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Blow up of Solutions for a Class of Nonlinear Wave Equations

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Abstract

In this work, we study the blow up of solutions to the initial boundary value problem for a class of nonlinear wave equations with a damping term.

1 Introduction

In this work, we study the blow up of solutions of initial boundary value problem for a class of nonlinear wave equations with a damping term:

$$u_{tt} = \operatorname{div} \sigma(\nabla u) + \Delta u_t - \Delta^2 u \quad \text{in} \quad \Omega \times (0, +\infty), \tag{1}$$

$$u|_{\partial\Omega} = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0 \quad \text{on} \quad (0, +\infty), \tag{2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega, \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$, ν is the outward normal to the boundary and $\sigma(s)$ are given nonlinear functions.

The study of nonlinear evolution equations with linear damping or dissipative term has been considered by many authors; see [1]–[7]. In our study, we establish a blow up result for solutions with negative energy. The proof of our technique is similar to the one in [7].

2 Blow up of solution

For this purpose, we define

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\Delta u(t)\|_2^2 + \int_\Omega \int_{[0,\nabla u]} \sigma(s) \cdot ds\, dx, \quad t \ge 0, \tag{4}$$

where

$$\sigma(s) = \nabla\omega(s),\ \omega(s) \in C^1(\mathbb{R}^n), \quad s \in \mathbb{R}^n, \quad \sigma(s)\cdot s \le k\int_{[0,s]} \sigma(\tau)\cdot d\tau \le -k\beta|s|^{m+1}, \tag{5}$$

$\cdot$ denotes the dot product in $\mathbb{R}^n$, the integrals in (4) and (5) are line integrals along arbitrary curves connecting 0 and ∇u (respectively 0 and s) in $\mathbb{R}^n$, $k > 2$ and $\beta > 0$ are constants, also $1 < m \le 3$.

Theorem 1 *Let u be the solution of problem* (1) – (3)*. Assume that the following conditions are valid:*

$$u_0 \in H_0^2(\Omega), \quad u_1 \in L_2(\Omega),$$

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\Delta u_0\|_2^2 + \int_\Omega\int_{[0,\nabla u_0]} \sigma(s)\cdot ds\,dx < 0. \tag{6}$$

Then the solution u blows up in finite time

$$T \le \begin{cases} \left[t_1^{\frac{3-m}{2}} + \frac{3-m}{2C_8(\alpha-1)y^{\alpha-1}(t_1)}\right]^{\frac{2}{3-m}}, & m < 3, \\ t_1\cdot\exp\frac{1}{C_8(\alpha-1)y^{\alpha-1}(t_1)}, & m = 3, \end{cases}$$

where t_1 and y will be defined respectively by (17) *and* (18)*, C_8 and $\alpha > 1$ are constants to be defined later.*

Proof. By multiplying equation (1) by u_t and integrating the new equation over Ω, we obtain

$$E'(t) + \|\nabla u_t(t)\|_2^2 = 0, \tag{7}$$

$$E(t) \le E(0) < 0, \quad t \ge 0.$$

Let

$$F(t) = \|u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2\,d\tau, \tag{8}$$

then

$$F'(t) = 2(u,u_t) + \|\nabla u(t)\|_2^2, \tag{9}$$

$$\begin{aligned} F''(t) &= 2\left(\|u_t(t)\|_2^2 - \|\Delta u(t)\|_2^2 - \int_\Omega \sigma(\nabla u)\cdot\nabla u\,dx\right) \\ &\ge 2\left(\|u_t(t)\|_2^2 - \|\Delta u(t)\|_2^2 - k\int_\Omega\int_{[0,\nabla u]} \sigma(s)\cdot ds\,dx\right) \\ &\ge 2\left(2\|u_t(t)\|_2^2 - (k-2)\int_\Omega\int_{[0,\nabla u]} \sigma(s)\cdot ds\,dx - 2E(0)\right) \\ &\ge 2\left(2\|u_t(t)\|_2^2 + (k-2)\beta\|\nabla u(t)\|_{m+1}^{m+1} - 2E(0)\right), \quad t > 0, \end{aligned} \tag{10}$$

where the assumption (5) and the fact that

$$k\int_\Omega\int_{[0,\nabla u]}\sigma(s)\cdot ds\,dx \le 2E(0)-\|u_t(t)\|_2^2+\|\Delta u(t)\|_2^2+(k-2)\int_\Omega\int_{[0,\nabla u]}\sigma(s)\cdot ds\,dx$$

have been used. Taking the inequality (10) and integrating this, we obtain

$$F'(t)\ge 2(k-2)\beta\int_0^t\|\nabla u(\tau)\|_{m+1}^{m+1}d\tau-4E(0)t+F'(0),\quad t>0. \tag{11}$$

After this calculation, we could add the inequalities (10) with (11), then we get

$$F''(t)+F'(t)\ge 4\|u_t(t)\|_2^2+2(k-2)\beta\left(\|\nabla u(t)\|_{m+1}^{m+1}+\int_0^t\|\nabla u(\tau)\|_{m+1}^{m+1}d\tau\right)$$
$$-4E(0)(1+t)+F'(0)=g(t),\qquad t>0. \tag{12}$$

Take $p=\frac{m+3}{2}$, obviously $2<p<m+1$ and $p'=\frac{m+3}{m+1}$ (<2). By using the Young inequality and the Sobolev-Poincaré inequality,

$$\begin{aligned}|(u,u_t)| &\le \frac{1}{p}\|u(t)\|_p^p+\frac{1}{p'}\|u_t(t)\|_{p'}^{p'}\\ &\le C_1\left[\left(\|\nabla u(t)\|_{m+1}^{m+1}\right)^\mu+\left(\|u_t(t)\|_2^2\right)^\mu\right],\\ |(u,u_t)|^{1/\mu} &\le C_2\left[\|\nabla u(t)\|_{m+1}^{m+1}+\|u_t(t)\|_2^2\right],\quad t>0,\end{aligned} \tag{13}$$

where in this inequality and in the sequel C_i $(i=1,2,\ldots)$ denote positive constants independent of t, $\mu=\frac{m+3}{2(m+1)}$ (<1). By the Sobolev-Poincaré inequality and the Hölder inequality

$$\|\nabla u(t)\|_{m+1}^{m+1} \ge C_3\left(\|u(t)\|_2^2\right)^{\frac{m+1}{2}},\quad t>0, \tag{14}$$

$$\int_0^t\|\nabla u(\tau)\|_{m+1}^{m+1}d\tau \ge C_4t^{\frac{1-m}{2}}\left(\int_0^t\|\nabla u(\tau)\|_2^2\,d\tau\right)^{\frac{m+1}{2}}. \tag{15}$$

By using the inequalities (13)–(15), we obtain

$$g(t)\ge C_5\left(3\|\nabla u(t)\|_{m+1}^{m+1}+\|u_t(t)\|_2^2+\int_0^t\|\nabla u(\tau)\|_{m+1}^{m+1}d\tau\right)-4E(0)t+F'(0)$$

$$\ge C_6\left(|(u,u_t)|^{1/\mu}+\left(\|u(t)\|_2^2\right)^{\frac{m+1}{2}}+\left(\|\nabla u(t)\|_2^2\right)^{\frac{m+1}{2}}+t^{\frac{1-m}{2}}\left(\int_0^t\|\nabla u(\tau)\|_2^2\,d\tau\right)^{\frac{m+1}{2}}\right)$$
$$-4E(0)t+F'(0)$$

$$\geq C_7 t^{\frac{1-m}{2}} \left(|(u,u_t)|^\alpha + \left(\|u(t)\|_2^2\right)^\alpha + \left(\|\nabla u(t)\|_2^2\right)^\alpha + \left(\int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \right)^\alpha \right)$$
$$-4E(0)t + F'(0) - C_7 t^{\frac{1-m}{2}}, \qquad t \geq 1, \tag{16}$$

where in this inequality and in the sequel $\alpha = \frac{1}{\mu} > 1$. Since $-4E(0)t + F'(0) - C_7 t^{\frac{1-m}{2}} \to \infty$ as $t \to \infty$, there must be a $t_1 \geq 1$ such that

$$-4E(0)t + F'(0) - C_7 t^{\frac{1-m}{2}} \geq 0 \quad \text{as} \quad t \geq t_1. \tag{17}$$

Let

$$y(t) = F'(t) + F(t), \tag{18}$$

then from the inequality (11) and the equality (8) we obtain $y(t) > 0$ as $t \geq t_1$. By using the inequality

$$(a_1 + \cdots + a_\ell)^n \leq 2^{(n-1)(\ell-1)}(a_1^n + \cdots + a_\ell^n),$$

where $a_i \geq 0$ $(i = 1, \ldots, \ell)$ and $n > 1$ are real numbers, by virtue of (17) and using the inequality (16), we get

$$g(t) \geq C_8 t^{\frac{1-m}{2}} y^\alpha(t), \quad t \geq t_1. \tag{19}$$

So combining (12) with (19) gives

$$y'(t) \geq C_8 t^{\frac{1-m}{2}} y^\alpha(t), \quad t \geq t_1. \tag{20}$$

Therefore, there exists a positive constant

$$T = \begin{cases} \left[t_1^{\frac{3-m}{2}} + \frac{3-m}{2C_8(\alpha-1)y^{\alpha-1}(t_1)} \right]^{\frac{2}{3-m}}, & m < 3, \\ t_1 \cdot \exp \frac{1}{C_8(\alpha-1)y^{\alpha-1}(t_1)}, & m = 3, \end{cases} \tag{21}$$

such that

$$y(t) \to \infty \quad \text{as} \quad t \to T^-. \tag{22}$$

By using (8), (9) and (22), we obtain

$$2\|u(t)\|_2^2 + \|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \geq F'(t) + F(t) \to \infty \text{ as } t \to T^-. \tag{23}$$

So (23) implies

$$\|u(t)\|_2^2 + \|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \to \infty \text{ as } t \to T^-.$$

This completes the proof.

Example 1 Take $\sigma(s) = a|s|^{m-1}s$, where $a < 0$, $1 < m < 3$ are real numbers. Obviously $\sigma(s) = \nabla\omega(s)$, where $\omega(s) = \frac{a}{m+1}|s|^{m+1} \in C^1(\mathbb{R}^n)$, $s \in \mathbb{R}^n$. A simple verification shows that

$$\sigma(s) \cdot s = k \int_{[0,s]} \sigma(\tau) \cdot d\tau = -k\beta|s|^{m+1},$$

where $k = m + 1 > 2$, $\beta = -\frac{a}{m+1} > 0$. If $u_0 \in H_0^2(\Omega)$, $u_1 \in L_2(\Omega)$ are such that $E(0) < 0$, then by Theorem 1 the solution of the corresponding problem (1) – (3) blows up in finite time.

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