the numbers $b_n > 0$ are such that $b_n \uparrow \infty$ as $n \to \infty$ and*

$$\sum_{i=1}^{\infty} \frac{\theta_i^2}{b_i^2} < \infty \tag{6}$$

Then a.s.

$$\frac{S_n - \mathbf{E}S_n}{b_n} \to 0. \tag{7}$$

The second result can be easily proved.

Lemma 1. *Let $\{\phi_j\}_{j\in\mathbb{N}}$ be an increasing sequence of reals, $\phi_n \to \infty$ as $n \to \infty$, and $\{b_j\}_{j\in\mathbb{N}}$ be a sequence satisfying $\sum_{j=0}^{\infty} b_j < \infty$. Then*

$$\frac{1}{\phi_n} \sum_{j=0}^{n} b_j \phi_j \to 0,$$

as $n \to \infty$.

3 Homogeneous equation

We put

$$\kappa_i = \ln\Big|1 + \xi_i\Big|, \quad S_n = \sum_{i=1}^{n} \kappa_i, \quad a = \mathbf{E}\kappa_i, \quad \theta^2 = \mathbf{Var}\kappa_i.$$

Applying Kolmogorov's Theorem with $b_i = i$, we get

$$\frac{S_n - \mathbf{E}S_n}{n} = \frac{\sum_{i=1}^n \kappa_i - na}{n} = \frac{\sum_{i=1}^n \kappa_i}{n} - a \to 0. \tag{8}$$

almost surely.

Theorem 2. *Assume that $\{\xi_{n+1}\}_{n\in\mathbb{N}}$ are independent and identically distributed random variables and $a \neq 0$. Then*
a) $\lim_{n\to+\infty} X_n = 0$ holds $\mathbb{P}$-a.s. for the solution $\{X_n\}_{n\in\mathbb{N}}$ to equation (2) if and only if

$$a = \mathbf{E}\ln\Big|1+\xi_i\Big| < 0. \tag{9}$$

b) $\lim_{n\to+\infty} |X_n| = \infty$ holds $\mathbb{P}$-a.s. for the solution $\{X_n\}_{n\in\mathbb{N}}$ to equation (2) if and only if

$$a = \mathbf{E}\ln\Big|1+\xi_i\Big| > 0. \tag{10}$$

Proof. Case (a), sufficiency. If $a < 0$, from (8) for $\varepsilon = -\frac{a}{2} > 0$ we can find $N_1 = N_1(\omega, \epsilon)$ such that for $n > N_1$ we have

$$\frac{\sum_{i=1}^n \kappa_i - na}{n} \le -\frac{a}{2},$$

and, therefore,

$$\sum_{i=1}^n \kappa_i \le \frac{a}{2} n \to -\infty,$$

when $n \to \infty$. Now the result is immediately obtained from (4).

Necessity. Suppose that $\lim_{n\to+\infty} X_n = 0$ which, according to (4), is equivalent to $\sum_{i=1}^n \kappa_i \to -\infty$.

Assume the contrary, *i.e.*, that $a > 0$. Then there is $N_2 = N_2(\omega)$ such that for $n > N_2$.

$$\frac{\sum_{i=1}^n \kappa_i - na}{n} \ge -\frac{a}{2}.$$

Then

$$\frac{an}{2} \le \sum_{i=1}^n \kappa_i \to -\infty \quad \text{as} \quad n \to \infty,$$

which is a contradiction to our assumption.

Case (b), sufficiency. If $a > 0$, from (8) for $\varepsilon = \frac{a}{2} > 0$ we can find $N_1 = N_1(\omega, \epsilon)$ such that for $n > N_1$ we have

$$\frac{\sum_{i=1}^n \kappa_i - na}{n} \ge -\frac{a}{2},$$

and, therefore,

$$\sum_{i=1}^{n} \kappa_i \geq \frac{a}{2} n \to \infty,$$

as $n \to \infty$. The result now follows from (5).

Necessity. Suppose that $\lim_{n\to+\infty} X_n = \infty$ which, according to (4), is equivalent to $\sum_{i=1}^{n} \kappa_i \to \infty$.

Assume the contrary, *i.e.*, that $a < 0$. Then there is $N_2 = N_2(\omega)$ such that for $n > N_2$.

$$\frac{\sum_{i=1}^{n} \kappa_i - na}{n} \leq -\frac{a}{2}.$$

Then

$$\frac{an}{2} \geq \sum_{i=1}^{n} \kappa_i \to \infty \quad \text{as} \quad n \to \infty,$$

which contradicts our assumption. □

The example below shows that ξ_n does not have to be bounded (in particular by 1) to satisfy (9).

Example 1. Let $\{\xi_{n+1}\}_{n\in\mathbb{N}}$ be independent and identically distributed random variables such that

$$\xi_n = \begin{cases} -1.5, & \text{with probability } 0.05, \\ -0.5, & \text{with probability } 0.45, \\ \ \ 0.6, & \text{with probability } 0.5. \end{cases}$$

We note that for any $n = 1, 2, \ldots$ the random variable ξ_n takes the value -1.5 with probability 0.05. However, $\mathbf{E}\xi_n = 0$ and $\mathbf{E} \ln|1 + \xi_n| = 0.5 \ln(0.8) < 0$.

The following remark shows that when ξ_n are bounded, in modulus, by 1, condition (9) is automatically fulfilled.

Remark 1. Suppose that $\mathbf{E}\xi_n^2 > 0$ and there exists some $k \in (0, 1)$ such that for any n

$$|\xi_n| \leq k. \tag{11}$$

Then $\mathbf{E} \ln\left|1 + \xi_i\right| < 0$.

Indeed, from (11) we have

$$0 < 1 - k \leq 1 + \xi_n \leq 1 + k,$$

so $\ln|1+\xi_n| = \ln(1+\xi_n)$. Expanding $\ln(1+u)$ in Taylor series we get

$$\ln(1+\xi_n) = \xi_n - \frac{\xi_n^2}{2(1+\theta_n)^2},$$

where $|\theta| \in (0, |\xi_n|)$. Using the estimates

$$1+\theta \le 1+k, \quad -\frac{1}{(1+\theta)^2} \le -\frac{1}{(1+k)^2}$$

we arrive at

$$\mathbf{E}\ln|1+\xi_n| = \mathbf{E}\left(\xi_n - \frac{\xi_n^2}{2(1+\theta_n)^2}\right) \le \mathbf{E}\left(\xi_n - \frac{\xi_n^2}{2(1+k)^2}\right) = -\frac{\mathbf{E}\xi_n^2}{2(1+k)^2} < 0.$$

4 Nonhomogeneous case

In this section we consider nonhomogeneous difference equations

$$X_{n+1} = X_n\Big(1+\xi_{n+1}\Big) + S_n, \quad n \ge 0. \tag{12}$$

Theorem 3. *Assume that $\{\xi_{i+1}\}_{i\in\mathbb{N}}$ are independent and identically distributed random variables,*

$$a = \mathbf{E}\ln\Big|1+\xi_i\Big| < 0, \tag{13}$$

and there exists $\delta > 0$ such that

$$\sum_{j=0}^{\infty} |S_j| e^{\delta j} < \infty. \tag{14}$$

Then $\lim_{n\to+\infty} X_n = 0$ holds $\mathbb{P}$-a.s. for the solution $\{X_n\}_{n\in\mathbb{N}}$ to equation (12).

Proof. We have recursively:

$$\begin{aligned}
X_2 &= X_1\Big(1+\xi_2\Big)+S_1 = X_0\Big(1+\xi_1\Big)\Big(1+\xi_2\Big)+S_0\Big(1+\xi_2\Big)+S_1\\
&= X_0\prod_{i=1}^{2}\Big(1+\xi_i\Big)+S_0\Big(1+\xi_2\Big)+S_1,\\
X_3 &= X_2\Big(1+\xi_3\Big)+S_2\\
&= \Big(X_0\Big(1+\xi_1\Big)\Big(1+\xi_2\Big)+S_0\Big(1+\xi_2\Big)+S_1\Big)\Big(1+\xi_3\Big)+S_2\\
&= X_0\prod_{i=1}^{3}\Big(1+\xi_i\Big)+S_0\prod_{i=2}^{3}\Big(1+\xi_i\Big)+S_1\prod_{i=3}^{3}\Big(1+\xi_i\Big)+S_2,\\
&\vdots\\
X_{n+1} &= X_0\prod_{i=1}^{n+1}\Big(1+\xi_i\Big)+\sum_{j=0}^{n}S_j\prod_{i=j+2}^{n+1}\Big(1+\xi_i\Big).
\end{aligned}$$

Then we can estimate

$$\begin{aligned}
|X_n| &\le |X_0|\prod_{i=1}^{n+1}\Big|1+\xi_i\Big|+\sum_{j=0}^{n}|S_j|\prod_{i=j+2}^{n+1}\Big|\Big(1+\xi_i\Big)\Big|\\
&= |X_0|\exp\left\{\sum_{i=1}^{n+1}\kappa_i\right\}+\sum_{j=0}^{n}|S_j|\exp\left\{\sum_{i=j+2}^{n+1}\kappa_i\right\},
\end{aligned}\tag{15}$$

where $\kappa_i = \ln\Big|1+\xi_i\Big|$. Since $\{\kappa_i\}$ form a sequence of independent identically distributed random variables, we apply Kolmogorov's Theorem to find $N_1 = N_1(\omega)$ such that for all $n > N_1$ we have

$$\left(a-\frac{\delta}{2}\right)n \le \sum_{i=1}^{n+1}\kappa_i \le \left(a+\frac{\delta}{2}\right)n, \tag{16}$$

where without loss of generality we assume $\delta < 2|a|$ so that $a+\delta/2 < 0$.

Then, continuing (15) for $n > N_1$, we have

$$
\begin{aligned}
|X_n| &\le \exp\left\{\sum_{i=1}^{n+1}\kappa_i\right\}\left(|X_0| + \sum_{j=0}^{n}|S_j|\exp\left\{-\sum_{i=1}^{j+1}\kappa_i\right\}\right)\\
&\le \exp\left\{\left(a+\frac{\delta}{2}\right)n\right\}\left(|X_0| + \sum_{j=0}^{N_1}|S_j|\exp\left\{-\sum_{i=1}^{j+1}\kappa_i\right\}\right.\\
&\qquad\qquad\qquad\qquad \left. + \sum_{j=N_1+1}^{n}|S_j|\exp\left\{-\sum_{i=1}^{j+1}\kappa_i\right\}\right)\\
&\le \exp\left\{\left(a+\frac{\delta}{2}\right)n\right\}\left(|X_0| + C + \sum_{j=N_1+1}^{n}|S_j|\exp\left\{-\left(a-\frac{\delta}{2}\right)j\right\}\right)\\
&= \exp\left\{\left(a+\frac{\delta}{2}\right)n\right\}\left(|X_0| + C + \sum_{j=N_1+1}^{n}|S_j|e^{\delta j}\exp\left\{-\left(a+\frac{\delta}{2}\right)j\right\}\right).
\end{aligned}
$$

The first two terms tend to zero since they are constants multiplied by an exponentially small factor. To estimate the last term we use condition (14) and Lemma 1 with $b_j = |S_j|e^{\delta j}$ and $\phi_j = \exp\left\{-\left(a+\frac{\delta}{2}\right)j\right\}$. □

Remark 2. It is important to note that by selecting different b_n in our application of Kolmogorov's Theorem we can somewhat relax condition (14). For example, instead of (14) we can demand that there is $\delta > 0$ and $\varepsilon > 0$ such that

$$\sum_{j=0}^{\infty}|S_j|e^{j^{1/2+\varepsilon}\delta} < \infty. \tag{17}$$

5 Remarks on non-identically distributed ξ_{n+1}

The results of the previous two sections can be generalized to the case when $\{\xi_{n+1}\}_{n\in\mathbb{N}}$ are just independent, but not identically distributed, with

$$\kappa_i = \ln\Big|1+\xi_i\Big|, \quad a_i = \mathbf{E}\kappa_i, \quad \theta_i^2 = \mathbf{Var}\kappa_i.$$

In this section we are going to mention these generalizations briefly without strong proof. We just note that for the proof of these results we apply the fact that if

$$\sum_{i=1}^{n}\theta_i^2 \to \infty,$$

then for every $\varepsilon > 0$ a.s.

$$\frac{\sum_{i=1}^{n} \kappa_i - \sum_{i=1}^{n} a_i}{b_n} \to 0, \tag{18}$$

where we introduced the notation $b_n = \left(\sum_{i=1}^{n} \theta_i^2\right)^{\frac{1}{2}+\varepsilon}$.

Theorem 4. *Assume that $\{\xi_{n+1}\}_{n\in\mathbb{N}}$ are independent random variables and the conditions*

$$\sum_{i=1}^{n} \theta_i^2 \to \infty, \tag{19}$$

$$b_n = \left(\sum_{i=1}^{n} \theta_i^2\right)^{\frac{1}{2}+\varepsilon} \le C \left|\sum_{i=1}^{n} a_i\right| \tag{20}$$

are fulfilled for some $\varepsilon > 0$ and C. Let $\{X_n\}_{n\in\mathbb{N}}$ be a solution to equation (2). Then a) $\lim_{n\to+\infty} X_n = 0$ holds $\mathbb{P}$-a.s. if and only if

$$\sum_{i=1}^{n} a_i \to -\infty. \tag{21}$$

b) $\lim_{n\to+\infty} |X_n| = \infty$ holds $\mathbb{P}$-a.s. if and only if

$$\sum_{i=1}^{n} a_i \to \infty. \tag{22}$$

We note that in the case when $\{\xi_{n+1}\}_{n\in\mathbb{N}}$ are identically distributed with mean $a \neq 0$ and variance $\theta^2 \neq 0$, conditions (19) and (20) are obviously fulfilled.

Now we consider the non-homogeneous equation (12) in the case when $\{\xi_{n+1}\}_{n\in\mathbb{N}}$ are not identically distributed. It can be proved that if condition (19) is fulfilled and there exists $\varepsilon > 0$ and $\delta > 0$ such that

$$\delta b_n + \sum_{i=1}^{n} a_i \to -\infty \tag{23}$$

monotonely and

$$\sum_{i=0}^{\infty} |S_i| e^{\varepsilon b_i} < \infty, \tag{24}$$

then $\lim_{n\to+\infty} X_n = 0$ holds $\mathbb{P}$-a.s. for the solution $\{X_n\}_{n\in\mathbb{N}}$ to equation (12).

It is worth noting that, when $\{\xi_{n+1}\}_{n\in\mathbb{N}}$ are not identically distributed, condition (24) on the growth of S_n can be less restrictive than condition (14). This will happen if $\theta_i \to 0$, that is the noise is getting smaller which compensates for the fact that S_i are allowed to be larger.

References

[1] Arnold L., *Stochastic Differential Equations: Theory and Applications*, Wiley, 1974.

[2] Higham D. J., *Mean-square and asymptotic stability of the stochastic Theta method*, SIAM J. Numer. Anal., **3**8 (2001), No. 3, 753–769.

[3] Kolmanovskii V. and Shaikhet L., *General method of Lyapunov functionals construction for stability investigation of stochastic difference equations*, in: Dynamical Systems and Applications, World Sci. Ser. Appl. Anal. **4**, World Sci. Publishing, 1995, pp. 397–439.

[4] Kolmanovskii V. and Shaikhet L., *Some peculiarities of the general method of Lyapunov functionals construction*, Appl. Math. Lett., **1**5 (2002), No. 3, 355–360.

[5] Liptser R. Sh. and Shiryaev A. N., *Theory of Martingales*, Kluwer Academic Publishers, Dordrecht, 1989.

[6] Mao X., *Stochastic Differential Equations & Applications*, Horwood Publishing, Chichester, 1997.

[7] Neveu J., *Discrete-Parameter Martingales*, North-Holland & American Elsevier, 1975.

[8] Protter P., *Stochastic Integration and Differential Equations*, Springer, 1990.

[9] Rodkina A., *On asymptotic behaviour of solutions of stochastic difference equations*, in: Proceedings of the Third World Congress of Nonlinear Analysts (WCNA, Catania, 2000). Nonlinear Analysis, **4**7 (2001), 4719–4730.

[10] Rodkina A., Mao X. and Kolmanovskii V., *On asymptotic behaviour of solutions of stochastic difference equations with Volterra type main term*, Stochastic Anal. Appl., **1**8 (2000), No. 5, 837–857.

[11] Rodkina A. and Schurz H., *On global asymptotic stability of solutions to some in-arithmetic-mean-sense monotone stochastic difference equations in* $\mathbb{R}^1$, Journal of Numerical Analysis and Modelling, (to appear).

[12] Rodkina A. and Schurz H., *Global asymptotic stability of solutions to cubic stochastic difference equations*, Advances in Difference Equations, **3** (2004), 249–260.

[13] Schurz H., *Numerical analysis of SDEs without tears*, in: Handbook of Stochastic Analysis and Applications (D. Kannan, V. Lakshmikantham, eds.), Marcel Dekker, 2002, pp. 237–359.

[14] Schurz H., *Stability of numerical methods for SDEs along Lyapunov-type and other functions with variable step sizes*, in: Lecture Notes, Conference on Numerics for SPDEs (Edinburgh, April 2003), Preprint m-03-025, Department of Mathematics, Southern Illinois University, Carbondale, 2003, 1–23.

[15] Shiryaev A. N., *Probability* (2nd edition), Springer, Berlin, 1996.