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On the Number of Large Families in a Branching Population

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Abstract

In the paper a method for investigation of some functionals of branching stochastic processes related to their genealogy will be discussed. The method is based on the construction and study of a special form of random sum of independent vectors of indicator functions. Using the results obtained for the random sum, limit distributions for the number of large families (generalized reduced processes) and for the number of excesses generated by productive ancestors in large populations have been found. This problem is considered for discrete time single and multi-type processes in critical, subcritical and supercritical cases. Possibilities of applications of the method in more general models of branching processes will also be discussed.

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1 Introduction

We consider a sequence of random vectors which is defined as following. Let $\{\xi_{ij}(k,m),\ j \geq 1\},\ i = 1,2,\ldots,n$, for any pair $(k,m) \in \mathbb{N}_0^2,\ \mathbb{N} = \{1,2,\ldots\},\ \mathbb{N}_0 = \{0\}\bigcup\mathbb{N}$, be n independent sequences of random variables and $\{\nu_{ik},\ k \in \mathbb{N}_0\},\ i = 1,2,\ldots,n$, be n sequences of (not necessarily independent) random variables taking values $0,1,\ldots$ and independent of the family $\{\xi_{ij}(k,m)\}$. We consider the family of random vectors

$$\mathbf{W}(k,m) = (W_1(k,m),\ldots,W_n(k,m)), \quad W_i(k,m) = \sum_{j=1}^{\nu_{ik}} \xi_{ij}(k,m). \qquad (1)$$

Assume that $\xi_{ij}(k,m),\ j = 1, 2, \ldots$, for any fixed k, m and i are independent and identically distributed Bernoulli random variables with parameter $P_{km}^{(i)}$ (*i.e.*, have distribution $b(1, P_{km}^{(i)})$).

We shall study the asymptotic behavior of $\mathbf{W}(k,m)$ as $k, m \to \infty$ under some assumptions on the random variables ν_{ik} and $\xi_{ij}(k,m)$ in different cases of relationship between the parameters k and m.

Random sums of independent random variables or random vectors have been considered by many authors. First it is because of the interest in extending classic limit theorems of the probability theory to a more general situation and to discover new properties of the random sums caused by "randomness" of the number of summands. On the other hand, many problems in different areas of probability can be connected with a sum of random number of random variables. A rather full list of publications on random sums can be found in the recent monograph by Gnedenko and Korolev (1996). Transfer theorems for the random sum of independent random variables can also be seen in Gnedenko (1997).

The relationship between random sums and branching stochastic processes is well known. Starting from early studies (see Harris (1966), for example) including the recent publications, the fact that the number of particles in a model of branching process can be represented as a random sum has been mentioned. Some of the investigations show that using this relationship in the study of branching models makes it possible to investigate new variables related to the genealogy of the process, to study more general modifications of branching processes and to consider different characteristics of the process from a unique point of view. So, limit distributions for the number of pairs of individuals at time τ having the same number of descendants at time $t,\ t > \tau$ are found in [7]. A more general variable of this kind, describing the number of individual pairs having "relatively close" number of descendants was considered in the paper [8] (see also [9, Ch. IV]). Using this relationship limit theorems for different models of branching processes with immigration which may depend on the reproduction processes of particles are also proved. This kind of problems are systematically studied in the above mentioned monograph [9]. Investigations of the maximum family size in a population by Arnold and Villasenor (1996), Rahimov and Yanev (1999) and by Yanev and Tsokos (2000) are also based on this kind of a relationship.

Here we consider the relationship of the random sum of random vectors and multitype branching processes. Although $\mathbf{X}(t)$, the number of individuals of different types at time t, is the main object of investigation in the theory of multitype branching processes, there are many other variables related to the population which are of interest as well. One example of such a variable is the time to the closest common ancestor of the entire population observed at certain time. For a single-type Galton-Watson process this variable was considered by Zubkov (1975), who proved that, if the process is critical, the time is asymptotically uniformly distributed. Later, it turned out that the time to the closest common ancestor may be treated as a functional of the so called reduced branching processes. This process was introduced by Fleischmann and Siegmund-Schultze (1977) as a process that counts only individuals at a given time τ having descendants at time $t,\ t > \tau$. They demonstrated that in the critical single-type case the reduced process can well be approximated by a nonhomogeneous pure birth process. Later a number of studies extended their results to general single and multitype models of branching processes (see [14, 15] and [12], for example).

It turned out that, if one uses theorems proved for the random sums defined in (1), one may study a generalized model of multitype reduced processes. Let $\boldsymbol{\theta}(t) = (\theta_1(t), \ldots, \theta_n(t))$ be a vector of nonnegative functions, τ and $t, \tau < t$ be two times of observation. We define process $\mathbf{X}(\tau, t) = (X_1(\tau, t), \ldots, X_n(\tau, t))$, where $X_i(\tau, t)$ is the number of type T_i individuals at time τ, whose number of descendants at time t of at least one type is greater than corresponding level, given by vector $\theta(t - \tau)$. It is clear that $\mathbf{X}(\tau, t)$ counts only "relatively productive" individuals at time τ. We also note that $\mathbf{X}(\tau, t)$ is a usual n-type reduced process if $\boldsymbol{\theta}(t) = \mathbf{0}$ for all $t \in \mathbb{N}_0$. In the project limit distributions for process $\mathbf{X}(\tau, t)$ as $t, \tau \to \infty$ in different cases of relationship between observation times τ and t for critical processes have been obtained. For single type processes limit theorems are also proved in subcritical and supercritical cases. Asymptotic behavior of expected number of individuals counted in generalized reduced process is also studied. Possibilities of applications in more general models branching processes and in other processes are investigated.

Here we consider as a large family one, whose number of members of at least one type is greater than a given level. One may consider the reduced processes with different sets of types effecting productivity of the individuals. For instance, a family is large, if the number of its members of types $T_i,\ i \in \nabla$, is greater than a given level. Here ∇ is a subset of $\{1, 2, \ldots, n\}$. Another example is the case when the sum of the numbers of its members of some types is large. Further investigations in this direction are in the progress. Below we provide main results obtained in this work without a proof. The proofs will be published elsewhere.

2 Results and discussion

2.1 Convergence of the random sum

For n-dimensional vectors $\mathbf{x} = (x_1, \ldots, x_n),\ \mathbf{y} = (y_1, \ldots, y_n)$ we denote $\mathbf{x} \oplus \mathbf{y} = (x_1 y_1, \ldots, x_n y_n),\ \mathbf{x}^{\mathbf{y}} = (x_1^{y_1}, \ldots, x_n^{y_n}),\ \mathbf{x}/\mathbf{y} = (\frac{x_1}{y_1}, \cdots, \frac{x_n}{y_n}),\ (\mathbf{x}, \mathbf{y}) = x_1 y_1 + \cdots +$

$x_n y_n$, $\sqrt{\mathbf{x}} = (\sqrt{x_1}, \ldots, \sqrt{x_n})$, and $\mathbf{x} \geq \mathbf{y}$ or $\mathbf{x} > \mathbf{y}$ if $x_i \geq y_i$ or $x_i > y_i$ respectively.

The first theorem obtained in the paper concerning the vector (1) covers the case when normalized vector $\boldsymbol{\nu}_k = (\nu_{ik}, i = 1, \ldots, n)$ has a limit distribution. Namely we assume that there exists a sequence of positive vectors $\mathbf{A_k} = (A_{ik}, i = 1, \ldots, n)$ such that $A_{ik} \to \infty, k \to \infty$,

$$\left\{ \frac{\boldsymbol{\nu}_\mathbf{k}}{\mathbf{A_k}} \middle| \boldsymbol{\nu}_\mathbf{k} \neq \mathbf{0} \right\} \to \mathbf{Y} = (Y_1, \ldots, Y_n) \tag{2}$$

in distribution and for the vector $\mathbf{P}(k, m) = (P_{km}^{(i)},\ i = 1, \ldots, n)$

$$\mathbf{P}(k, m) \oplus \mathbf{A_k} \to \mathbf{a} = (a_1, \ldots, a_n), \tag{3}$$

where the components of the vector $\mathbf{a}$ may be $+\infty$.

Theorem 1 *If conditions* (2) *and* (3) *are satisfied, then*

$$\left\{ \frac{\mathbf{W}(k, m)}{\mathbf{P}(k, m) \oplus \mathbf{A}_k} \middle| \nu_k \neq \mathbf{0} \right\} \to \mathbf{W}$$

in distribution and $Ee^{(\lambda, \mathbf{W})} = \varphi(\lambda^*)$, *where* $\varphi(\lambda)$ *is the Laplace transform of the vector* $\mathbf{Y}$, $\lambda^* = (\lambda_1^*, \ldots, \lambda_n^*)$ *and* $\lambda_i^* = \lambda_i$ *if* $a_i = \infty$, *and* $\lambda_i^* = a_i(1 - e^{-\lambda_i / a_i})$ *if* $a_i < \infty$.

The family of vectors (1) is eventually a sum of independent vectors if the vectors $\nu_k = (\nu_{ik},\ i = 1, \ldots, n)$ have degenerate distributions. Therefore one may expect to obtain a normal limit distribution under some natural assumptions. The next theorem obtains the conditions under which the limit of vector $\mathbf{W}(k, m)$ is a mixture of the normal and a given distribution. Assume

C1. For a given sequence of positive vectors $\mathbf{A}_k$ there exists a sequence $\mathbf{l}_k = (l_{ik},\ i = 1, \ldots, n)$, $k \geq 1$ such that $A_{ik}/l_{ik} \to \infty$, $k \to \infty$, $i = 1, \ldots, n$, and

$$\mathbf{l}_k \oplus \mathbf{P}(k, m) \oplus (1 - \mathbf{P}(k, m)) \to \mathbf{C}$$

as $k, m \to \infty$, where $\mathbf{C} = (C_i,\ i = 1, \ldots, n)$ is a positive vector of constants.

Theorem 2 *If conditions* (2) *and C1 are satisfied, then*

$$\left\{ \frac{\mathbf{W}(k, m) - \nu_{ik} \oplus \mathbf{P}(k, m)}{\sqrt{\mathbf{A}_k \oplus \mathbf{C}/\mathbf{l}_k}} \middle| \nu_k \neq \mathbf{0} \right\} \to \mathbf{W}$$

as $k, m \to \infty$, *where*

$$P\{\mathbf{W} \leq \mathbf{x}\} = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n \Phi\left(\frac{x_i}{\sqrt{y_i}}\right) dT(y_1, \ldots, y_n),$$

$\Phi(x)$ *is the standard normal distribution and* $T(x_1, \ldots, x_n)$ *is the distribution of the vector* $\mathbf{Y}$ *in* (2).

2.2 Construction of the generalized reduced process

Now we give a rigorous definition of the generalized reduced process $\mathbf{X}(\tau, t)$. We use the following notation for individuals participating in the process. Let the process start with a single ancestor at time $t = 0$ of type $T_i,\ i = 1, \ldots, n$. We denote it by T_i and consider as zero-th generation. The direct offspring of the initial ancestor we denote as (T_i, T_j, m_j), where $T_j,\ j = 1, \ldots, n$ is the type of the direct descendant and $m_j \in \mathbb{N},\ \mathbb{N} = \{1, 2, \ldots\}$ is the label (the number) of the descendant in the set of all immediate descendants of T_i. Thus the m_{k+1}-st direct descendant of the type $T_{i_{k+1}}$ of the individual $\alpha = (T_i, T_{i_1}, m_1, \ldots, T_{i_k}, m_k)$ will be denoted as $\alpha' = (\alpha, T_{i_{k+1}}, m_{k+1})$. Here and later on for any two vectors $\alpha = (i_1, \ldots, i_k)$ and $\beta = (j_1, \ldots, j_m)$ we will understand the ordered pair (α, β) as the $k+m$ dimensional vector $(i_1, \ldots, i_k, j_1, \ldots, j_m)$.

If we use the above notation, the set $\Re_t \in E$, where E is the space of all finite subsets of

$$\bigcup_{k=1}^{\infty} N_1^k, N_1^k = N_1^{k-1} \times N_1, \quad N_1 = \{T_i\} \times \{T_1, \ldots, T_n\} \times \mathbb{N},$$

corresponds to the population of the t-th generation. It is clear that $\Re_t$ can be decomposed as $\Re_t = \cup_{i=1}^n \Re_t^{(i)}$, where $\Re_t^{(i)}$ is the population of the type T_i individuals of the t-th generation. Consequently, the components of the process $\mathbf{X}(t)$ are found as $X_i(t) = \text{card}\,\{\Re_t^{(i)}\},\ t \in \mathbb{N}_0$, and for any τ and t such that $\tau < t$ we have

$$\mathbf{X}(t) = \sum_{i=1}^{n} \sum_{\alpha \in \Re_\tau^{(i)}} \mathbf{X}^{(\alpha)}(t - \tau),$$

where $\mathbf{X}^{(\alpha)}(t) = (X_1^{(\alpha)}(t), \ldots, X_n^{(\alpha)}(t))$ is the n-type branching process generated by the individual α.

Let $\Im_i([\boldsymbol{\theta}], \tau, t)$ be the set of individuals in $\Re_\tau^{(i)}$ having at least one type of descendants at time t more than the corresponding component of $\boldsymbol{\theta}(t - \tau)$. It is not difficult to see that it can be described as follows:

$$\Im_i([\boldsymbol{\theta}], \tau, t) = \{\alpha \in \Re_\tau^{(i)} :\ \text{for at least one } j\ \exists\ \text{ more than } \theta_j(t - \tau)$$

$$\beta \text{ -sets such that } (\alpha, \beta) \in \Re_t^{(j)}\},$$

where $\alpha \in N_1^\tau,\ \beta \in N_1^{t-\tau}$. Thus the generalized reduced process is defined as $\mathbf{X}(\tau, t) = (X_i(\tau, t), i = 1, ..., n)$ with $X_i(\tau, t) = \text{card}\{\Im_i([\boldsymbol{\theta}], \tau, t)\}$.

In particular, if $\theta(t) = \mathbf{0}$ for all t, then $\Im_i([\mathbf{0}], \tau, t)$ contains all individuals of type T_i only living in the τ-th generation and having descendants (at least of one type) in generations $\tau+1, \tau+2, \ldots, t$. Consequently, in this case $\mathbf{X}(\tau, t),\ 0 < \tau < t$, is the n-type usual reduced branching process.

It follows from the definition of the reduced process that $\mathbf{X}(\tau, t),\ 0 < \tau < t$, is the number of productive individuals at time τ who have a large number of descendants at time t. Let us now consider families in the population $\Re_t$ at time t. A family is the set of individuals (of all types) at time t who have a common ancestor at time τ. The family is large, if the number of its members of at least one type is greater than the value of corresponding function in the "level vector" $\boldsymbol{\theta}(t-\tau)$. Then, it is clear that $\mathbf{X}(\tau, t)$ is the vector describing the number of "large families" in $\Re_t$ where $X_i(\tau, t)$ being generated by an individual of type T_i in $\Re_\tau$.

2.3 Limit behavior of the reduced process

We denote by $P^i_\alpha, \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0$, the offspring distribution of the process $\mathbf{X}(t)$, *i.e.*,

$$P^i_\alpha = P\{\mathbf{X}(1) = \alpha | \mathbf{X}(0) = \boldsymbol{\delta}_i\}$$

is the probability that an individual of type T_i generates the total number $\boldsymbol{\alpha}$ of new individuals. Here $\delta_i = (\delta_{ij},\ j = 1, \ldots, n)$, $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$. We also denote

$$F^i(\mathbf{S}) = \sum_{\alpha \in \mathbb{N}^n_0} P^i_\alpha S_1^{\alpha_1} \cdots S_n^{\alpha_n}, \quad \mathbf{F}(\mathbf{S}) = (F^1(\mathbf{S}), \ldots, F^n(\mathbf{S})),$$

$$Q^i(t) = P\{\mathbf{X}(t) \neq \mathbf{0} | \mathbf{X}(0) = \delta_i\}, \quad \mathbf{Q}(\mathbf{t}) = (Q^1(t), \ldots, Q^n(t)).$$

Let for $i, j, k = 1, 2, \ldots, n$,

$$a^j_i = \frac{\partial F^j(\mathbf{S})}{\partial S_i} |_{\mathbf{S}=\mathbf{1}}\ , \qquad b^j_{ik} = \frac{\partial^2 F^j(\mathbf{S})}{\partial S_i \partial S_k} |_{\mathbf{S}=\mathbf{1}}\ ,$$

$\mathbf{A} = \left\| a^j_i \right\|$ be the matrix of expectations, ρ be its Perron root and the right and the left eigenvectors $\mathbf{U} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{V} = (v_1, v_2, \ldots, v_n)$ corresponding to the Perron root be such that

$$\mathbf{AU} = \rho \mathbf{U}, \quad \mathbf{VA} = \rho \mathbf{V}, \qquad (\mathbf{U}, \mathbf{V}) = 1, \quad (\mathbf{U}, \mathbf{1}) = 1.$$

If $\mathbf{A}$ is indecomposable, aperiodic and $\rho = 1$, the process $\mathbf{X}(t)$ is called critical indecomposable multitype branching process. We assume that the generating function $\mathbf{F}(\mathbf{S})$ satisfies the following representation

$$x - \sum_{j=1}^{n} v_j(\mathbf{1} - F^j(\mathbf{1} - \mathbf{U}x)) = x^{1+\alpha} L(x), \tag{4}$$

where $0 < x \leq 1$, $\alpha \in (0,1]$, and $L(x)$ is a slowly varying function as $x \downarrow 0$. Note that in this case $\rho = 1$, *i.e.*, the process is critical and the second moments of the offspring distribution b_{ik}^j, $i, j, k = 1, \ldots, n$, may not be finite. Under this assumption the following limit theorem for the process $\mathbf{X}(t)$ holds (see Vatutin (1977)).

Proposition 1 *If the offspring generating function* $\mathbf{F}(\mathbf{S})$ *satisfies representation* (4)*, then we have*
a)

$$Q^i(t) \sim u_j t^{-1/\alpha} L_1(t)$$

as $t \to \infty$, *where* $L_1(t)$ *is a slowly varying as* $t \to \infty$ *function;*
b)

$$\lim_{t\to\infty} P\left\{\mathbf{X}(t)q(t) \leq \mathbf{x} \oplus \mathbf{V} \mid \mathbf{X}(t) \neq \mathbf{0}, \mathbf{X}(0) = \boldsymbol{\delta}_i\right\} = \pi(\mathbf{x}),$$

where $q(t) = \sum_{j=1}^n v_j Q^j(t)$ *and* $\pi(\mathbf{x}) = \pi(x_1, x_2, \ldots, x_n)$ *is a distribution having the Laplace transform*

$$\phi(\boldsymbol{\lambda}) = \int_{\mathbb{R}_+^n} e^{-(\mathbf{x},\lambda)} d\pi(\mathbf{x}) = 1 - (1 + \bar{\lambda}^{-\alpha})^{-1/\alpha}, \quad \bar{\lambda} = (\boldsymbol{\lambda}, \mathbf{1}). \tag{5}$$

Now we are in a position to state our first result about $\mathbf{X}(\tau, t)$. Let $\boldsymbol{\theta} = (\theta_1, ..., \theta_n) \in \mathbb{R}_+^n$, $\mathbb{R}_+ = [0, \infty)$, $\mathbf{C} = (C_1, \ldots, C_n) \in \mathbb{R}_+^n$ be some nonnegative vectors.

Theorem 3 *If condition* (4) *is satisfied,* $\theta(t) = \theta \oplus \mathbf{V}/q(t)$ *and* $t, \tau \to \infty$, $t - \tau \to \infty$ *so that* $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau) \to \mathbf{C}$, *then*

$$P\{\mathbf{X}(\tau, t) = \mathbf{k} | \mathbf{X}(\tau) \neq \mathbf{0},\ \mathbf{X}(0) = \delta_i\} \to P_{\mathbf{k}}^*,$$

where $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ *and the probability distribution* $\{P_{\mathbf{k}}^*,\ \mathbf{k} \in \mathbb{N}_0^n\}$ *has the generating function* $\phi^*(\mathbf{S}) = \phi(\mathbf{a})$ *with* $\mathbf{a} = b\mathbf{C} \oplus \mathbf{U} \oplus \mathbf{V} \oplus (\mathbf{1} - \mathbf{S})$, $b = 1 - \pi(\theta)$, $\mathbf{S} = (S_1, \ldots, S_n)$ *and* $\phi(\boldsymbol{\lambda})$ *is the Laplace transform defined in* (5)*.*

Remark It is clear that the vector $\mathbf{C}$ in the condition $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau) \to \mathbf{C}$ necessarily has the form $\mathbf{C} = C\mathbf{1}$, where $C \geq 0$ is some constant.

Example 1 Let $\mathbf{F}(\mathbf{S})$ satisfy condition (4) with $\alpha = 1$. We shall note here that in this case the second moments of the offspring distribution still may be infinite. For this kind of a process the limit distribution $\pi(\theta)$ is exponential and the generating function $\phi^*(\mathbf{S})$ has the form $\phi^*(\mathbf{S}) = (1+d)^{-1}$, where $d = bC\sum_{j=1}^n u_j v_j(1-S_j),\ b = e^{-\theta^*},\ \theta^* = \min\{\theta_1, \dots, \theta_n\}$. We represent it as follows

$$\phi^*(\mathbf{S}) = \frac{1}{1+Ce^{-\theta^*}}\left(1 - \frac{Ce^{-\theta^*}}{1+Ce^{-\theta^*}}\sum_{i=1}^n u_i v_i S_i\right)^{-1}. \tag{6}$$

What is the distribution having the last probability generating function? To answer this question we consider a sequence of independent random variables $X_1, X_2, \dots$ such that $P\{X_i = j\} = p_j,\quad j = 0, 1, 2, \dots, n,\quad \sum_{j=0}^n p_j = 1$, where $p_0 = (1+Ce^{-\theta^*})^{-1},\quad p_j = Ce^{-\theta^*}u_j v_j/(1+Ce^{-\theta^*}),\quad j = 1, 2, ..., n$. Let Δ_1 be the number of $1's$, Δ_2 be the number of $2's$ and so on Δ_n be the number of $n's$ observed in the sequence $X_1, X_2, \dots$ before the first zero is obtained. Then it follows from the formula for the generating function of generalized multivariate geometric distribution in Ch. 36.9 of Johnson *et al.* (1997) that the vector $(\Delta_1, \dots, \Delta_n)$ has the probability generating function given by (6), *i. e.*,

$$E\left(S_1^{\Delta_1} S_2^{\Delta_2} \cdots S_n^{\Delta_n}\right) = \phi^*(\mathbf{S}).$$

Hence we have the following result.

Corollary 1 *If the assumptions of Theorem 3 are satisfied with* $\alpha = 1$*, then the probability distribution* $\{P^*_{\mathbf{k}},\ \mathbf{k} \in \mathbb{N}_0^n\}$ *is a multivariate geometric distribution defined by the generating function (5) such that*

$$P^*_{\mathbf{k}} = P\{\Delta_i = k_i,\ i = 1, \dots, n\}.$$

It is clear that, if $n = 1$, the distribution is geometric, *i.e.*, $P^*_k = pq^k, k = 0, 1, \dots$ with $p = (1+Ce^{-\theta_1})^{-1},\ q = Ce^{-\theta_1}(1+Ce^{-\theta_1})^{-1}$.

Example 2 Let the assumptions of Theorem 3 be satisfied and $\tau = [\varepsilon t],\ 0 < \varepsilon < 1$. Using the asymptotic behavior of $\mathbf{Q}(\mathbf{t})$ and the uniform convergence theorem for the slowly varying functions we obtain that as $t \to \infty$

$$\frac{\mathbf{Q}(t-\tau)}{\mathbf{Q}(\tau)} \to \left(\frac{\varepsilon}{1+\varepsilon}\right)^{1/\alpha}\mathbf{1}.$$

Consequently, in this case the limit distribution has the generating function $\phi^*(\mathbf{S})$ with $C = (\varepsilon/(1+\varepsilon))^{1/\alpha}$. In particular, we have the following result.

Corollary 2 *If the assumptions of Theorem 3 are satisfied and* $\tau = o(t)$*, then*

$$\lim_{t\to\infty} P\{\mathbf{X}(\tau, t) = \mathbf{k} | \mathbf{X}(\tau) \neq \mathbf{0}, \mathbf{X}(0) = \delta_i\} = 0$$

for all $\mathbf{k} \in \mathbb{N}_0^n$ *and* $\mathbf{k} \neq \mathbf{0}$.

It is known that in the critical case the process $\mathbf{X}(t)$ goes to extinction with probability 1. Corollary 2 shows that, if $\tau = o(t)$, even a conditioned process $\mathbf{X}(\tau,t)$ given $\mathbf{X}(\tau) \neq \mathbf{0}$ vanishes with a probability approaching 1.

Theorem 3 gives a limit distribution for $\mathbf{X}(\tau,t)$ when the times of observation $\tau \to \infty$ and $t \to \infty$ so that $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau)$ has a finite limit. Now we consider the case when this limit is not finite. Let $T_i(\tau,t) = Q^i(t-\tau)/Q^i(\tau)$ and $\mathbf{T}(\tau,t) = (T_1(\tau,t),\ldots,T_n(\tau,t))$.

Theorem 4 *If condition* (4) *holds,* $\theta(t) = \theta \oplus \mathbf{V}/q(t)$ *and* $t,\tau \to \infty,\ t-\tau \to \infty$ *so that* $T_i(\tau,t) \to \infty,\ i = 1,2,\ldots,n,$ *then*

$$P\left\{\frac{\mathbf{X}(\boldsymbol{\tau},\mathbf{t})}{\mathbf{T}(\boldsymbol{\tau},\mathbf{t})} \leq \mathbf{x} | \mathbf{X}(\tau) \neq \mathbf{0}, \mathbf{X}(0) = \delta_i\right\} \to \pi(\frac{1}{b}\mathbf{x}),$$

where $\pi(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}_+^n$, *is the distribution from Proposition 2 and* $b = 1 - \pi(\boldsymbol{\theta})$.

Remark It follows from the asymptotic behavior of $Q^i(t)$ that, if $T_i(\tau,t) \to \infty$ for at least one i, then it holds for each $i = 1,2,\ldots,n$.

Example 3 If the matrix $\mathbf{A}$ is indecomposable, aperiodic, $\rho = 1$ and $b^i_{jk} < \infty,\ i,j,k = 1,\ldots,n$, then (4) is satisfied with $\alpha = 1,\ L(x) \to \text{const},\ x \to 0$. In this case $Q^i(t) \sim 2u_i/\sigma^2 t, \quad i = 1,\ldots,n$ as $t \to \infty$, where $\sigma^2 = \sum_{j,m,k=1}^n v_j b^j_{mk} u_m u_k$. Consequently,

$$q(t) = \sum_{j=1}^n Q^j(t) v_j \sim \frac{2}{\sigma^2 t}, \quad t \to \infty$$

and $\boldsymbol{\theta}(t) \sim \sigma^2 t \boldsymbol{\theta} \oplus \mathbf{V}/2$. On the other hand, $b = e^{-\theta^*}, \theta^* = \min\{\theta_1,\ldots,\theta_n\}$ and $T_j(\tau,t) \sim \tau/(t-\tau),\ j = 1,\ldots,n$. Thus $T_j(\tau,t) \to \infty$ if, for example, $\tau \sim t$ and we obtain the following result from Theorem 4.

Corollary 3 *If* $\rho = 1,\ 0 < \sigma^2 < \infty$ *and* $t,\tau \to \infty,\ t-\tau \to \infty$ *so that* $\tau \sim t$, *then*

$$P\left\{\frac{t-\tau}{\tau}\mathbf{X}(\tau,t) \leq \mathbf{x} | \mathbf{X}(\tau) \neq \mathbf{0}, \mathbf{X}(0) = \delta_i\right\} \to 1 - \exp\{-\frac{x^*}{b^*}\},$$

where $\mathbf{x} \in \mathbb{R}_+^n,\ x^* = \min\{x_1,\ldots,x_n\}, \quad b^* = \exp\{-\min\{\theta_1,\ldots,\theta_n\}\}$.

The above two theorems describe the asymptotic behavior of $\mathbf{X}(\tau,t)$ when $t - \tau \to \infty$. Now we consider the case $\tau = t - \Delta$, where $\Delta \in (0,\infty)$ is a constant.

Theorem 5 *If condition* (4) *is satisfied,* $t, \tau \to \infty$ *so that* $t - \tau = \Delta \in (0, \infty)$ *and* $\boldsymbol{\theta}(\mathbf{t}) = \boldsymbol{\theta} = (\theta_1, ..., \theta_n) \in \mathbb{R}_+^n$*, then*

$$P\{\mathbf{X}(\tau, t) \oplus \mathbf{Q}(\tau) \le \mathbf{x} | \mathbf{X}(\tau) \ne \mathbf{0}, \mathbf{X}(0) = \delta_i\} \to \pi(\frac{\mathbf{x}}{\mathbf{R}(\Delta)}),$$

where $\mathbf{x} \in \mathbb{R}_+^n$ *and*

$$\mathbf{R}(\Delta) = (R^1(\Delta), \dots, R^n(\Delta)),\ R^i(\Delta) = P\{\bigcup_{j=1}^{n} \{X_j(\Delta) > \theta_j\} | \mathbf{X}(0) = \delta_i\}.$$

Remark It follows from Proposition 1 that

$$\frac{Q^i(\tau)}{Q^i(t)} \sim \left(\frac{t}{t - \Delta}\right)^{1/\alpha} \frac{L_1(t - \Delta)}{L_1(t)}$$

which shows that $Q^i(\tau) \sim Q^i(t)$ as $t, \tau \to \infty,\ t - \tau = \Delta$, for each $i = 1, \dots, n$. Therefore the vector of normalizing functions $\mathbf{Q}(\tau)$ in Theorem 5 can be replaced by $\mathbf{Q}(t)$.

2.4 The number of productive ancestors

Now we consider a population containing at time $t = 0$ a random number $\nu_i(t),\ i = 1, \dots, n,\ t \in \mathbb{N}_0$, of individuals (ancestors) of n different types $T_1, \dots, T_n$. Each of these individuals generates a discrete time indecomposable n-type branching stochastic process. Let $\theta(t) = (\theta_1(t), \dots, \theta_n(t))$ be a vector of nonnegative functions. In how many processes generated by these ancestors the number of descendants at time t of at least one type will exceed the corresponding level given by $\theta(t)$? To answer the question we investigate the process $\mathbf{Y}(t) = \mathbf{Y}([\theta], t) = (Y_1(t), \dots, Y_n(t))$, where $Y_i(t)$ is the number of initial individuals of type T_i whose number of descendants at time t of at least one type is greater than the corresponding component of the vector $\theta(t)$. It is clear that $\mathbf{Y}(t)$ takes into account only "relatively productive" ancestors regulated by the family of levels $\theta(t),\ t \in \mathbb{N}_0$.

A process $\mathbf{Y}(t)$ may be associated with the following scheme describing the growth of n-type trees in a forest. Suppose at time zero we have $\nu_i(t),\ i = 1, \dots, n$, one-branch trees of types T_i. Each of these trees will grow and give new branches of types $T_1, \dots, T_n$ according to independent, indecomposable n-type branching processes. Then the process $\mathbf{Y}(t) = (Y_1(t), \dots, Y_n(t))$ will count the number of "big trees": $Y_i(t)$ is the number of big trees of type T_i having more than $\theta_j(t)$ new branches at time t for at least one $j,\ j = 1, \dots, n$.

It is not difficult to see that the components of the process $Y_i(t)$ can be presented as

$$Y_i(t) = \sum_{j=1}^{\nu_i(t)} \xi_{ij}(t), \tag{7}$$

where $\xi_{ij}(t) = \chi(\bigcup_{l=1}^{n}\{X_{il}^{j}(t) > \theta_l(t)\})$ and $X_{il}^{j}(t)$ is, as before, the number of individuals of type T_l at time t in the process initiated by the j-th ancestor of type T_i. Consequently, theorems proved for the random sum (1) may be applied to this process.

Let all assumptions from Part §2.3 on n-type branching process $\mathbf{X}(t),\ t \in \mathbb{N}_0$, be satisfied and the generating function corresponding to the probability distribution $P_{\alpha}^{i},\ \alpha \in \mathbb{N}_0^n$, satisfy equation (4).

Theorem 6 *Let condition* (4) *be satisfied and* $\theta(t) = \theta \oplus \mathbf{V}/q(t),\ \theta \in \mathbb{R}_+^n$. *If condition* (2) *is satisfied and for the normalizing coefficients in* (2)

$$A_{it}Q^i(t) \to \infty \tag{8}$$

as $t \to \infty$ *for* $i = 1, \ldots, n$, *then*

$$P\left\{ \frac{Y_i(t) - \nu_{it}a_i(t)}{\sqrt{\nu_{it}a_i(t)}} \le x_i,\ i = 1, ..., n \,\middle|\, \nu \neq \mathbf{0} \right\} \to L(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^n,\ a_i(t) = bQ^i(t),\ b = 1 - \pi(\theta),\ \theta \in \mathbb{R}_+^n$ *and* $L(\mathbf{x})$ *defined in Theorem 2.*

2.5 Noncritical processes

We now assume that the initial branching process $X(t)$ is single-type, *i.e.*, there are individuals of one type. Let the offspring distribution of a single individual be $p_k = P(X(1) = k \mid X(0) = 1)$ and

$$f(s) = E[s^{X(1)} \mid X(0) = 1], \quad f_{t+1}(s) = f(f_t(s)), \quad f_1(s) = f(s), \quad A = f'(1).$$

It is not difficult to see from the definition of the process $X(\tau, t)$ that it can be written as follows

$$X(\tau, t) = \sum_{j=1}^{X(\tau)} \chi\{X_j^{(\tau)}(t - \tau) > \theta(t - \tau)\}, \tag{9}$$

where $\chi(A)$ is the indicator of the event A and $X_j^{(\tau)}(t)$ is the process generated by the j-th individual existing at time τ. Hence we can apply theorems on the random sum $W_k^{(n)}$ to process $X(\tau, t)$ with $\nu_\tau = X(\tau)$ and

$$\xi_{j\tau}^{(t)} = \chi\{X_j^{(\tau)}(t-\tau) > \theta(t-\tau)\}.$$

The generating function $f(s)$ is strictly increasing, so it has an inverse $g(s)$. Let $g_t(s)$, $t \in \mathbb{N}$, be the t-th functional iteration of $g(s)$. Then for $q < s < 1$ we put $k_t(s) = -\ln g_t(s)$. It is well known that when $A > 1$, the following limit theorem holds for $X(t)$ (see Jagers (1975), pp. 31–34, for example).

Theorem A *Let $X(t)$ be a supercritical process, i.e., $A > 1$.*
a) There exists a sequence $\{k_t\}$ of positive numbers, which can be chosen as $k_t = k_t(s)$, $q < s < 1$, so that $k_t X(t)$ converges almost surely as $t \to \infty$ to a nondegenerate, finite and nonnegative random variable $W(s)$.
b) If $EX(1)\ln X(1) < \infty$, then k_t can be chosen as $k_t = A^{-t}$.
c) The Laplace transform of the limit random variable $B(\lambda) = Ee^{-\lambda W(s)}$, $\lambda \geq 0$, satisfies the equation $B(A\lambda) = f(B(\lambda))$.

Using limit theorems for W_n, we obtained the following results for supercritical single type processes.

Theorem 7 *Let $A > 1$, $\theta(t) = \theta/k_t$, $\theta \in (0,\infty)$, $k_\tau \to 0$. If $t, \tau \to \infty$ so that $t - \tau \to \infty$, then*

$$P\{X(\tau,t)k_\tau \leq x \mid X(\tau) > 0\} \to \pi\left(\frac{x}{P(W > \theta)}\right),$$

where the distribution $\pi(x)$ has the Laplace transform $(B(\lambda) - q)/(1-q)$ and $B(\lambda)$ satisfies the equation $B(A\lambda) = f(B(\lambda))$.

Now we consider the case when $t, \tau \to \infty$ but $t - \tau \equiv$ constant.

Theorem 8 *Let $A > 1$, $\theta(t) \equiv \theta \in (0,\infty)$. If $t, \tau \to \infty$ so that $t - \tau = t_0 \in (0,\infty)$, then*

$$P\{X(\tau,t)k_\tau \leq x \mid X(\tau) > 0\} \to \pi\left(\frac{x}{R(\theta,t_0)}\right),$$

where $R(\theta, t_0) = P(X(t_0) > \theta)$.

Example 1 Let the offspring distribution be positive geometric, *i.e.*, $p_k = p(1-p)^{k-1},\ k \geq 1,\ 0 < p < 1$ and $p_0 = 0$. It is clear that in this case the process is supercritical with

$$f(s) = \frac{sp}{1 - s(1-p)}, \quad A = p^{-1} > 1.$$

The equation for the Laplace transform in Theorem A will have the form

$$B(A\lambda) = \frac{pB(\lambda)}{1 - (1-p)B(\lambda)}.$$

It is not difficult to see that the function $B(\lambda) = p/(p+\lambda)$ satisfies the above equation. On the other hand, $q = 0,\ k_\tau = A^{-\tau}$ and the distribution $\pi(x)$ is exponential of the parameter p. Thus we have from Theorem 3 the following result in this case.

Corollary 1 *If the offspring distribution is positive geometric with parameter* $p,\ 0 < p < 1,\ \theta(t) = \theta p^{-\tau},\ \theta \in (0,\infty)$, *then*

$$P\{X(\tau,t)p^\tau \leq x \mid X(t) > 0\} \to 1 - \exp\{-pe^{\theta p}x\},\ x \geq 0.$$

Now we consider the subcritical case. In this case we use the following result which is known as Yaglom's theorem (see Jagers(1975), p. 29, for example).

Theorem B *If* $A < 1$, *then there exists*

$$\lim_{t\to\infty} P(X(t) = k \mid X(t) > 0) = P_k^*, \quad k \in \mathbb{N},$$

the probability generating function $F^*(s)$ *of* $\{P_k^*,\ k \in \mathbb{N}\}$ *satisfies the functional equation*

$$1 - F^*(f(s)) = A(1 - F^*(s)). \tag{10}$$

In this case limit theorems for random sums give the following result.

Theorem 9 *Let* $A < 1$ *and* $\theta(t) \equiv \theta \in [0,\infty)$.
a) If $t, \tau \to \infty$ *so that* $t - \tau \to \infty$, *then*

$$P\{X(\tau,t) = k \mid X(\tau) > 0\} \to 0$$

for each $k \geq 1$.
b) If $t, \tau \to \infty$ *so that* $t - \tau = t_0 \in (0,\infty)$, *then*

$$P\{X(\tau,t) = k \mid X(\tau) > 0\} \to q_k^*, \quad k \in \mathbb{N}_0$$

and $\{q_k^*,\ k \in \mathbb{N}_0\}$ *has the probability generating function* $F^*(1 - R(\theta,t_0)(1-s))$.

2.6 Behavior of the expected number of particles

Let $X(\tau, t)$ be a single-type generalized reduced process. Using identity (9) we obtained the following exact formula for the expected number of particles.

$$EX(\tau, t) = EX(\tau)P\{X(t-\tau) > \theta(t-\tau)\}. \tag{11}$$

The assumption (4) will have the following form in this case

$$f(s) = s + (1-s)^{1+\alpha}L(1-s), \tag{12}$$

The following results give the asymptotic behavior of the expectation.

Proposition 1 *If* (12) *is satisfied,* $\theta(t) = \theta/Q(t)$, $\theta \in [0, \infty)$, *and* $t, \tau \to \infty$, $t - \tau \to \infty$ *so that* $Q(t-\tau)/Q(\tau) \to C \in [0, \infty)$, *then*

$$EX(\tau, t) \sim (1 - \pi(\theta))C\frac{N(\tau)}{\tau^{1/\alpha}}$$

where $N(t)$ *is a slowly varying function such that* $N^{\alpha}(t)L(N(t)/t^{1/\alpha}) \to \alpha^{-1}$.

The next proposition gives the asymptotic behavior of the expectation in the case of supercritical processes.

Proposition 2 *Let* $A > 1$, $\theta(t) = \theta/k_t$, $\theta \in [0, \infty)$, *where* k_t *is the same sequence of normalizing constants from Theorem A.*
a) If $t, \tau \to \infty$, $t - \tau \to \infty$, *then*

$$EX(\tau, t) \sim A^{\tau}P\{W > \theta\}.$$

b) If $t, \tau \to \infty$, $t - \tau = \Delta \in (0, \infty)$, *then*

$$EX(\tau, t) \sim A^{\tau}R(\Delta, \theta).$$

Asymptotic formulas for the expectation in the subcritical case have been also obtained.

3 Applications

3.1 More general models of branching processes

So far we considered the subpopulation of productive individuals in a single or n-type Galton-Watson processes. It turned out that the methods developed here could be used in the study of more general models of branching processes. It can be seen from the proofs of the main theorems that the assumptions on the initial branching process lead us to use known results about asymptotic behavior of the "non-extinction" probability and limit theorems for the initial process. For branching models such as continuous-time Markov branching processes, Bellman-Harris processes, the Sevastyanov model and Crump-Mode-Jagers processes this kind of results are well known. So to study the generalized reduced processes for these models one needs slightly modify Theorems 1 and 2 and use known limit theorems for one or another model of branching processes. Thus we can conclude that our approach developed here allows to consider different models of branching processes from a unique point of view.

3.2 Other stochastic processes

The theorems obtained for random sums of indicators may also be used in other stochastic processes, when one needs to count the number of some events related to the process in a time interval. Let us consider a single server queue system in which the arrival and service times are independent and the queue discipline is first come first served. Let $X(t)$ be the number of customers in the queue at time t, ξ_{it}, $i = 1, 2, \ldots$, be the service times of these $X(t)$ customers. Then

$$\nu(t) = \sum_{i=1}^{X(t)} \chi(\xi_{it} > \theta(t))$$

is the number of customers which need a "long time" service, $\theta(t)$ is the minimum time required to serve each of those customers. It is clear that the results obtained here for random sums of indicators allow to study the process $\nu(t)$. It can be done in the cases of stationary and nonstationary service times.

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