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The Lyapunov Exponent and the Long-Term Stability of the Natural Systems

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Abstract

Our recent interest is focused on establishing the necessary and sufficient conditions that guarantee a long-term stable evolution of both natural and artificial systems that exert fluctuations. One already established necessary condition is the boundedness of the fluctuations, namely: the fluctuations should not exceed the thresholds of stability of the system. Another necessary condition is the boundedness of the increments. This condition means that the amount of energy and/or matter that the system exchanges currently with the environment is also bounded. This provokes our interest in studying the properties of the bounded irregular sequences (BIS).

A certain class of rather chaotic properties of the BIS'es are set on the strong parallel between the deterministic chaos brought about by simple dynamical systems and the BIS'es. This parallel is provided by the boundedness, since the available phase space volume of any system exhibiting deterministic chaos is always finite. Thus, the created trajectories are BIS'es. The major mechanism that brings about chaos in simple dynamical systems is stretching and folding. From the viewpoint of a BIS, the presence of folding is another necessary condition that provides a long-term stable evolution. By the use of reformulated in terms of BIS'es Lyapunov exponent we find out the condition for presence of folding mechanism that is insensitive to the particularities of the boundary conditions imposed by the thresholds of stability. The target condition is a certain relation among the 3 characteristics of every BIS: the threshold of stability, the variance and a parameter set on the incremental statistics.

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Introduction

The major goal of a series of our papers [1, 2, 3] is to establish explicitly the necessary and sufficient conditions that guarantee a long-term stable evolution of both natural and artificial systems that exert fluctuations. One of the most important problems is whether these conditions are insensitive to the nature of the fluctuation sequence, to the details of its incremental statistics and to its length. So far, we have found out two necessary conditions that have such general properties. The first one is that the fluctuations that a system exerts should not exceed the thresholds of stability of the system. In other words, the fluctuations should be permanently bounded. Another necessary condition is the boundedness of the increments. This condition means that the amount of energy and/or matter that the system exchanges currently with the environment is also bounded. The next general condition is that the incremental statistics should have finite memory so that its size be much smaller than the thresholds of stability. Then, each bounded irregular sequence (BIS) exhibits properties set on the boundedness and the finite size of the incremental memory that are insensitive to the particularities of the incremental statistics and to the length of the sequence [3]. Some of them are the following:

(i) The power spectrum uniformly fits the shape $1/f^{\alpha(f)}$, where $\alpha(f) \to 1$ as $f \to 1/T$ (T is the length of the sequence) and $\alpha(f)$ monotonically increases to $p > 2$ as $f \to \infty$.

(ii) the phase space attractor has a non-integer correlation dimension $\nu(X)$ that monotonically decreases from $\nu(X) = d$ at the mean value to $\nu(X) = 0$ at the boundaries of the attractor; d is the embedding (topological) dimension.

(iii) the Kolmogorov entropy is finite.

So far, these properties have been exclusive for the deterministic chaos — a phenomenon that occurs in the dynamics of simple deterministic systems. It is associated with unpredictability and great sensitivity to the initial conditions introduced by stretching and folding mechanism [4, 5]. However, the deterministic chaos also exhibits boundedness: the folding is provided by the fact that the dynamics of the discussed systems is confined to a finite volume of phase space. The stretching happens along the unstable directories and gives rise to the unpredictability.

Yet, because of the confinement to a finite phase space volume, every time series produced by the stretching and folding mechanism creates a BIS. Dynamical systems whose set of unstable hyperbolic points has a non-zero measure give rise to BIS'es whose incremental memory has a finite size. Then, it is to be expected that the chaotic properties listed above are rather generic for the BIS'es than being the hallmarks of only the deterministic chaos. The leading role of the boundedness poses the question how to relate the characteristics of the deterministic chaos to those of the BIS'es.

Why is it important to reformulate the hallmarks of the deterministic chaos in terms of BIS'es? Our particular aim is to substantiate explicitly the characteristics of the stretching and folding mechanism for the BIS'es. In particular, our attention is focused on the folding because its presence sustains the evolution of a chaotic system permanently bounded in a finite phase volume. Intuitively it seems that for the BIS'es the folding is ensured by the presence of the thresholds of the stability. However, it is to be expected that the particularities of the boundary conditions imposed by the thresholds of stability make the folding mechanism sensitive to them and thus not universal. The question now becomes whether there is a folding mechanism insensitive to the details of the boundary conditions. We consider this problem together with the problem about the existence of a folding mechanism viewed as a necessary condition for keeping the evolution in the phase space bounded arbitrarily long time. We find out that a universal folding mechanism does exist when a certain relation among the thresholds of stability, a parameter set on the incremental statistics and the variance of the BIS holds. In turn, this relation is another necessary condition that provides a long-term stable evolution of a BIS.

It is obvious that from the viewpoint of the deterministic chaos the folding mechanism is associated with a negative value of the Lyapunov exponent. On the other hand, from the point of view of BIS'es, it is to be associated with the largest fluctuations, namely those whose amplitude is of the order of the thresholds of stability. Thus, our task is to define the Lyapunov exponent in terms of the large scaled fluctuations and to show explicitly the dependence of its value and sign on the their characteristics. A great advantage in doing this is the universal behavior of the large-scaled fluctuations in the case when the incremental statistics have finite-size memory [3]. It has been found out that namely the large scaled fluctuations are the bearers of the chaotic properties listed above. The finite size of the incremental memory renders the influence of the incremental statistics over the properties of the large-scaled fluctuations to be manifested by a single parameter. The latter is the one that appears in the target relation among the characteristics of BIS'es that provides the folding mechanism. It should be stressed that we look for a folding mechanism that does not involve any specific boundary condition(s) imposed by the thresholds of stability.

Thus, our study is focused on deriving an explicit definition of the Lyapunov exponent in terms of BIS'es. This is done in the next section.

The measure for the unpredictability of the deterministic chaos is the value of the Lyapunov exponent. Figuratively, it is the average measure how fast a trajectory deviates under an arbitrarily small perturbation of the initial conditions. Its rigorous definition is:

$$\xi = \lim_{t\to\infty} \frac{1}{t} \log |U(t)|, \tag{1a}$$

where

$$U(t) = X(t) - X^*(t), \tag{1b}$$

$X^*(t)$ is the unperturbed trajectory and $U(t)$ is the average deviation from it. So $|U(t)|$ is the measure of all the available deviations from a given point $X^*(t)$.

Now we have to rewrite eq. (1) by using the characteristics of a BIS. Evidently, the essential deviations of any trajectory come out from the large fluctuations. So, to construct an explicit expression for the Lyapunov exponent in terms of large-scaled fluctuations we need those properties of a BIS that are brought about by the boundedness and finite-size incremental memory and at the same time preserve the chaoticity.

1 Properties of large-scaled fluctuations of BIS'es [3]

Very recently it has been obtained [3] that the large scale fluctuations of every BIS appear as a sequence of separated by non-zero intervals successive excursions. Each excursion is characterised by its amplitude, duration and embedding interval. An excursion is a trajectory of a walk originating at the mean value of a given sequence at the moment t and returning to it for the first time at the moment $t+\Delta$. The separation of the successive excursions means embedding of each of them in a larger interval so that no other excursions can be found in that interval. The duration of the "embedding" interval is a multi-valued function whose properties are strongly related to the duration of the embedded excursion Δ itself: the range and the values of the selection are set on Δ; the realisation of any excursion is always associated with the realisation of its embedding interval. Since the duration of each embedding interval is a multi-valued function, its successive performances permanently introduce stochasticity through the random choice of one selection among all available. Thus, this induced stochastisity breaks any possible long-range periodicity (*i.e.*, large-size memory) and helps the large-scaled excursion sequence to preserve the chaotic properties listed in the Introduction.

The relation amplitude-duration of each excursion is a result of the "blob" structure of the underlying incremental walk. The finite size of the incremental memory renders that the incremental walk can be considered as a discrete symmetric random walk with a step equal to the blob size. The implication that the incremental memory has a finite size renders that the incremental walks that creats blobs has also a finite length m; the particularities of the incremental statistics determines an exponent β, such that the $\sqrt{m.s.d.}$ of the blob creating walks equals m^β. Thus, the relation between the amplitude X (where X^2 is the m.s.d.=*mean square deviation* of the incremental walk) and the duration Δ of an excursion reads

$$X^2 \propto m^{\beta} \frac{\Delta}{m} = \Delta^{\alpha(\Delta)}, \tag{2}$$

where $\alpha(\Delta)$ is a monotonic function that tends to 1 on increasing the number of steps of the underlying incremental walk N regardless to what the value of β is. The latter is determined by the specific properties of the incremental statistics but it is confined in the range $[\beta_{\min}, 2]$. The non-zero bounds of β are due to the boundedness of the increments. Indeed, any boundedness introduces certain artificial correlations among increments. In turn, these correlations manifest themselves in a finite value of β. The value of the upper bound is brought about by the largest possible correlation among the increments. The finite value of β means that the size, duration and the structure of the blobs strongly depends on the incremental statistics and thus are specific to the BIS. To compare, suppose $\beta = 0$. In this case, the size and the duration of the blobs are independent of one another and appear as parameters. On the contrary, any $\beta \neq 0$ provides a diffeomorphism between the amplitude of an excursion X and its duration Δ. Indeed, along with eq. (2) there holds the following relation:

$$\Delta \propto \frac{1}{m^{\beta}} \frac{m}{N} = X^{\mu(X)}, \tag{3}$$

where $\mu(X)$ is diffeomorphic to $\frac{1}{\alpha(\Delta)/2}$.

A major property of any BIS is the existence of mean and variance that is guaranteed by the Lindeberg theorem [6]. This brings about two very important consequences. The first one is that the amplitudes of the excursions are normally distributed. The second one is that the excursion sequence is a stationary process. Indeed, the boundedness and the finite-size memory render a uniform convergence of the average to the mean of every BIS. In turn, it provides the stationarity of the excursion sequence. Thus, the frequency of occurrence of an excursion of any size X is time-independent. Following [2], it reads:

$$P(X) = cX^{\mu(X)} \frac{1}{\sigma} \exp\left(-X^2/\sigma^2\right). \tag{4}$$

The required probability $P(x)$ is given by the duration $\Delta = X^{\mu(X)}$ of an excursion of amplitude A weighted by the probability for appearance of an excursion of that size. As it has been already established, the fluctuation sizes are normaly distributed. σ is the variance of the BIS and in the present consideration is a parameter. $c = \sigma^{-1/\beta(\sigma)}$ is the normalizing term. The stationarity of the excursion appearance ensures that

$P(X)$ has the same value at every point of the sequence. It is worth noting that the variance is a characteristic of the full BIS, not only of its fine or coarse-grained structure.

2 Stretching and folding of the excursion sequences

Now we are ready to write down an asymptotic explicit expression for the average deviation from a trajectory that starts at X^*. The corresponding $|U(t)|$ set on in terms of the excursions reads:

$$|U(t)| = \int_{X^*}^{X_{tr}} XP(X)\,dX + \int_{X_{cgr}}^{X^*} XP(X)\,dX. \tag{5}$$

X_{cgr} is the level of coarse-graining, *i.e.*, averaging over all scales smaller than X_{cgr}. This "smooths out" all the excursions whose sizes are smaller than X_{cgr} and renders their contribution to the Lyapunov exponent zero.

The separation into two terms each of which represents the deviations from X^* to larger and smaller amplitudes is formal. It is made only to elucidate the idea that starting at any point of the attractor one can reach any other through a sequence of excursions. Thus, the Lyapunov exponent ξ reads:

$$\xi = \log \int_{X_{cgr}}^{X_{tr}} XP(X)\,dX. \tag{6}$$

Here we come to the same result as the Oseledec theorem states [7], namely: the Lyapunov exponent for the chaotic systems does not depend on the initial point of the trajectory.

On the other hand, the stationarity of the excursion process makes the set of excursion sequence a dense set of periodic orbits. Moreover, it renders its transitivity as well: starting anywhere in the attractor a sequence of excursions can reach any other point in it. Some authors [8] list these properties as a definition of the chaos. Here they appear as a result of the boundedness and finite-size incremental memory of the BIS'es. It is worth noting that the chaotic properties listed in the Introduction are also a result of the boundedness and the finite-size memory of the incremental statistics. It supports our suggestion about the paramount role of the boundedness in defining the chaoticity.

It should be stressed that the chaotic properties listed in the Introduction are derived under the condition that all scales larger than the blob size contribute uniformly to the stochastic properties of any BIS. The above chaotic properties also do not involve any specific scale larger than the blob size. However, it seems that it brings about a contradiction: how the scale-free process of excursion occurrence interferes with the boundary conditions imposed by the presence of the thresholds of stability. The contradiction is solved in the presence of folding since the latter makes the approach to a boundary a tangent U-turn. Thus, the folding being a neccessary condition for keeping the evolution of a BIS permanently confined in a finite attractor ensures that the chaoticity produced by the stretching and folding is a scale-free process.

It is to be expected that the size of an excursion determines its contribution to the stretching or folding of a trajectory. Indeed, the small size excursions are random walks whose frequency is essentially high (eq. (4)). Thus, figuratively speaking, they "hold" any trajectory permanently deviated from the mean. So, the small size excursion most probably contributes to the stretching of the trajectories. On the contrary, the largest excursions are rather occasional and any trajectory subjected to them spends most of its time closest to the mean. So, they would provide the folding. The explicit revealing of the role of small and large excursions is made by the use of the coarse-graining: the role of the excursion size is carried out by scanning the ratio X_{cgr}/σ.

The ratio X_{cgr}/σ has two extreme cases:

(ii) $X_{cgr} \ll \sigma$. By the use of the steeepest descent method eq. (6) yields:

$$\xi \approx \log \sigma. \tag{7}$$

Eq. (7) says that asymptotically any trajectory visits any point of the attractor so that the mean deviation from any initial point is the same for every trajectory and is bounded by the thresholds of the attractor itself. This makes the value of ξ positive. The latter justifies our speculation that the small size excursions contribute to the stretching.

(ii) $X_{cgr} \gg \sigma$. In this case eq. (6) yields

$$\xi \approx (\mu(X_{cgr}) + 1) \log \frac{X_{cgr}}{\sigma} + \log \sigma - \frac{X_{cgr}^2}{\sigma^2}. \tag{8}$$

While ξ from eq. (7) is always positive which provides stretching, eq. (8) opens the alternative for ξ being both positive or negative setting on the relation among μ, σ and X_{cgr}. The permanent presence of folding is necessary for keeping the evolution bounded in a finite size attractor arbitrary long time. The natural measure of the folding is the negative value of the Lyapunov exponent. Thus we come to the condition: the largest size fluctuations provide folding when X_{tr}, $\mu(X_{tr})$ and σ are such that $\xi < 0$. So our target relation reads:

$$\xi \approx (\mu(X_{tr}) + 1) \log \frac{X_{tr}}{\sigma} + \log \sigma - \frac{X_{tr}^2}{\sigma^2} < 0. \quad (9)$$

It is worth noting that the eq. (9) is derived under the condition that the excursions do not "feel" the boundaries at $X = X_{tr}$. Thus, the folding is insensitive to the details of the threshold of stability and the way it is approached. It is worth noting that the folding is a broader notion than a tangent approach to the boundary. Both the folding and the tangent approach produce the same effect — they contribute to the convergence of a trajectory making it depart from the threshold. Yet, the tangent approach itself is a property of the random walk that creates the excursions along with appropriate boundary conditions imposed, while the folding is provided by eq. (9) not involving any boundary conditions.

Conclusions

The present paper elucidates the interrelation between the boundeness as a necessary condition for a BIS to demonstrate chaotic properties and the presence of stretching and folding mechanism for keeping the evolution of a BIS confined in a finite attractor arbitrarily long time. In order to account for the leading role of the boundedness for exhibiting chaos we reformulate the major characteristics of the stretching and folding mechanism, the Lyapunov exponent, in terms of BIS'es. On the other hand, an unlimited in the time stable evolution of a BIS is possible if and only if the folding does not "feel" the boundary conditions imposed by the presence of the tresholds of stability. Otherwise, the evolution stongly depends on the way the excursion approaches the threshold.

The new definition of the Lyapunov exponent is set on the properties of the large-scaled fluctuations. Previously, it has been established that the large-scaled fluctuations of every BIS occur as a sequence of well separated by non-zero quiescent intervals excursions [3]. Each excursion is characterised by 3 interrelated parameters: amplitude, duration and embedding interval. This structure of the large-scaled fluctuations sequence is a result of 3 assumptions: (i) boundedness of fluctuations; (ii) finite size of the incremental memory and (iii) a uniform contribution of all scales larger than the size of the incremental memory to the properties of a BIS. Then, the excursion sequence is the bearer of the chaotic properties listed in the Introduction.

Now we have proved that when a certain relation (eq. (9)) among the major characteristics of any BIS holds, there exists a stretching and folding mechanism that meets the above assumptions. Actually, eq. (9) ensures the presence of folding.

In turn, the latter renders the largest possible excursions to make a tangent U-turn at the threshold of stability. As a result, the presence of the thresholds is unperceptible which ensures that all scales, thresholds included, uniformly participate in setting on the chaoticity of a BIS. In turn, the uniform participation of all scales, thresholds included, ensures an unlimited confinement of the BIS evolution in a finite attractor.

Indeed, eqs. (5)–(6) justify the insensitivity of the Lyapunov exponent to any scale and point in the attractor. This insensitivity along with the stationarity of the excursion occurence makes the coarse-grained structure of the BIS attractor a transitive dense set of periodic orbits. Consequently, our attractor has steady properties insensitive to the development of any specific trajectory and/or the way of approaching the boundaries.

On the other hand, the universality of the large-scaled fluctuations occurrence as a sequence of excursions renders that the coarse-grained structure of every BIS is also universal, namely: on meeting the relation (9) each BIS attractor is a transitive dense set of periodic orbits.

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