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Improved Results Regarding Some Multidimensional Reaction-Diffusion Systems

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Abstract

We shall generalize some results concerning the boundedness and the blow-up of solutions to some reaction-diffusion systems. Firstly, we consider systems of m unknown functions rather than two, and secondly, we weaken the hypotheses to allow for a large class of nonlinearities.

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1 Introduction

Let Ω be an open and bounded subset of $\mathbb{R}^n$ of class C^1. In this talk we shall be interested in some quantitative properties such as boundedness, exponential decay and blow-up of the nonnegative solutions to the following reaction-diffusion system

$$\begin{cases} \dfrac{\partial u_j}{\partial t} - \Delta u_j = \sum\limits_{i=1}^{m} a_{ij} u_i + f_j\left(t, u\right), \quad t > 0,\ x \in \Omega, \\ \qquad\qquad\qquad \text{for } j = 1, \ldots, m, \end{cases} \tag{1}$$

where u_j stands for $u_j\left(x, t\right)$, with $(x, t) \in \Omega \times (0, \infty)$ and $u = (u_1, \ldots, u_m)$. The constants $\left\{a_{ij}\right\}_{1 \leq i,j \leq m}$ are nonnegative real numbers and the real-valued functions f_j are defined and continuous on the set $(0, \infty) \times \mathbb{R}^m$.

We assume that the solution u is subject to the initial conditions

$$u_j\left(x, 0\right) = u_{0j}\left(x\right),\ x \in \Omega, \quad \text{for } j = 1, \ldots, m, \tag{2}$$

the data u_{0j} for $j = 1, \ldots, m$ are supposed to be continuous and bounded on the set Ω.

We assume further that u satisfies the following boundary condition:

1) **Dirichlet** boundary condition

$$u_j(x,t) = 0,\ t \geq 0,\ x \in \partial\Omega, \quad \text{for } j = 1, \ldots, m, \tag{B1}$$

It is not hard to carry out the same task with either

2) **Neumann** boundary condition

$$\frac{\partial u_j}{\partial \eta}(x,t) = 0,\ t \geq 0,\ x \in \partial\Omega, \quad \text{for } j = 1, \ldots, m \tag{B2}$$

(η being the outer normal vector to $\partial\Omega$), or

3) **Robin** boundary condition

$$\frac{\partial u_j}{\partial \eta}(x,t) = -\sigma(x)\, u_j(x,t),\ t \geq 0,\ x \in \partial\Omega, \quad \text{for } j = 1, \ldots, m. \tag{B3}$$

Because of the importance of the study of quantitative properties of solutions to the above problem we shall generalize, on the one hand, the ideas proposed in [5] to a higher number of unknown functions and, on the other hand, weaken our hypotheses to obtain blow-up and boundedness of the nonnegative solution.

The success of these generalizations lies on deriving some Bernoulli's inequality whose solution can be estimated by some known quantities.

We say that the solution $(u_1, \ldots, u_m)$ of problem $(1), (2), (\mathbf{Bi})$ *blows up at a finite time* $T < \infty$ if its largest domain of existence is the cylinder $\Omega \times [0, T)$ and

$$\lim_{t \to T} \sup_{x \in \Omega} \left(\sum_{i=1}^{m} |u_i(x,t)| \right) = +\infty.$$

If, for each positive and finite T_0 the solution $(u_1, \ldots, u_m)$ remains bounded on the set $\Omega \times [0, T_0)$, then we have a *global existence* of the solution in $\Omega \times [0, \infty)$.

Next, denote respectively by λ and $\phi(x)$ the smallest positive eigenvalue and its corresponding positive eigenfunction satisfying the problem

$$\Delta\phi + \lambda\phi = 0 \ \text{ in } \ \Omega \ \text{ and } \ \phi = 0 \ \text{ on } \ \partial\Omega, \tag{3}$$

with $\int_\Omega \phi(x)\, dx = 1$.

We define throughout this work the function

$$\begin{aligned} J &: [0, T) \to \mathbb{R}, \\ J(t) &= e^{\lambda t} \sum_{j=1}^{m} \int_\Omega \phi u_j \, dx \end{aligned}$$

and $J(0) = \sum_{j=1}^{m} \int_\Omega \phi(x)\, u_{0j}(x)\, dx$.

We shall repeatedly use the following useful discrete inequality, namely,

Lemma 1 *Let m be a positive integer and p a positive real number. If $\{a_i\}_{i=1}^m$ is a set of nonnegative real numbers, then*

$$\left(\sum_{i=1}^m a_i\right)^p \le \left(\max\left(1, m^{p-1}\right)\right)\left(\sum_{i=1}^m a_i^p\right) \quad \text{if } p > 0,$$

and

$$\left(\sum_{i=1}^m a_i^p\right) \le m^{1-p}\left(\sum_{i=1}^m a_i\right)^p \quad \text{if } 0 < p \le 1.$$

To prove the lemma, it suffices to apply the integral Jensen's Inequality with the counting measure and the convex function $\varphi(x) = x^p$, if $p > 1$, and $\varphi(x) = x^{1/p}$, if $0 < p < 1$. Finally, the inequality $\left(\sum_{i=1}^m a_i\right)^p \le \sum_{i=1}^m a_i^p$, for $0 < p < 1$ can be treated by induction on m.

2 Main results

Here is our first result:

Proposition 2 *If*

$$0 \le \sum_{j=1}^m f_j(t,u) \le c_0 \sum_{j=1}^m u_j + \sum_{k=1}^l c_k t^{\alpha_k} \exp(\beta_k t), \ \forall u = (u_1, \ldots, u_m) \in \mathbb{R}_+^m, t \ge 0,$$

for some real constants $\{c_k\}_{k=0}^m$, $\{\alpha_k\}_{k=1}^l$ and $\{\beta_k\}_{k=1}^l$ such that

1) *$0 < c_0 \le \lambda$ and $\beta_k < c_0 - \lambda$ for $k = 1, \ldots, l$,*

2) *$\{c_k\}_{k=1}^m \subset \mathbb{R}^+$ and $0 < \alpha_k + 1$ for $k = 1, \ldots, l$,*

then, the nonnegative solution to system $(1), (2), (\mathbf{B1})$ satisfies the estimate

$$\sum_{j=1}^m \int_\Omega \phi(x) u_j(x,t)\, dx \le [C + J(0)]\, e^{-(\lambda - c_0)t} \quad \text{for all } t \ge 0,$$

where

$$C = \sum_{k=1}^l c_k (c_0 - \lambda - \beta_k)^{-(\alpha_k+1)}\, \Gamma(\alpha_k + 1) \ \ (\ \Gamma(x) \text{ being the gamma function}).$$

Proof. Let $u = (u_1, \ldots, u_m)$ be a nonnegative solution to (1), (2), $(\mathbf{B1})$. Define

$$J(t) = e^{\lambda t} \sum_{j=1}^m \int_\Omega \phi u_j\, dx, \ t \ge 0,$$

then, by differentiation we obtain

$$J'(t) = e^{\lambda t}\int_{\Omega}\phi\sum_{j=1}^{m} f_j(t,u)\,dx \geq 0,$$

so that

$$\begin{aligned} J'(t) &\leq c_0\sum_{j=1}^{m} e^{\lambda t}\int_{\Omega}\phi u_j\,dx + e^{\lambda t}\int_{\Omega}\phi\sum_{k=1}^{l} c_k t^{\alpha_k}\exp(\beta_k t)\,dx \\ &\leq c_0 J(t) + \sum_{k=1}^{l} c_k t^{\alpha_k}\exp(\lambda+\beta_k)\,t, \end{aligned}$$

which gives

$$\left(Je^{-c_0 t}\right)' \leq \sum_{k=1}^{l} c_k t^{\alpha_k}\exp(\lambda+\beta_k-c_0)\,t.$$

Integrating both sides of the above inequality from 0 to t, we get

$$\begin{aligned} J(t)\,e^{-c_0 t} - J(0) &\leq \sum_{k=1}^{l} c_k\int_0^{\infty} t^{\alpha_k}\exp(\lambda+\beta_k-c_0)\,t\,dt \\ &\leq \sum_{k=1}^{l} c_k\left(c_0-\lambda-\beta_k\right)^{-(\alpha_k+1)}\Gamma(\alpha_k+1) = C, \end{aligned}$$

giving

$$J(t) \leq (C+J(0))\,e^{c_0 t}.$$

Therefore,

$$\sum_{j=1}^{m}\int_{\Omega}\phi(x)\,u_j(x,t)\,dx \leq [C+J(0)]\,e^{-(\lambda-c_0)t} \quad \text{for all} \quad t\geq 0. \qquad \square$$

Here is another result of boundedness of the nonnegative solution which is established in the following Theorem:

Theorem 3 *Assume that $u_{0j} \geq 0$ in Ω, for $j = 1,\ldots,m$. Let*

$$f_j = \alpha_j u_j - u_j\sum_{i=1}^{m}\beta_i u_i \quad \textit{for} \quad j = 1,\ldots,m,$$

and

$$a_{ij} = 0 \ \ for \ \ i, j = 1, \ldots, m,$$

where $\{\alpha_j\}_{j=1}^m, \{\beta_j\}_{j=1}^m \subset (0, \infty)$.

Let $\alpha = \max_{1 \le j \le m} \{\alpha_j\}$ *and* $\beta = \min_{1 \le j \le m} \{\beta_j\}$. *If*

$$J(0) = \sum_{j=1}^m \int_\Omega \phi u_{0j}(x)\, dx > \frac{\alpha - \lambda}{\beta},$$

then the nonnegative solution u *to* (1), (2), (**B1**) *satisfies*

$$\sum_{i=1}^m \int_\Omega \phi u_j(t, x)\, dx \le J(0) \quad for \ \ t \ge 0.$$

Proof. Let a and b be positive real numbers satisfying

$$\beta J(0) < a \le 2\beta J(0) + \lambda - \alpha, \tag{4}$$

we set $b = \beta J(0) + \varepsilon$ for $0 < \varepsilon < a - \beta J(0)$. It follows that

$$\beta J(0) < b < a < 2b + \lambda - \alpha. \tag{5}$$

Next, define the function $S(t)$ by

$$S(t) = e^{at} \int_\Omega \phi \left(b - \beta \sum_{j=1}^m u_j \right) dx, \ t \ge 0,$$

then

$$S(0) = \int_\Omega \phi \left(b - \beta \sum_{j=1}^m u_{0j}(x) \right) dx = b - \beta J(0) = \varepsilon > 0.$$

A differentiation with respect to t yields

$$\begin{aligned} S'(t) &= e^{at} \int_\Omega \phi \left\{ ab - \beta \sum_{j=1}^m (a + \alpha_j - \lambda) u_j + \beta \sum_{i=1}^m \sum_{j=1}^m \beta_i u_i u_j \right\} dx \\ &\ge e^{at} \int_\Omega \phi \left\{ ab - \beta \sum_{j=1}^m (a + \alpha - \lambda) u_j + \beta^2 \left(\sum_{j=1}^m u_j \right)^2 \right\} dx. \end{aligned}$$

On the other hand, we have by virtue of (5) the estimates

$$b^2 < ab \text{ and } a + \alpha - \lambda < 2b,$$

from which we get

$$\begin{aligned} & ab - \beta\,(a + \alpha - \lambda)\sum_{j=1}^{m} u_j + \beta^2 \left(\sum_{j=1}^{m} u_j\right)^2 \\ \geq\ & b^2 - 2b\beta \sum_{j=1}^{m} u_j + \left(\beta \sum_{j=1}^{m} u_j\right)^2 \\ =\ & \left(b - \beta \sum_{j=1}^{m} u_j\right)^2 . \end{aligned}$$

Therefore,

$$\begin{aligned} S'(t) &\geq e^{at} \int_\Omega \phi \left(b - \beta \sum_{j=0}^{m} u_j\right)^2 dx \\ &\geq e^{at} \left(\int_\Omega \phi \left(b - \beta \sum_{j=0}^{m} u_j\right) dx\right)^2 \\ &= e^{-at} S^2\,(t)\,, \end{aligned}$$

from which we obtain

$$S(t) \geq \frac{aS\,(0)\,e^{at}}{(a - S\,(0))\,e^{at} + S\,(0)},\ t \geq 0,$$

and accordingly,

$$\begin{aligned} \sum_{j=0}^{m} \int_\Omega \phi u_j\,dx &\leq \frac{1}{\beta}\left\{b - \frac{a\varepsilon e^{at}}{(a-\varepsilon)e^{at} + \varepsilon}\right\} \\ &< \frac{b}{\beta} = \frac{1}{\beta}\left(\beta J\,(0) + \varepsilon\right). \end{aligned}$$

Now, since ε is arbitrary, then

$$\sum_{j=0}^{m} \int_\Omega \phi u_j\,dx \leq J\,(0)\,, \quad t \geq 0,$$

which completes the proof. □

In what follows we shall get rid of some unnecessary integral condition on the initial data which is widely used in previous works in order to obtain the blow-up of the solution.

Theorem 4 *Let* $\{q_{ij}\}_{1\le i,j\le m} \subset (1,\infty)$, $p = \min_{1\le i,j\le m}(q_{ij})$ *and* $q = \max_{1\le i,j\le m}(q_{ij})$, *and let* $\{a_{ij}\}_{1\le i,j\le m}$, $\{b_{ij}\}_{1\le i,j\le m}$, $\{\gamma_{ij}\}_{1\le i,j\le m} \subset [0,\infty)$ *and* $f_j(t,u) = \sum_{i=1}^{m} b_{ij} e^{-\gamma_{ij} t} u_i^{q_{ij}}$, *with*

$$\begin{aligned} \alpha &= \min_{1\le i\le m}\left(\sum_{j=1}^{m} a_{ij}\right) > 0, \quad \beta = \min_{1\le i\le m}\left(\sum_{j=1}^{m} b_{ij}\left(\int_\Omega \phi u_{0i}\,dx\right)^{q_{ij}-p}\right) > 0 \\ \text{and } \gamma &= \max_{1\le i,j\le m}\left(\gamma_{ij}\right). \end{aligned}$$

Then the nonnegative solution $u = (u_1,\ldots,u_m)$ *of problem* (1), (2), (**B1**) *blows up in finite time in each of the following cases:*

1) If $\Delta = \alpha p - \alpha - \lambda q + \lambda - \gamma = 0$ *and* $J(0) > 0$ *, then* u *blows up at a finite time* T_0 *with*

$$T_0 \le \frac{1}{\beta(p-1)}\left(\frac{m}{J(0)}\right)^{p-1}$$

and it satisfies in $[0,T_0)$ *the estimate*

$$\left\|\sum_{j=1}^{m} u_j(t,\cdot)\right\|_\infty \ge mJ(0)\,e^{(\alpha-\lambda)t} / \left\{m^{p-1} - \beta(p-1)J^{p-1}(0)\,t\right\}^{1/(p-1)}.$$

2) If $\Delta > 0$ *and* $J(0) > 0$, *then the solution blows up in a finite time* T_1 *with*

$$T_1 \le \frac{1}{\Delta}\ln\left[1 + \tfrac{\Delta}{\beta(p-1)}\left(\tfrac{m}{J(0)}\right)^{p-1}\right]$$

and satisfies the estimate

$$\left\|\sum_{j=1}^{m} u_j(t,.)\right\|_\infty \ge \frac{\Delta^{1/(p-1)} e^{-\lambda t}}{K^{1/(p-1)}(t)} \quad \text{for } t \in [0,T_1),$$

where

$$\begin{aligned} K(t) &= \left\{\Delta J^{1-p}(0) + \beta(p-1)\,m^{1-p}\right\} e^{-(\alpha p-\alpha)t} \\ &\quad -\beta(p-1)\,m^{1-p} e^{-(q\lambda-\lambda+\gamma)t}. \end{aligned}$$

3) If $\Delta < 0$ *and* $J(0) > m\left(\frac{-\Delta}{\beta(p-1)}\right)^{1/(p-1)}$*, then the solution blows up at a finite time* T_2 *with*

$$T_2 \leq \frac{1}{\Delta} \ln\left[1 + \tfrac{\Delta}{\beta(p-1)}\left(\tfrac{m}{J(0)}\right)^{p-1}\right],$$

and satisfies the estimate

$$\left\| \sum_{j=1}^{m} u_j(t,.) \right\|_\infty \geq \frac{(-\Delta)^{1/(p-1)} e^{-\lambda t}}{(-K)^{1/(p-1)}(t)} \quad \textit{for} \ \ t \in [0, T_2).$$

Proof. Let $u = (u_1, \ldots, u_m)$ be a nonnegative solution to (1), (2), (**B1**) and $J(t)$ be as above. By a differentiation of J we obtain

$$\begin{aligned} J'(t) &= \sum_{i=1}^{m} e^{\lambda t} \int_\Omega \phi \left\{ \left(\sum_{j=1}^{m} a_{ij}\right) u_j + \left(\sum_{j=1}^{m} b_{ij} e^{-\gamma_{ij} t}\right) u_i^{q_{ij}} \right\} dx \\ &\geq \alpha \sum_{i=1}^{m} e^{\lambda t} \int_\Omega \phi u_i \, dx + \sum_{i=1}^{m} \sum_{j=1}^{m} b_{ij} e^{(\lambda-\gamma)t} \int_\Omega \phi u_i^{q_{ij}} \, dx. \end{aligned}$$

On the other hand, we observe that for any $j = 1, \ldots, m$ we have

$$\left(e^{\lambda t} \int_\Omega \phi u_j dx\right)' = \sum_{i=1}^{m} e^{\lambda t} \int_\Omega \phi \left\{ a_{ij} u_i + b_{ij} e^{-\gamma_{ij} t} u_i^{q_{ij}} \right\} dx \geq 0,$$

which gives

$$e^{\lambda t} \int_\Omega \phi u_j dx \geq \int_\Omega \phi u_{0j} \, dx \ \ \text{for} \ \ j = 1, \ldots, m,$$

so that

$$\left(e^{\lambda t} \int_\Omega \phi u_j \, dx\right)^{q_{ij}} \geq \left(\int_\Omega \phi u_{0j} \, dx\right)^{q_{ij}-p} \left(e^{\lambda t} \int_\Omega \phi u_j \, dx\right)^p,$$

for $i, j = 1, \ldots, m$.

Setting $c_j = \int_\Omega \phi u_{0j}\,dx$ for $j = 1, \ldots, m$, and taking into account Lemma 1, we get

$$\begin{aligned}
J'(t) &\geq \alpha \sum_{i=1}^{m} e^{\lambda t} \int_\Omega \phi u_i\,dx + \sum_{i=1}^{m}\sum_{j=1}^{m} b_{ij} e^{(\lambda-\gamma)t} \int_\Omega \phi u_i^{q_{ij}}\,dx \\
&\geq \alpha \sum_{i=1}^{m} e^{\lambda t} \int_\Omega \phi u_i\,dx + \sum_{i=1}^{m}\sum_{j=1}^{m} b_{ij} e^{(\lambda-\gamma-q\lambda)t} \left(e^{\lambda t}\int_\Omega \phi u_i\,dx\right)^{q_{ij}} \\
&\geq \alpha \sum_{i=1}^{m} e^{\lambda t} \int_\Omega \phi u_i\,dx + e^{(\lambda-\gamma-q\lambda)t} \sum_{i=1}^{m} \left(\sum_{j=1}^{m} b_{ij} c_i^{q_{ij}-p}\right) \left(e^{\lambda t}\int_\Omega \phi u_i\,dx\right)^{p} \\
&\geq \alpha J\left(t\right) + \beta m^{1-p} e^{(\lambda-\gamma-q\lambda)t} J^p\left(t\right).
\end{aligned}$$

Putting $\psi\left(t\right) = J^{1-p}\left(t\right)$, then the previous differential inequality becomes

$$\psi' + \alpha(p-1)\psi \leq \beta(1-p) m^{1-p} e^{(\lambda-\gamma-q\lambda)t},$$

so that

$$\left(\psi e^{(\alpha p-\alpha)t}\right)' \leq -\beta\left(p-1\right) m^{1-p} e^{(\alpha p-\alpha+\lambda-\gamma-\lambda q)t}. \tag{6}$$

Set $\Delta = \alpha p - \alpha + \lambda - \gamma - \lambda q$, and consider the following cases:

1) If $\Delta = 0$, then the differential inequality (6) gives

$$\psi\left(t\right) = \frac{1}{J^{p-1}\left(t\right)} \leq e^{-\alpha(p-1)t}\left\{J^{1-p}\left(0\right) - \beta\left(p-1\right) m^{1-p} t\right\},$$

this is meaningful whenever $t < T_0^* = \dfrac{1}{\beta\left(p-1\right)}\left(\dfrac{m}{J\left(0\right)}\right)^{p-1}$.

Therefore, for $0 \leq t < T_0^*$, one has

$$J(t) \geq e^{\alpha t} / \left\{J^{1-p}\left(0\right) - \beta\left(p-1\right) m^{1-p} t\right\}^{1/(p-1)}$$

and, accordingly, the solution must blow up at a finite time $T_0 \leq T_0^*$, and

$$\sum_{j=1}^{m} \int_\Omega \phi u_j dx \geq mJ\left(0\right) e^{(\alpha-\lambda)t} / \left\{m^{p-1} - \beta\left(p-1\right) J^{p-1}\left(0\right) t\right\}^{1/(p-1)}$$

for all $t \in \left[0, T_0\right)$, from which we infer

$$\left\|\sum_{j=1}^{m} u_j\left(t,\cdot\right)\right\|_\infty \geq mJ\left(0\right) e^{(\alpha-\lambda)t} / \left\{m^{p-1} - \beta\left(p-1\right) J^{p-1}\left(0\right) t\right\}^{1/(p-1)}$$

for $0 \leq t < T_0$.

2) Suppose that $\Delta > 0$.

Integrating (6) we find

$$\psi \leq \frac{\left\{\Delta J^{1-p}(0) + \beta(p-1) m^{1-p}\right\} e^{-(\alpha p-\alpha)t} - \beta(p-1) m^{1-p} e^{-(q\lambda-\lambda+\gamma)t}}{\Delta}.$$

Denote by $K(t)$ the numerator of the right-hand side of the above estimate. Since $\psi \geq 0$, then $K(t)$ must be nonnegative. On the other hand, we observe that $K(t)$ vanishes if and only if there is $T_1^* > 0$ such that

$$\frac{e^{-(q\lambda-\lambda+\gamma)T_1^*}}{e^{-(\alpha p-\alpha)T_1^*}} = \frac{\Delta J^{1-p}(0) + \beta(p-1)m^{1-p}}{\beta(p-1)m^{1-p}},$$

that is,

$$e^{\Delta T_1^*} = 1 + \frac{\Delta}{\beta(p-1)} \left(\frac{m}{J(0)}\right)^{p-1},$$

giving

$$T_1^* = \frac{1}{\Delta} \ln \left[1 + \frac{\Delta}{\beta(p-1)} \left(\frac{m}{J(0)}\right)^{p-1}\right].$$

Thus, $K(t)$ is nonnegative if and only if $t \in [0, T_1^*]$. Next, since

$$J^{1-p}(t) \leq \frac{K(t)}{\Delta},$$

then

$$J(t) \geq \left(\frac{\Delta}{K(t)}\right)^{1/(p-1)} \quad \text{for} \quad t \in [0, T_1^*).$$

This shows that the solution must blow up at a finite time $T_1 \leq T_1^*$, and we have

$$\left\| \sum_{j=1}^{m} u_j(t, \cdot) \right\|_\infty \geq \frac{\Delta^{1/(p-1)} e^{-\lambda t}}{K^{1/(p-1)}(t)} \quad \text{for} \quad t \in [0, T_1).$$

3) Suppose now that $\Delta < 0$.

Integrating (6) we find as above

$$\psi \leq \frac{K(t)}{\Delta} = \frac{-K(t)}{-\Delta}.$$

Since $\psi \geq 0$ and $\Delta < 0$, then $-K(t)$ must be nonnegative. On the other hand, $K(t)$ vanishes at $T_2^* > 0$ given by

$$e^{\Delta T_2^*} = 1 + \frac{\Delta}{\beta(p-1)} \left(\frac{m}{J(0)}\right)^{p-1},$$

that is,

$$T_2^* = \frac{1}{\Delta} \ln \left[1 + \tfrac{\Delta}{\beta(p-1)} \left(\tfrac{m}{J(0)} \right)^{p-1} \right],$$

provided that $J(0) > m \left(\frac{-\Delta}{\beta(p-1)} \right)^{1/(p-1)}$.

Hence, $-K(t)$ is nonnegative if and only if $t \in [0, T_2^*]$. Next, since

$$J^{1-p}(t) \leq \frac{-K(t)}{-\Delta},$$

then

$$J(t) \geq \left(\frac{-\Delta}{-K(t)} \right)^{1/(p-1)} \quad \text{for} \quad t \in [0, T_2^*).$$

This shows that the solution blows up at a finite time $T_2 \leq T_2^*$; moreover, we have the estimate

$$\left\| \sum_{j=1}^{m} u_j(t, \cdot) \right\|_\infty \geq \frac{(-\Delta)^{1/(p-1)} e^{-\lambda t}}{(-K)^{1/(p-1)}(t)} \quad \text{for} \quad t \in [0, T_2). \qquad \square$$

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