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Effect of the Thickness of Tube on Wave Propagation in Elastic Tubes Filled with Viscous Fluid

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Abstract

In order to understand the effect of initial stress and the thickness of the tube on flow in elastic tubes, the propagation of time harmonic waves in a prestressed elastic, isotropic tube filled with a viscous fluid is studied. The thick walled tubes are important to understand the blood flow through the arteries. Although the blood is known to be a non-Newtonian fluid, for simplicity in the mathematical analysis it is assumed to be a viscous fluid.

After obtaining the field equations, for the sake of better understanding of the effect of thickness, a finite difference method is used to solve the equations. The variations of primary and secondary wave speeds and the transmission coefficients are investigated with respect to the thickness ratio and Womersley parameter by considering long wave approximation. Also a cut-off frequency which could not have been obtained in the previous works that used a truncated power series method is obtained.

Key words: Harmonic waves, Thick tube, Initial stress.

1 Introduction

The importance of fluid mechanics and mathematical models in understanding the blood flow process in human and animal bodies has been reviewed by Rudinger [1]. Witzig [2] is the first one who took the viscosity into account but ignored the effects of Poisson's ratio and obtained the propagation constants as a function of the frequency and viscosity. Morgan and Kiely [3] studied the same problem by assuming the artery as an elastic tube and the blood as a viscous but incompressible fluid and obtained the dispersion relation in which the effects of Poisson's ratio is also included. Womersley [4] in his pioneering work treated the artery as a thick walled shell and blood as incompressible viscous fluid. In reality, the artery is subject to average pressure which is about 100 mm Hg and axial stretch which is

1.5 under physiological conditions. The initial stress (or deformation) of arterial wall had been first taken into consideration by Atabek and Lew [5], and the effects of initial stresses and tethering were numerically analyzed. The effects of initial deformation were properly taken into account by Rachev [6], but he simply treated artery as a membrane. However, physiological studies on arteries show that the ratio of thickness to mean radius of the artery (large blood vessels in general) changes from $\frac{1}{6}$ to $\frac{1}{4}$. This means that the arterial wall is not thin enough to use the membrane theory and to take the constant initial stresses distributions. Demiray and Antar [7] have taken into account the initial stresses and the thickness of artery, but to obtain solution of equations governing the solid body, they have used a truncated power series in terms of thickness ratio.

In the present work the propagation of harmonic waves in an initially stressed isotropic, elastic cylindrical tube filled with an incompressible viscous fluid is studied. Considering the arterial wall as an elastic, isotropic and incompressible material subjected to a large initial static deformation, the governing differential equations in cylindrical coordinates are obtained for fluid flow and solid body. Although a closed form solution can be obtained for equations governing the fluid body, due to the variability of coefficients of differential equations of solid body, such a closed form solution is not possible. Therefore the solution is obtained numerically by using the finite difference method.

2 Equations of motion

2.1 Equations of fluid

Although the blood is known to be a non-Newtonian fluid, the incremental behavior of blood in arteries can be treated as Newtonian fluid. For an axially symmetric motion, the equations of fluid are given by

$$\begin{aligned} -\frac{\partial \bar{p}}{\partial r} + \mu\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right) - \bar{\rho}\frac{\partial \bar{u}}{\partial t} = 0, \\ -\frac{\partial \bar{p}}{\partial z} + \mu\left(\frac{\partial^2 \bar{w}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}}{\partial r} + \frac{\partial^2 \bar{w}}{\partial z^2}\right) - \bar{\rho}\frac{\partial \bar{w}}{\partial t} = 0, \end{aligned} \quad (1)$$

where $\bar{\rho}$ is the mass density of the fluid, μ is the viscosity, $\bar{p}$ is the incremental inner pressure, the velocity field is given as

$$\mathbf{u} = (\bar{u}, 0, \bar{w})$$

and the equation of incompressibility is

$$\frac{\partial \bar{u}}{\partial r} + \frac{\bar{u}}{r} + \frac{\partial \bar{w}}{\partial z} = 0. \quad (2)$$

The components of stress tensor which we need in using the boundary conditions are given by

$$\bar{\sigma}_{rr} = -\bar{p} + 2\mu \frac{\partial \bar{u}}{\partial r}, \qquad \bar{\sigma}_{rz} = \mu \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial r} \right). \tag{3}$$

2.2 Equations of solid body

In this work the arterial wall material is assumed to be incompressible, isotropic and elastic. The cylindrical tube is subject to a large static deformation under the effects of inner pressure P_i and the axial force N. The governing field equations for such a deformation may be given by (see Eringen and Suhubi [8])

$$\sigma_{kl,l} = \rho \frac{\partial^2 u_k}{\partial t^2}, \tag{4}$$

where ρ is the mass density of solid body, $\mathbf{u}$ is the incremental displacement field and the incremental stress tensor σ_{kl} is defined by

$$\sigma_{kl} = t_{kl} + m_{kl}, \tag{5}$$

here

$$m_{kl} = u_{k,m} t^{\circ}_{ml} \tag{6}$$

and t_{kl} is the incremental stress tensor referred to the final deformed area.

The material is assumed to be incompressible neo-Hookean. So the incompressibility condition and the constitutive relations become

$$u_{k,k} = 0,$$
$$t^{\circ}_{kl} = P^{\circ} g_{kl} + \alpha c_{kl}, \qquad W = \frac{\alpha}{2}(I_1 - 3), \tag{7}$$

where P° is the hydrostatic pressure, g_{kl} is the metric tensor of the spatial frame, c_{kl} the Finger deformation tensor, I_1 is the first invariant of c_{kl}, and α is a material constant to be determined from experimental measurements. Assuming the geometry of the tube is a cylindrical thick shell, the stress distribution in cylindrical coordinates may be given as follows (Demiray and Dost [9])

$$\begin{aligned}
t^{\circ}_{rr} &= \frac{\alpha}{\lambda^2} \left(\frac{x^2 - x_o^2}{2} + \lambda \ln \frac{x}{x_o} \right), \\
t^{\circ}_{\theta\theta} &= \frac{\alpha}{x^2} - \frac{\alpha}{\lambda^2} \left(\frac{x^2 + x_o^2}{2} - \lambda \ln \frac{x}{x_o} \right), \\
t^{\circ}_{zz} &= \alpha\lambda^2 - \frac{\alpha}{\lambda^2} \left(\frac{x^2 + x_o^2}{2} - \lambda \ln \frac{x}{x_o} \right), \qquad x \equiv \frac{R}{r}, \\
P^{\circ} &= -\frac{\alpha}{\lambda^2} \left(\frac{x^2 + x_o^2}{2} - \lambda \ln \frac{x}{x_o} \right), \quad P_i = \frac{\alpha}{\lambda^2} \left(\frac{x_o^2 - x_i^2}{2} + \lambda \ln \frac{x_o}{x_i} \right),
\end{aligned} \tag{8}$$

where P_i is the inner pressure, λ is the axial stretch. R and r are the radial coordinates of a material point before and after deformation and the subscripts (i) and (o) stand for the values of a quantity evaluated on the inner and outer surfaces, respectively. On the other hand, the incremental stress tensor is given by [8]

$$t_{kl} = pg_{kl} - 2P^\circ e_{kl}, \tag{9}$$

where e_{kl} is defined as $e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$. In order to obtain the complete field equations, one must also know the initial stress distribution t°_{kl} through the elastic wall. The tube that we shall study here is subject to the inner pressure P_i and axial stretch λ.

After writing the explicit form of the incremental stress tensor from (9) and the components of m_{kl} from (6) by considering (7), the governing equations can be written in the following form:

$$\begin{aligned}
&\frac{\partial p}{\partial r} + \bar{\beta}_1(r)\frac{\partial^2 u}{\partial r^2} + \bar{\beta}_2(r)\frac{\partial u}{\partial r} - \bar{\beta}_3(r)u + \bar{\beta}_4(r)\frac{\partial^2 u}{\partial z^2} = \rho\frac{\partial^2 u}{\partial t^2},\\
&\frac{\partial p}{\partial z} + \bar{\beta}_1(r)\frac{\partial^2 w}{\partial r^2} + \bar{\beta}_5(r)\frac{\partial w}{\partial r} + \bar{\beta}_4(r)\frac{\partial^2 w}{\partial z^2}\\
&\qquad +(\bar{\beta}_2(r) - \bar{\beta}_5(r))\frac{\partial u}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2},
\end{aligned} \tag{10}$$

where the coefficients $\bar{\beta}_i(r)$ $(i = 1, 2, \ldots, 5)$ are defined by

$$\begin{aligned}
\bar{\beta}_1(r) &= t^\circ_{rr} - P^\circ, \quad \bar{\beta}_2(r) = \frac{t^\circ_{\theta\theta}}{r} - 2\frac{dP^\circ}{dr} - \frac{P^\circ}{r}, \quad \bar{\beta}_3(r) = \frac{t^\circ_{\theta\theta}}{r^2} - \frac{P^\circ}{r^2},\\
\bar{\beta}_4(r) &= t^\circ_{zz} - P^\circ, \quad \bar{\beta}_5(r) = \frac{t^\circ_{\theta\theta}}{r} - \frac{dP^\circ}{dr} - \frac{P^\circ}{r}.
\end{aligned} \tag{11}$$

These differential equations are to be supplemented by the boundary conditions given by

$$\begin{aligned}
t_{rr}(r_i) &= \bar{\sigma}_{rr}(r_i), \quad t_{rz}(r_i) = \bar{\sigma}_{rz}(r_i), \quad t_{rr}(r_o) = t_{rz}(r_o) = 0,\\
\frac{\partial u}{\partial t}(r_i) &= \bar{u}, \quad \frac{\partial w}{\partial t}(r_i) = \bar{w},
\end{aligned} \tag{12}$$

where r_i and r_0 represent the inner and outer radius of the tube, respectively. On the other hand, the effects of tethering on the outer boundary have been neglected here.

3 Solution of the governing equations

In the present work we shall seek a harmonic wave type of solution to the field equations. For that purpose we set

$$(\bar{u}, \bar{w}, \bar{p}) = [\bar{U}(r), \bar{W}(r), \bar{P}(r)] \exp[i(\omega t - kz)], \tag{13}$$

where ω is the angular frequency, k is the wave number and $\bar{U}(r), \bar{W}(r), \bar{P}(r)$ are the complex amplitudes to be determined from the solution of the field equations and the boundary conditions. Substituting (13) into (1) and (2), the solution of the resulting equations for $\bar{U}(r)$, $\bar{W}(r)$ and $\bar{P}(r)$ can be obtained:

$$\begin{aligned} \bar{U} &= k[I_1(kr)\bar{A} + J_1(sr)\bar{B}], \\ \bar{W} &= -i[kI_0(kr)\bar{A} + sJ_0(sr)\bar{B}], \\ \bar{P} &= -i\bar{\rho}\omega I_0(kr)\bar{A}, \end{aligned} \tag{14}$$

where we defined $s^2 = -\left(k^2 + \frac{i\bar{\rho}\omega}{\mu}\right)$ and $J_n(sr)$, $I_n(kr)$ are the first kind and modified Bessel functions of order n, respectively, $\bar{A}$ and $\bar{B}$ are integration constants to be determined by using the boundary conditions.

In order to obtain the solution of the field equations of solid body, we shall seek again a harmonic type solution which can be written as

$$(u, w, p) = [\hat{U}(r), \hat{W}(r), \hat{P}(r)] \exp[i(\omega t - kz)]. \tag{15}$$

Considering the incompressibility condition and substituting this solution into the eqs. (10), we have

$$\begin{aligned} &\frac{d\hat{P}}{dr} + \bar{\beta}_1 \frac{d^2\hat{U}}{dr^2} + \bar{\beta}_2 \frac{d\hat{U}}{dr} + (\rho\omega^2 - \bar{\beta}_3 - k^2\bar{\beta}_4)\hat{U} = 0, \\ &-ik\hat{P} + \bar{\beta}_1 \frac{d^2\hat{W}}{dr^2} + \bar{\beta}_5 \frac{d\hat{W}}{dr} + (\rho\omega^2 - k^2\bar{\beta}_4)\hat{W} - ik(\bar{\beta}_2 - \bar{\beta}_5)\hat{U} = 0, \\ &\frac{d\hat{U}}{dr} + \frac{1}{r}\hat{U} - ik\hat{W} = 0. \end{aligned} \tag{16}$$

By defining a function $\phi(r)$ as

$$\phi(r) = \int_0^r \xi \hat{W}(\xi)\, d\xi, \tag{17}$$

from the last equation of (16) the following can be obtained:

$$\hat{U} = \frac{ik}{r}\phi(r), \qquad \hat{W}(r) = \frac{1}{r}\frac{d\phi}{dr}. \tag{18}$$

By using this function, the first two equations of (16) can be written in the following form:

$$\begin{aligned} &\frac{dP}{d\xi} + i\eta \frac{\beta_1}{\xi}\frac{d^2\bar{\phi}}{d\xi^2} + \frac{i\eta}{\xi}\left(\beta_2 - \frac{2\beta_1}{\xi}\right)\frac{d\bar{\phi}}{d\xi} + \frac{i\eta}{\xi}\left(\frac{2\beta_1}{\xi^2} - \frac{\beta_2}{\xi} + \Omega^2 - 7\beta_3 - \eta^2\beta_4\right)\bar{\phi} = 0, \\ &-i\eta P + \frac{\beta_1}{\xi}\frac{d^3\bar{\phi}}{d\xi^3} + \left(-\frac{2\beta_1}{\xi^2} + \frac{\beta_5}{\xi}\right)\frac{d^2\bar{\phi}}{d\xi^2} \\ &\qquad + \left[\frac{2\beta_1}{\xi^3} - \frac{\beta_5}{\xi^2} + \frac{1}{\xi}(\Omega^2 - \eta^2\beta_4)\right]\frac{d\bar{\phi}}{d\xi} + \frac{\eta^2}{\xi}(\beta_2 - \beta_5)\bar{\phi} = 0, \end{aligned} \tag{19}$$

where we used the following non-dimensionalized quantities

$$\bar{\beta}_1 = \alpha\beta_1, \quad \bar{\beta}_2 = \frac{\alpha}{\bar{r}}\beta_2, \quad \bar{\beta}_3 = \frac{\alpha}{\bar{r}}\beta_3, \quad \bar{\beta}_4 = \alpha\beta_4, \quad \bar{\beta}_5 = \frac{\alpha}{\bar{r}}\beta_5,$$
$$\hat{P} = \alpha P, \quad r = \bar{r}\xi, \quad \phi = \bar{r}^3\bar{\phi}, \quad \eta = k\bar{r}, \quad c_0^2 = \frac{\alpha}{\rho}, \quad \gamma = \bar{r}s,$$
$$\Omega = \frac{\omega\bar{r}}{c_0}, \quad \alpha_0 = \frac{i\bar{\rho}\omega\bar{r}^2}{2\mu}, \quad f = \frac{\eta I_1(\eta)}{I_0(\eta)}, \quad g = \frac{\gamma J_0(\gamma)}{J_1(\gamma)}, \quad q = \frac{\rho}{\bar{\rho}}. \tag{20}$$

On the other hand, we obtain boundary conditions in the following form

$$\left[P - 2P^\circ \frac{i\eta}{\xi}\left(\frac{d\bar{\phi}}{d\xi} - \frac{\bar{\phi}}{\xi}\right)\right]_{\xi=\xi_i} = \left(\eta^2 - \frac{f}{\xi_i} + \alpha_0\right)A + \eta\left(g - \frac{1}{\xi_i}\right)B,$$
$$\left[\frac{P^\circ}{\xi}\left(-\eta^2\bar{\phi} - \frac{d^2\bar{\phi}}{d\xi^2} + \frac{1}{\xi}\frac{d\bar{\phi}}{d\xi}\right)\right]_{\xi=\xi_i} = -i\eta f A - i(\alpha_0 + \eta^2)B,$$
$$\left[P - 2P^\circ \frac{i\eta}{\xi}\left(\frac{d\bar{\phi}}{d\xi} - \frac{\bar{\phi}}{\xi}\right)\right]_{\xi=\xi_o} = 0,$$
$$\left[-\eta^2\bar{\phi} - \frac{d^2\bar{\phi}}{d\xi^2} + \frac{1}{\xi}\frac{d\bar{\phi}}{d\xi}\right]_{\xi=\xi_o} = 0,$$
$$\frac{i\Omega^2}{\xi_i}\bar{\phi}(\xi_i) + q\alpha_0\left(\frac{f}{\eta}A + B\right) = 0,$$
$$\frac{i\Omega^2}{\xi_i}\left(\frac{d\bar{\phi}}{d\xi}\right)_{\xi=\xi_i} + q\alpha_0(\eta A + gB) = 0, \tag{21}$$

where

$$A = \frac{2\mu I_0(\eta)}{\alpha\bar{r}^2}\bar{A}, \qquad B = \frac{2\mu J_1(\gamma)}{\alpha\bar{r}^2}\bar{B} \tag{22}$$

and $\bar{r}$ is the midradius of the tube after deformation.

It is almost impossible to give a closed form analytical solutions to (19). To see the effects of thickness of the tube on wave propagation more clearly, we will investigate numerical solutions of the equations by using finite difference method. It is obvious that the results will be obtained by using this method, will be more realistic than the power series method in which only first order terms have been taken into account. To use the finite difference method we divide the thickness of the tube, $\bar{h} = r_0 - r_i$, into n equal intervals, thus we define

$$\xi_j = \xi_0 + jh, \qquad j = 0, 1, \ldots, n, \tag{23}$$

where

$$\xi_0 = \frac{r_i}{\bar{r}} = 1 - \frac{nh}{2}, \quad \xi_n = \frac{r_o}{\bar{r}}, \quad h = \frac{\bar{h}}{n\bar{r}}.$$

We characterize the value of a function at the point $\xi_j = \xi_0 + jh$ by a subscript j. By introducing appropriate finite-difference expressions for various derivatives, we obtain the following difference equations for the equations (21) and (19):

$$
\begin{aligned}
&-hP_j + hP_{j+1} + \frac{i\eta}{\xi_j}[\alpha_{1j} + h^2(\Omega^2 - \eta^2\beta_{4j})]\bar{\phi}_j \\
&\qquad + \frac{i\eta}{\xi_j}\alpha_{2j}\bar{\phi}_{j+1} + \frac{i\eta}{\xi_j}\beta_{1j}\bar{\phi}_{j+2} = 0, \\
&-i\eta h^3 P_j + \Big[\alpha_{3j} - \frac{h^2}{\xi_j}(\Omega^2 - \eta^2\beta_{4j}) + \frac{h^3\eta^2}{\xi_j}(\beta_{2j} - \beta_{5j})\Big]\bar{\phi}_j \\
&\qquad + \Big[\alpha_{4j} + \frac{h^2}{\xi_j}(\Omega^2 - \eta^2\beta_{4j})\Big]\bar{\phi}_{j+1} + \alpha_{5j}\bar{\phi}_{j+2} + \frac{\beta_{1j}}{\xi_j}\bar{\phi}_{j+3} = 0, \\
&hP_0 + \frac{2P_0^\circ i\eta}{\xi_0}\left(1 + \frac{h}{\xi_0}\right)\bar{\phi}_0 - \frac{2P_0^\circ i\eta}{\xi_0}\bar{\phi}_1 \\
&\qquad - h\left(\eta^2 - \frac{f}{\xi_0} + \alpha_0\right)A - h\eta\left(g - \frac{1}{\xi_0}\right)B = 0, \\
&-\frac{P_0^\circ}{\xi_0}\left(\eta^2h^2 + 1 + \frac{h}{\xi_0}\right)\bar{\phi}_0 + \frac{P^\circ}{\xi_0}\left(2 + \frac{h}{\xi_0}\right)\bar{\phi}_1 - \frac{P_0^\circ}{\xi_0}\bar{\phi}_2 \\
&\qquad + i\eta f h^2 A + ih^2(\alpha_0 + \eta^2)B = 0, \\
&\frac{i\Omega^2}{\xi_0}\bar{\phi}_0 + q\alpha_0\left(\frac{f}{\eta}A + B\right) = 0, \\
&\frac{i\Omega^2}{\xi_0}(\bar{\phi}_1 - \bar{\phi}_0) + q\alpha_0 h(\eta A + gB) = 0, \\
&h\xi_n P_n + 2P_n^\circ i\eta\left(1 + \frac{h}{\xi_n}\right)\bar{\phi}_n - 2P_n^\circ i\eta\bar{\phi}_{n+1} = 0, \\
&-\left(\eta^2h^2 + 1 + \frac{h}{\xi_n}\right)\bar{\phi}_n + \left(2 + \frac{h}{\xi_n}\right)\bar{\phi}_{n+1} - \bar{\phi}_{n+2} = 0,
\end{aligned} \tag{24}
$$

where

$$
\begin{aligned}
\alpha_{1j} &= \beta_{1j} - h\left(\beta_{2j} - \frac{2\beta_{1j}}{\xi_j}\right) + h^2\left(\frac{2\beta_{1j}}{\xi_j^2} - \frac{\beta_{2j}}{\xi_j} - \beta_{3j}\right), \\
\alpha_{2j} &= -2\beta_{1j} + h\left(\beta_{2j} - \frac{2\beta_{1j}}{\xi_j}\right), \\
\alpha_{3j} &= -\frac{\beta_{1j}}{\xi_j} + h\left(\beta_{5j}\xi_j - \frac{2\beta_{1j}}{\xi_j^2}\right) - h^2\left(\frac{2\beta_{1j}}{\xi_j^3} - \frac{\beta_{5j}}{\xi_j^2}\right),
\end{aligned}
$$

$$\alpha_{4j} = \frac{3\beta_{1j}}{\xi_j} - 2h\left(\frac{\beta_{5j}}{\xi_j} - \frac{2\beta_{1j}}{\xi_j^2}\right) + h^2\left(\frac{2\beta_{1j}}{\xi_j^3} - \frac{\beta_{5j}}{\xi_j^2}\right),$$

$$\alpha_{5j} = -\frac{3\beta_{1j}}{\xi_j} + h\left(\beta_{5j}\xi_j - \frac{2\beta_{1j}}{\xi_j^2}\right),$$

$$\beta_{1j} = \frac{x_j^2}{\lambda^2}, \qquad \beta_{2j} = \frac{1}{\xi_j}\left[\frac{1}{x_j^2} - \frac{2}{\lambda^2}\left(x_j - \frac{\lambda}{x_j}\right)^2\right], \qquad \beta_{3j} = \frac{1}{\xi_j^2 x_j^2}, \quad \beta_{4j} = \lambda^2,$$

$$\beta_{5j} = \frac{1}{\xi_j}\left[\frac{1}{x_j^2} - \frac{1}{\lambda^2}\left(x_j - \frac{\lambda}{x_j}\right)^2\right] \qquad P_j^{\circ} = -\frac{1}{\lambda^2}\left[\frac{x_j^2 + x_o^2}{2} - \lambda \ln \frac{x_j}{x_o}\right],$$

$$x_j = \frac{[\lambda(\bar{r}^2\xi_j^2 - r_i^2) + R_i^2]^{1/2}}{\bar{r}\xi_j}, \quad x_o = \frac{R_o}{r_o}.$$

In order to have a non-zero solution for the coefficients $P_j, \bar{\phi}_j$ $(j = 0, 1, \ldots, n)$, A and B, the determinant of the coefficients matrix must vanish.

4 Long wave approximation

Even for large arteries the wave length is very large as compared to the mean radius of the artery. Therefore, for this special case $|\eta| \ll 1$, f approaches $\frac{\eta^2}{2}$. To understand the effects of the thickness of the arterial wall on the wave characteristics parameter n defining the number of intervals through thickness must be greater than 1. For the values $n > 1$, the cut-off frequency which could not have been obtained in the previous works have been obtained from the dispersion relation by setting $\eta \to 0$. For instance, the cut-off frequency for small values of h and $n = 2$ is obtained as

$$\Omega_c^2 = \{h[\beta_{11}(2\beta_{10} - \beta_{50} - P_0^0 gq) + \beta_{10}(\beta_{51} - P_1^0)gq] + \beta_{11}\beta_{10}\}/(h^2\beta_{10}). \tag{25}$$

For the more general case, we will take $n = 4$, $\lambda = 1.3$, $R_i = 0.3$ cm and $P_i = 0.1$. Moreover, experimental studies indicate that the density of the arterial wall tissue is quite close to that of blood, therefore, we may approximate q as $q \cong 1$ and decompose the non-dimensional phase velocity c into real and imaginary parts as

$$c = X + iY. \tag{26}$$

The speed of propagation v and the transmission coefficient χ are defined by (Atabek and Lew [5])

$$v = \frac{X^2 + Y^2}{X}, \qquad \chi = \exp(-2\pi Y/X). \tag{27}$$

The speeds of propagation and transmission coefficients are evaluated for various values of thickness ratio and Womersley parameter.

Figure 1 shows the variation of primary wave speed with Womersley parameter and the thickness ratio. It shows that speed increases with Womersley parameter and thickness ratio. This result indicates the effects of the thickness on the wave speeds. The variation of secondary wave speed is depicted in Figure 2. The variations of transmission coefficients of primary wave are depicted in Fig. 3. Examination of this figure indicates that the coefficient decreases very fast with Womersley parameter at the beginning but stars to increase after a certain value of the parameter and with the thickness ratio. Finally, the variation of transmission coefficient of secondary wave with the same parameters is depicted in Figure 4. As might be seen from the figure, the coefficient increases with Womersley parameter but decreases with the thickness ratio.

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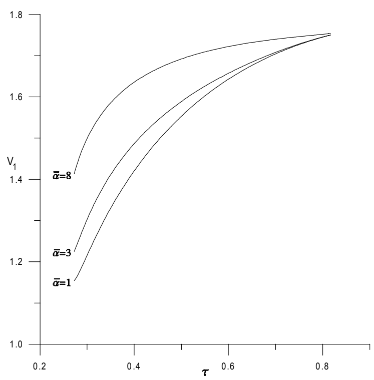

Figure 1: Variation of the primary wave speed with respect to the thickness ratio

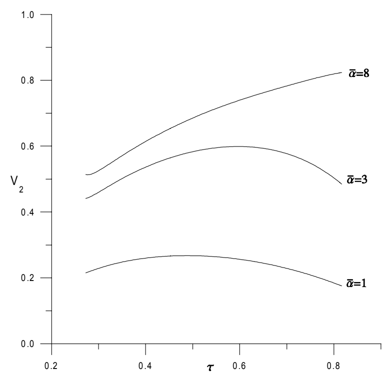

Figure 2: Variation of the secondary wave speed with respect to the thickness ratio

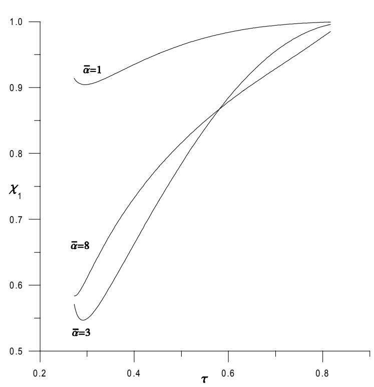

Figure 3: Variation of the transmission coefficient of the primary wave speed

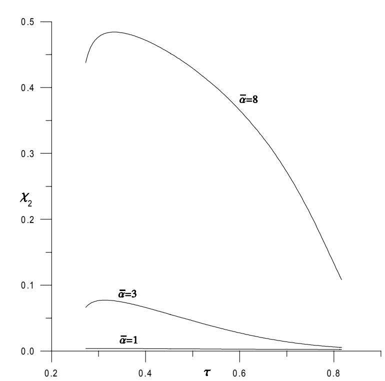

Figure 4: Variation of the transmission coefficient of the secondary wave speed