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Nonoscillations in Retarded Equations

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1 Introduction

This note regards the existence of nonoscillatory solutions of the linear difference retarded functional equation

$$x\left(t\right)=\int_{-1}^{0}x\left(t-r\left(\theta\right)\right)dq(\theta), \tag{1}$$

where $x\left(t\right)\in\mathbb{R}$, $r\left(\theta\right)$ is a positive real continuous function on $[-1,0]$ and $q(\theta)$ is a function of bounded variation on $[-1,0]$, normalized in a manner such that $q\left(-1\right)=0$.

In the case where $q(\theta)$ is a step function, with a number p of jump points, we obtain the important class of delay difference equations

$$x\left(t\right)=\sum_{j=1}^{p}a_{j}x\left(t-r_{j}\right), \tag{2}$$

where the a_j are nonzero real numbers and each r_j is a positive real number $(j=1,\ldots,p)$.

Considering the value $\|r\|=\max\left\{r\left(\theta\right):\ -1\leqslant\theta\leqslant0\right\}$, by a *solution* of (1) we mean a continuous function $x:\ [-\|r\|,+\infty[\ \to\mathbb{R}$, which satisfies (1) for every $t\geq0$. A solution is said to be *oscillatory* whenever it has an infinite number of zeros; otherwise it will be said to be *nonoscillatory*. When all solutions are oscillatory, the equation (1) is called *oscillatory*. If (1) has at least one nonoscillatory solution, then the equation will be said to be *nonoscillatory*.

We will say that a function $\phi:\ [-1,0]\to\mathbb{R}$ is increasing (decreasing) on $J\subset[-1,0]$, if ϕ is nonconstant on J and for every $\theta_1,\theta_2\in J$ such that $\theta_1<\theta_2$, one has $\phi(\theta_1)\leq\phi(\theta_2)$ (respectively, $\phi(\theta_2)\leq\phi(\theta_1)$). For a given $\theta\in[-1,0]$, if for every $\varepsilon>0$, sufficiently small, ϕ is increasing (decreasing) in $[\theta-\varepsilon,\theta+\varepsilon]$ ($[-\varepsilon,0]$ if $\theta=0$,

$[-1,-1+\varepsilon]$ if $\theta=-1$), θ will be said to be a *point of increase* of ϕ (respectively, a *point of decrease* of ϕ).

Letting $\theta_0 \in [-1,0]$ be such that $r(\theta_0)=\|r\|$, in [1] it is shown that if $r(\theta) < r(\theta_0)$ for every $\theta \neq \theta_0$ and θ_0 is a point of increase, then (1) is nonoscillatory. With respect to (2), the same holds provided that $a_k > 0$, where the integer k is determined by the relation $r_k = \max\{r_j : j=1,\ldots,p\}$.

In Section 2 we will show that under the situation of having $r(\theta)$ differentiable on $[-1,0]$, (1) and (2) can be nonoscillatory in a different framework. Fot that purpose we will denote by D^+ the family of all positive real functions which are differentiable on $[-1,0]$.

2 Nonoscillations

It is well known that any solution of (1) exhibits an integral exponential boundedness (see [3]). Therefore by [4] we can conclude that (1) is oscillatory if and only if

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, dq(\theta) < 1,$$

for every real λ (see [1]). So, (1) is nonoscillatory if and only if there exists a real λ such that

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, dq(\theta) \geq 1.$$

Assuming that $-1 \leq \alpha \leq \beta \leq 0$, let $R(\alpha,\beta)$ be the family of all delay functions $r(\theta)$, in D^+, which are increasing on $[-1,\alpha]$, constant on $[\alpha,\beta]$ and decreasing on $[\beta,0]$.

Theorem 1 *Let $r(\theta)$ be any delay function in $R(\alpha,\beta)$. If $q(0)>0$ and*

$$q(\theta) \leqslant 0 \qquad \text{for every } \theta \in [-1,\alpha], \tag{3}$$

$$q(\theta) \geqslant 0 \qquad \text{for every } \theta \in [\beta,0], \tag{4}$$

then (1) is nonoscillatory.

Proof. Using the integral decomposition

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, dq(\theta) = \int_{-1}^{\alpha} \exp(-\lambda r(\theta))\, dq(\theta) + \int_{\alpha}^{\beta} \exp(-\lambda r(\theta))\, dq(\theta) + \int_{\beta}^{0} \exp(-\lambda r(\theta))\, dq(\theta),$$

and integrating by parts each one of the integrals in the right-hand side of this equation, one has

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, dq(\theta) = \exp(-\lambda r(0))\, q(0) + \int_{-1}^{\alpha} \lambda \exp(-\lambda r(\theta))\, q(\theta)\, dr(\theta) + \int_{\beta}^{0} \lambda \exp(-\lambda r(\theta))\, q(\theta)\, dr(\theta).$$

Since $\lambda r(\theta) \exp(-\lambda r(\theta)) \leqslant e^{-1}$, for every real λ and $r \in R(\alpha, \beta)$, the assumptions (3), (4) imply that

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, dq(\theta) \geqslant \exp(-\lambda r(0))\, q(0) + e^{-1} \left[\int_{-1}^{\alpha} q(\theta)\, d\ln r(\theta) + \int_{\beta}^{0} q(\theta)\, d\ln r(\theta) \right].$$

As $q(0) > 0$, one easily sees that the function

$$f(\lambda) = e\left[1 - \exp(-\lambda r(0))\, q(0)\right],$$

is increasing, $f(\lambda) \to -\infty$ as $\lambda \to -\infty$ and $f(\lambda) \to e$ as $\lambda \to +\infty$. Thus, there exists $\lambda_0 \in \mathbb{R}$ such that

$$\int_{-1}^{\alpha} q(\theta)\, d\ln r(\theta) + \int_{\beta}^{0} q(\theta)\, d\ln r(\theta) = e\left[1 - \exp(-\lambda_0 r(0))\, q(0)\right]$$

and, by consequence,

$$\int_{-1}^{0} \exp(-\lambda_0 r(\theta))\, dq(\theta) \geqslant 1.$$

Hence (1) is nonoscilatory. ■

With $\theta_0 \in [-1, 0]$, denote by $R(\theta_0)$ the family of all delay functions, $r(\theta)$, which are increasing on $[-1, \theta_0]$ and decreasing on $[\theta_0, 0]$. By putting in the Theorem 1, $\alpha = \beta = \theta_0$, the following corollary is obtained.

Corollary 2 *Let $r(\theta)$ be any delay function in $R(\theta_0)$. If $q(0) > 0$ and*

$$q(\theta) \leqslant 0 \qquad \text{for every } \theta \in [-1, \theta_0],$$
$$q(\theta) \geqslant 0 \qquad \text{for every } \theta \in [\theta_0, 0],$$

then (1) *is nonoscillatory.*

By choosing $\theta_0 = -1$, we obtain an important particular case of the Corollary 2.

Corollary 3 *If* $q(\theta) \geqslant 0$ *for every* $\theta \in [-1,0]$, *with* $q(0) > 0$, *then* (1) *is nonoscillatory for all delay functions,* $r(\theta)$ *in* D^+, *which are decreasing on* $[-1,0]$.

The conditions of Corollary 2 cannot be fulfilled when $\theta_0 = 0$, that is when $r(\theta)$ is increasing on $[-1,0]$, since in that case one has necessarily $q(0) = q(\theta_0) = 0$. However, in this situation the following theorem holds:

Theorem 4 *Let* $q(\theta) \leqslant 0$ *for every* $\theta \in [-1,0]$, *with* $q(0) = 0$. *Then* (1) *is nonoscillatory for all delay functions,* $r(\theta)$ *in* D^+, *which are increasing on* $[-1,0]$.

Proof. Integrating by parts directly, we obtain

$$\int_{-1}^{0} e^{-\lambda r(\theta)} dq(\theta) = \int_{-1}^{0} \lambda e^{-\lambda r(\theta)} q(\theta)\, dr(\theta). \tag{5}$$

Taking $\lambda < 0$, since $r(\theta)$ is increasing, we have

$$\int_{-1}^{0} e^{-\lambda r(\theta)}\, dq(\theta) \geqslant \lambda e^{-\lambda r(-1)} r(-1) \int_{-1}^{0} q(\theta)\, d\ln r(\theta).$$

Letting for $\lambda \in]-\infty, 0[$,

$$f(\lambda) = \frac{\exp(\lambda r(-1))}{\lambda r(-1)},$$

one easily sees that f is decreasing, $f(\lambda) \to 0$ as $\lambda \to -\infty$ and $f(\lambda) \to -\infty$ as $\lambda \to 0^-$. So there exists $\lambda_0 < 0$ such that

$$\int_{-1}^{0} q(\theta)\, d\ln r(\theta) = \frac{e^{\lambda_0 r(-1)}}{\lambda_0 r(-1)}$$

and, consequently,

$$\int_{-1}^{0} e^{-\lambda_0 r(\theta)}\, dq(\theta) \geqslant 1,$$

which achieves the proof. ∎

The results above are illustrated by the following examples.

Example 5 Let

$$q(\theta) = \begin{cases} -1-\theta & \text{if} \quad -1 \leqslant \theta < -\frac{2}{5}, \\ \ \ 1-\theta & \text{if} \quad -\frac{2}{5} \leqslant \theta \leqslant 0, \end{cases}$$

and $r \in R\left(-\frac{3}{5}, -\frac{2}{5}\right)$. As

$$q(-1) = 0, \qquad q(0) = 1,$$

and

$$q(\theta) \leqslant 0 \qquad \text{for every } \theta \in \left[-1, -\frac{3}{5}\right],$$

$$q(\theta) \geqslant 0 \qquad \text{for every } \theta \in \left[-\frac{2}{5}, 0\right],$$

by Theorem 1, the corresponding equation (1) is nonoscillatory.

Example 6 Let

$$q(\theta) = \begin{cases} -2(1+\theta) & \text{if} \quad -1 \leqslant \theta < -\frac{1}{2}, \\ 0 & \text{if} \quad \theta = -\frac{1}{2}, \\ 5 - 3\theta & \text{if} \quad -\frac{1}{2} < \theta \leqslant 0, \end{cases}$$

and $r \in R\left(-\frac{1}{2}\right)$. Since

$$q(-1) = 0, \qquad q(0) = 5,$$

and

$$q(\theta) \leqslant 0 \qquad \text{for every } \theta \in \left[-1, -\frac{1}{2}\right],$$

$$q(\theta) \geqslant 0 \qquad \text{for every } \theta \in \left[-\frac{1}{2}, 0\right],$$

by Corollary 2, the corresponding equation (1) is nonoscillatory.

Example 7 The equation

$$x(t) = \int_{-1}^{0} x(t - r(\theta))\, d\left((\theta+1)(\theta-1)^2\right)$$

is nonoscillatory, by Corollary 3, for every delay function, $r(\theta)$ in D^+, decreasing on $[-1, 0]$.

Example 8 By Theorem 4, one easily verifies that the equation

$$x(t) = \int_{-1}^{0} x(t - r(\theta))\, d(\theta(\theta+1))$$

is nonoscillatory, for all delay functions, $r(\theta)$ in D^+, which are increasing on $[-1, 0]$.

Remark 9 If, aditionally, $q(\theta)$ is a step function, in Corollary 2 one easily can conclude that θ_0 is a point of increase of $q(\theta)$. The same holds with -1 in Corollary 3 and with 0 in Theorem 4. This means that the corresponding consequences which can be obtained for equation (2) are already covered by the results obtained in [1].

The main assumptions in the preceding results fall upon the function $q(\theta)$, implying then that (1) is nonoscillatory for a certain class of delay functions. The following theorems are of different kind since beyond having $r(\theta)$ monotonous, they involve a mixed type condition on $r(\theta)$ and $q(\theta)$.

Theorem 10 *Let* $r(\theta)$, *in* D^+, *be decreasing on* $[-1,0]$ *and* $q(\theta) \leqslant 0$ *for every* $\theta \in [-1,0]$, *with* $q(0) = 0$. *If*

$$\int_{-1}^{0} q(\theta)\, d\ln r(\theta) \geqslant \frac{r(-1)}{r(0)} e, \tag{6}$$

then the equation (1) *is nonoscillatory.*

Proof. Proceeding as in the proof of Theorem 4, from (5) we have for every $\lambda \in]0,+\infty[$

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, dq(\theta) \geqslant \lambda r(0) \exp(-\lambda r(-1)) \int_{-1}^{0} q(\theta)\, d\ln r(\theta),$$

since $r(\theta)$ is decreasing and $q(\theta) \leqslant 0$ on $[-1,0]$. For λ in $]0,+\infty[$, the function

$$g(\lambda) = \frac{\exp(\lambda r(-1))}{\lambda r(0)},$$

tends to $+\infty$ either as $\lambda \to +\infty$ or $\lambda \to 0^+$. Moreover,

$$g\left(\frac{1}{r(-1)}\right) = \frac{r(-1)}{r(0)} e$$

is the minimum of g on $]0,+\infty[$. So, by (6) there exists λ_0 such that

$$\lambda_0 r(0) \exp(\lambda_0 r(-1)) \int_{-1}^{0} q(\theta)\, d\ln r(\theta) = 1,$$

and therefore

$$\int_{-1}^{0} \exp(-\lambda_0 r(\theta))\, dq(\theta) \geqslant 1$$

which completes the proof. ∎

By changing in the proof of Theorem 10, $r(0)$ with $r(-1)$, we obtain, analogously, the following theorem.

Theorem 11 *Let $r(\theta)$, in D^+, be increasing on $[-1,0]$, and $q(\theta) \geqslant 0$ for every $\theta \in [-1,0]$, with $q(0) = 0$. If*

$$\int_{-1}^{0} q(\theta)\, d \ln r(\theta) \geqslant \frac{r(0)}{r(-1)} e, \tag{7}$$

then (1) *is nonoscillatory.*

The examples below illustrate these results.

Example 12 Applying Theorem 10, we can conclude that the equation

$$x(t) = \int_{-1}^{0} x(t - (2 - \theta))\, d(80\theta^2 + 80\theta)$$

is nonoscillatory, since

$$\int_{-1}^{0} q(\theta)\, d \ln r(\theta) = -\int_{-1}^{0} \frac{80\theta^2 + 80\theta}{-\theta + 2}\, d\theta \approx 5,3767$$

is larger than

$$\frac{r(-1)}{r(0)} e = \frac{3}{2} e \approx 4,0774.$$

We notice that by [2], Corollary 15, when $r(\theta)$, in D^+, is decreasing and $q(\theta) \leqslant 0$ on $[-1,0]$, with $q(0) = 0$, and

$$\int_{-1}^{0} q(\theta)\, d \ln r(\theta) < e, \tag{8}$$

then (1) is oscillatory. As in (6), $r(-1)/r(0) > 1$, one easily observes that verifying (6) we are in the opposite of (8). The same holds in Theorem 11.

The Theorems 10 and 11 applied to equation (2) give, respectively, the following corollaries.

Corollary 13 *If $r_1 > \cdots > r_p$,*

$$\sum_{j=1}^{p} a_j = 0, \quad \sum_{j=1}^{k} a_j \leqslant 0, \text{ for every } 1 \leqslant k \leqslant p-1,$$

and

$$\sum_{k=1}^{p-1} \left(\sum_{j=1}^{k} a_j \right) \ln \frac{r_{k+1}}{r_k} \geqslant \frac{r_1}{r_p} e, \tag{9}$$

then (2) *is nonoscillatory.*

Corollary 14 *If* $r_1 < \cdots < r_p$,

$$\sum_{j=1}^{p} a_j = 0, \quad \sum_{j=1}^{k} a_j \geqslant 0, \text{ for every } 1 \leqslant k \leqslant p-1,$$

and

$$\sum_{k=1}^{p-1} \left(\sum_{j=1}^{k} a_j \right) \ln \frac{r_{k+1}}{r_k} \geqslant \frac{r_p}{r_1} e, \tag{10}$$

then (2) *is nonoscillatory.*

Example 15 The equation

$$x(t) = -2x\left(t - \frac{5}{2}\right) - 6x\left(t - \frac{3}{2}\right) + 8x\left(t - \frac{1}{2}\right),$$

is nonoscillatory, since $\sum_{j=1}^{k} a_j$ is equal to $-2, -8$ and 0, for $k = 1, 2$ and 3, respectively, and

$$\sum_{k=1}^{p} \left(\sum_{j=1}^{k} a_j \right) \ln \frac{r_{k+1}}{r_k} = -2 \ln \frac{3}{5} - 8 \ln \frac{1}{3} \approx 9,8105$$

which is larger than

$$\frac{r_p}{r_1} e = \frac{1}{5} e \approx 0,543668.$$

Note that by [2], Corollary 18, when $r_1 > \cdots > r_p$, $\sum_{j=1}^{p} a_j = 0$, $\sum_{j=1}^{k} a_j \leqslant 0$ for every $1 \leqslant k \leqslant p-1$, and

$$\sum_{k=1}^{p-1} \left(\sum_{j=1}^{k} a_j \right) \ln \frac{r_{k+1}}{r_k} < e, \tag{11}$$

then (1) is oscillatory. Since $r_1/r_p > 1$, we notice that under (9) we are in the complementary of (11). A similar situation happens with (10) of Corollary 14.

References

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