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On the Approximation of Singular Integrals of Cauchy Type

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Abstract

The aim of this work is to approximate numerically the singular integral of Cauchy type on a piecewise smooth curve by expressions based on the cubic spline, which is one of the recent ideas in numerical analysis.

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Let Γ be a a piecewise regular curve, in other words, Γ consists of a finite number of smooth non-intersecting contours in a complex plane, where Γ can be represented in the form

$$t(s) = x(s) + iy(s), \quad a \le s \le b,\ a, b \in \mathbb{R},$$

where $x(s)$ and $y(s)$ are continuous functions in the interval $[a, b]$ with the following property:

The functions $x(s)$ and $y(s)$ have continuous first derivatives $x'(s)$ and $y'(s)$ within the interval $[a, b]$, including the endpoints, and these derivatives are never simultaneously zero.

Let $F(t_0)$ be a singular integral defined by

$$F(t_0) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(t)}{t - t_0}\, dt, \quad t_0 \in \Gamma. \tag{1}$$

For the existence of the principal value for a given density $\varphi(t)$, we will need more than mere continuity, in other words, the density $\varphi(t)$ has to satisfy the Hölder condition ($\varphi \in H(\mu)$) [2].

Let us now consider an arbitrary natural number N, generally we take it large enough, we divide the interval $[a,b]$ into N subintervals of $[a,b]=\{a=s_0<s_1<\cdots<s_N=b\}$ called I_1 to I_N, so that $I_{\sigma+1}=[s_\sigma,s_{\sigma+1}]$. Also define $h_{\sigma+1}=s_{\sigma+1}-s_\sigma$, noting that the subintervals need not be of equal length.

But, in our case and for reasons of programming one takes the subintervals of the same length, into N equal parts by the points

$$s_\sigma=a+\sigma\frac{l}{N},\quad l=b-a,\quad \sigma=0,1,2,\dots,N.$$

Denoting $t_\sigma=t(s_\sigma)$ and using the smoothness of Γ, we can take $h_{\sigma+1}=t_{\sigma+1}-t_\sigma$ [3, 7] and assuming that $\sigma,\nu=0,1,2,\dots,N-1$, we consider now that the point t_0 belongs to the arc $t_\nu t_{\nu+1}$, where $t_\nu t_{\nu+1}$ denotes the smallest arc with ends t_ν and $t_{\nu+1}$ [3, 6].

For the arbitrary numbers σ,ν from $1,2,\dots,N-1$ we define the function $\beta_{\sigma\nu}(\varphi;\,t,t_0)$ dependent on φ,t and t_0 by

$$\beta_{\sigma\nu}(\varphi;\,t,t_0)=(S_3(\varphi;\,t,\sigma)-S_3(\varphi;\,t_0,\nu))\frac{2(t-t_0)}{(t_\sigma-t_0)+(t_{\sigma+1}-t_0)},\qquad(2)$$

where the expression $S_3(\varphi;\,t,\sigma)$ denotes the cubic spline to the function density $\varphi(t)$ on the curve Γ given by the following formula

$$\begin{aligned}S_3(\varphi\ t,\sigma)\ &=\ \frac{M_\sigma(t_{\sigma+1}-t)^3}{6h_{\sigma+1}}+\frac{M_{\sigma+1}(t-t_\sigma)^3}{6h_{\sigma+1}}\\&\quad+\left(\varphi(t_\sigma)-\frac{M_\sigma h_{\sigma+1}^2}{6}\right)\frac{t_{\sigma+1}-t}{h_{\sigma+1}}\\&\quad+\left(\varphi(t_{\sigma+1})-\frac{M_{\sigma+1}h_{\sigma+1}^2}{6}\right)\frac{t-t_\sigma}{h_{\sigma+1}}\end{aligned}$$

and the density φ still represents a given function on the curve Γ of class $H(\mu)$.

Seeing that the equality $[(t_\sigma-t_0)+(t_{\sigma+1}-t_0)]=0$ is possible only when $\sigma=\nu$, in this case we take the function $\beta_{\sigma\sigma}(\varphi;\,t,t_0)$ omitting the expression $\frac{2(t-t_0)}{(t_\sigma-t_0)+(t_{\sigma+1}-t_0)}$, as given by

$$\beta_{\sigma\sigma}(\varphi;\,t,t_0)=S_3(\varphi;\,t,\sigma)-S_3(\varphi;\,t_0,\sigma).\qquad(3)$$

It is simple to see that, for N large enough, the limit of the expression $\frac{2(t-t_0)}{(t_\sigma-t_0)+(t_{\sigma+1}-t_0)}$ is equal to the unit. However, the expressions (2) and (3) are almost equal, so we can confirm that the function $\beta_{\sigma\nu}(\varphi;\,t,t_0)$ is defined for all values of the variables $t,t_0\in\Gamma$, and almost continuous at all points, for all $\sigma,\nu=0,1,\dots,N-1$.

Now we define the function

$$\psi_{\sigma\nu}(\varphi;\,t,t_0)=\begin{cases}\varphi(t_0)+\beta_{\sigma\nu}(\varphi;\,t,t_0),\ t\in\tau_\sigma\tau_{\sigma+1},\ t_0\in\tau_\nu\tau_{\nu+1},\\ \sigma=0,1,\dots,N-1;\ \nu=0,1,\dots,N-1.\end{cases}$$

It can be easily seen that the function $\beta_{\sigma\nu}(\varphi;\, t, t_0)$ contains $(t - t_0)$ as a factor, for all $\sigma, \nu = 0, 1, \ldots, N-1$, whence, the function $\psi_{\sigma\nu}(\varphi;\, t, t_0)$ admits the following representation

$$\psi_{\sigma\nu}(\varphi;\, t, t_0) = \varphi(t_0) + (t - t_0) Q_{\sigma\nu}(\varphi;\, t, t_0). \tag{4}$$

After this construction, one replaces the singular integral (1)

$$F(t_0) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(t)}{t - t_0}\, dt$$

by the following ones

$$S(\varphi;\, t_0) = \frac{1}{\pi i} \int_\Gamma \frac{\psi_{\sigma\nu}(\varphi;\, t, t_0)}{t - t_0}\, dt = \varphi(t_0) + \frac{1}{\pi i} \int_\Gamma Q_{\sigma\nu}(\varphi;\, t, t_0)\, dt. \tag{5}$$

Let us now cite the theorem concerning the accuracy of approximation of singular integrals (1) by expressions of the form (5).

Theorem *Let Γ be a rectifiable simple path of finite length and let φ be a density satisfying the Hölder condition $(H(\mu))$, then the following estimation*

$$\mid F(t_0) - S(\varphi;\, t_0) \mid \leq \frac{C_N}{N^\mu}, \quad N > 1,$$

holds, where the constant C_N depends only of the curve Γ. Furthermore, if we suppose that φ and its first derivatives are continuous and $\max_{t\in\Gamma} \mid \varphi^{(4)}(t) \mid = M$, then one has

$$\mid F(t_0) - S(\varphi;\, t_0) \mid \leq \frac{C_N}{N^{\mu+4}}, \quad N > 1.$$

For the sake of simplicity, we try to prove only the first estimate. Indeed, for $t \in t_\sigma t_{\sigma+1}$ and $t_0 \in t_\nu t_{\nu+1}$, we consider

$$\varphi(t) - \psi_{\sigma\nu}(\varphi;\, t, t_0) = \varphi(t) - \{\varphi(t_0) + \beta_{\sigma\nu}(\varphi;\, t, t_0)\}.$$

For the sake of simplicity we take the cubic spline as a polynomial of degree three characterized by its moments M_σ,

$$S_3(\varphi;\, t, \sigma) = \alpha_\sigma + \beta_\sigma(t - t_\sigma) + \gamma_\sigma(t - t_\sigma)^2 + \delta_\sigma(t - t_0)^3 \ \text{ for } t \in [t_\sigma, t_{\sigma+1}],$$

where

$$\begin{aligned}
\alpha_\sigma &= \varphi(t_\sigma), \\
\beta_\sigma &= \frac{\varphi(t_{\sigma+1}) - \varphi(t_\sigma)}{h_{\sigma+1}} - \frac{2M_\sigma + M_{\sigma+1}}{6} h_{\sigma+1}, \\
\gamma_\sigma &= \frac{M_\sigma}{2}, \\
\delta_\sigma &= \frac{M_{\sigma+1} - M_\sigma}{6h_{\sigma+1}}.
\end{aligned}$$

For all $t \in t_\sigma t_{\sigma+1}$ and $t_0 \in t_\nu t_{\nu+1}$, $\sigma \neq \nu$, we can write

$$\begin{aligned} & \varphi(t) - \psi_{\sigma\nu}(\varphi;\, t, t_0) = \varphi(t) - \varphi(t_0) \\ - \; & \{\varphi(t_\sigma) + \beta_\sigma(t - t_\sigma) + \gamma_\sigma(t - t_\sigma)^2 + \delta_\sigma(t - t_\sigma)^3 \\ & -\varphi(t_\nu) - \beta_\nu(t - t_\nu) - \gamma_\nu(t - t_\nu)^2 \\ & -\delta_\nu(t - t_\nu)^3\} \frac{2(t - t_0)}{(t_\sigma - t_0) + (t_{\sigma+1} - t_0)}. \end{aligned} \tag{6}$$

If $\sigma = \nu$, we can easily put our expression in the form

$$\begin{aligned} & \varphi(t) - \psi_{\sigma\nu}(\varphi;\, t, t_0) = \varphi(t) - \varphi(t_0) \\ - \; & \{\beta_\sigma + \gamma_\sigma((t - t_\sigma) + (t_0 - t_\sigma)) + \delta_\sigma((t - t_\sigma)^2 \\ + \; & (t - t_\sigma)(t_0 - t_\sigma) + (t_0 - t_\sigma)^2)\}(t - t_0). \end{aligned} \tag{7}$$

Taking into account the expressions (6), (7) above, we have

$$\begin{aligned} & \frac{1}{\pi i} \int_\Gamma \frac{\varphi(t) - \psi_{\sigma\nu}(\varphi;\, t, t_0)}{t - t_0}\, dt = \sum_{\substack{\sigma=0 \\ \sigma\neq\nu}}^{N-1} \frac{1}{\pi i} \int_\Gamma \left\{ \frac{\varphi(t) - \varphi(t_0)}{t - t_0} \right. \\ & - \left[\varphi(t_\sigma) + \beta_\sigma(t - t_\sigma) + \gamma_\sigma(t - t_\sigma)^2 + \delta_\sigma(t - t_\nu)^3 \right. \\ & -\varphi(t_\nu) - \beta_\nu(t - t_\nu) - \gamma_\nu(t - t_\nu)^2 \\ & \left.\left. -\; \delta_\nu(t - t_\nu)^3\right] \frac{1}{\frac{t_\sigma + t_{\sigma+1}}{2} - t_0} \right\} dt \\ & + \frac{1}{\pi i} \int_{t_\nu t_{\nu+1}} \left\{ \frac{\varphi(t) - \varphi(t_0)}{t - t_0} - [\beta_\nu + \gamma_\nu((t - t_\nu) + (t_0 - t_\nu)) \right. \\ & \left. +\; \delta_\nu((t - t_\nu)^2 + (t - t_\nu)(t_0 - t_\nu) + (t_0 - t_\nu)^2)] \right\} dt. \end{aligned} \tag{8}$$

Passing now to the estimation of the expression (8), we have for $t_0 \in t_\nu t_{\nu+1}$ and $\sigma \neq \nu$ the relation

$$\left| \begin{aligned} & \sum_{\substack{\sigma=0 \\ \sigma\neq\nu}}^{N-1} \int_{t_\sigma t_{\sigma+1}} \left\{ \tfrac{\varphi(t) - \varphi(t_0)}{t - t_0} - [\varphi(t_\sigma) - \varphi(t_\nu) + \beta_\sigma(t - t_\sigma) \right. \\ & \qquad \left. -\; \beta_\nu(t - t_\nu)] \, \tfrac{1}{\frac{t_\sigma + t_{\sigma+1}}{2} - t_0} \right\} dt \end{aligned} \right| = O(N^{-\mu}).$$

Naturally, this estimation given above is obtained using expressions of β_σ and $\varphi \in H(\mu)$ [2]. Besides, it is easy to see that

$$\left| \sum_{\substack{\sigma=0 \\ \sigma\neq\nu}}^{N-1} \int_{t_\sigma t_{\sigma+1}} \{\gamma_\sigma(t - t_\sigma)^2 - \gamma_\nu(t_0 - t_\nu)^2\} \frac{1}{\frac{t_\sigma + t_{\sigma+1}}{2} - t_0}\, dt \right| = O(N^{-2})$$

and

$$\left|\sum_{\substack{\sigma=0\\ \sigma\neq\nu}}^{N-1}\int_{t_\sigma t_{\sigma+1}}\{\delta_\sigma(t-t_\sigma)^3-\delta_\nu(t_0-t_\nu)^3\}\frac{1}{\frac{t_\sigma+t_{\sigma+1}}{2}-t_0}\,dt\right|=O(N^{-2}).$$

Further, using again the condition $\varphi\in H(\mu)$ and the condition of smoothness of Γ, we have

$$\left|\int_{t_\nu t_{\nu+1}}\frac{\varphi(t)-\varphi(t_0)}{t-t_0}\,dt\right|\le A\int_{s_\nu}^{s_{\nu+1}}|\,s-s_0\,|^{\mu-1}\;ds=O(N^{-\mu}).$$

And again on the base of $\varphi\in H(\mu)$ for the expression of β_ν, we can easily come to

$$\left|\begin{array}{c}\int_{t_\nu t_{\nu+1}}\{\beta_\nu+\gamma\ _\nu((t-t_\nu)+(t_0-t_\nu))\\ +\delta_\nu((t-t_\nu)^2+(t-t_\nu)(t_0-t_\nu)+(t_0-t_\nu)^2)\}\,dt\end{array}\right|=O(N^{-\mu}).$$

Numerical experiments: Using our approximation, we apply the algorithms to singular integrals and we present results concerning the accuracy of the calculations. In these numerical experiments each table I represents the exact value of the singular integral and $\widetilde{I}$ corresponds to the approximate calculation produced by our approximation at points of interpolation.

Example Consider the singular integral

$$I=F(t_0)=\frac{1}{\pi i}\int_\Gamma\frac{\varphi(t)}{t-t_0}\,dt,$$

where the curve Γ denotes the unit circle and the function density φ is given by the following expression

$$\varphi(t)=\frac{-2t^2+8t+12}{4t(t^2-t-6)}.$$

N	$\parallel I-\widetilde{I}\parallel_1$	$\parallel I-\widetilde{I}\parallel_2$	$\parallel I-\widetilde{I}\parallel_\infty$
20	1.8246599E−02	9.1665657E−03	5.0822943E−03
40	3.6852972E−03	1.9270432E−03	1.5697196E−03
60	2.3687426E−03	1.1880117E−03	6.6070486E−04

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