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Nonexistence of Solutions for Semilinear Equations and Systems in Cylindrical Domains

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Abstract

We establish an integral identity in $\Omega = \mathbb{R} \times]\alpha, \beta[$ which we use to prove nonexistence of nontrivial solutions in $H^2(\Omega) \cap L^\infty(\Omega)$ to some semilinear equations under some conditions on f and g. We then extend this method to systems of the form

$$\begin{cases} \lambda \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = g(v) \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+, \\ \lambda \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = f(u) \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+, \\ u = v = 0 \text{ on } \partial\Omega. \end{cases}$$

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1 Introduction and notations

The question of existence and the nonexistence of solutions for the semi-linear elliptic problem in bounded or unbounded domain Ω in $\mathbb{R}^N$

$$\begin{cases} -\Delta u + f(u) = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

was studied by several authors for different reasons. We quote by way of examples the works of **Esteban & Lions** [2], **Kirane, Nabana & Pohozaev** [5], **Pucci & Serrin** [11], **Pohožaev** [12] and **Van der Vorst** [13].

M. J. Esteban & **P.-L. Lions** show that the Dirichlet problem

$$\begin{cases} -\Delta u + f(u) = 0, \quad u \in C^2(\overline{\Omega}), \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

satisfying $\nabla u \in L^2(\Omega),\ F(u) = \int_0^u f(s)ds \in L^1(\Omega)$, where Ω is a connected unbounded domain of $\mathbb{R}^N$ such that

$$\exists \Lambda \in \mathbb{R}^N,\ |\Lambda| = 1,\ \langle n(x), \Lambda\rangle \geq 0 \ \text{ on } \ \partial\Omega,\ \langle n(x), \Lambda\rangle \neq 0$$

($n(x)$ is the outward normal to $\partial\Omega$ at the point x) does not have a solution.

The question which arises then is to know if this result is still valid for the Neumann problem

$$\begin{cases} -\Delta u + f(u) = 0, \\ \dfrac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

The answer to this question is negative. Indeed, **Berestycki**, **Gallouët** and **Kavian** established that the problem

$$-\Delta u - u^3 + u = 0, \qquad u \in H^2(\mathbb{R}^2),$$

admits a radial solution, see [1].

The same solution satisfies

$$\begin{cases} -\Delta u - u^3 + u = 0, \quad u \in H^2(]0, +\infty[\times\mathbb{R}), \\ \qquad\qquad\quad \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \{0\}\times\mathbb{R}. \end{cases}$$

To show the nonexistence of solutions of elliptic problems several methods exist, but for this work, we use integral identities.

We establish in the second section an integral identity in a cylindrical domain of $\mathbb{R}^2$ which shows that some semilinear elliptic as well as hyperbolic equations do not have nontrivial solutions in $H^2(\Omega) \cap L^\infty(\Omega)$.

In the third section, we illustrate our results by examples, namely we show that, under some assumptions on the nonlinearity, the **Klein–Gordon** equation does not have nontrivial solutions.

Finally, in the last section, we prove that with the help of two integral identities the following differential system

$$\begin{cases} \lambda\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = g(v) \ \text{ in } \ \Omega = \mathbb{R}\times\mathbb{R}^+, \\ \lambda\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = f(u) \ \text{ in } \ \Omega = \mathbb{R}\times\mathbb{R}^+, \\ u = v = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where f and g satisfy

$$\begin{cases} f, g \in C(\mathbb{R}), \\ f(0) = g(0) = 0, \\ F(u) \cdot G(v) \geq 0, \end{cases}$$

does not possess nontrivial solutions (u, v) in $H^2(\Omega) \cap L^\infty(\Omega) \times H^2(\Omega) \cap L^\infty(\Omega)$.

A nonexistence result for problems of the form

$$\begin{cases} \Delta^2 u = f(u) \text{ in } \Omega, \\ \Delta u = 0 \text{ on } \partial\Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

will follow as a particular case of the above system.

Let us denote by (x, y) a generic point of $\Omega = \mathbb{R}\times]\alpha, \beta[,\ \Gamma = \partial\Omega = \partial\left(\mathbb{R}\times]\alpha, \beta[\right) = \mathbb{R} \times (\alpha) \cup \mathbb{R} \times (\beta)$ and $n(x, y) = (n_1(x, y),\ n_2(x, y))$ the outward normal to Γ at the point (x, y). We consider a locally Lipschitzian real function

$$f :]\alpha, \beta[\times\mathbb{R} \to \mathbb{R},$$

such that $f(y, 0) = 0\ \forall y \in]\alpha, \beta[$, so that $u = 0$ is a solution of the problem

$$\begin{cases} \lambda \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} + f(y, u) = 0 \quad \text{in} \quad \Omega = \mathbb{R}\times]\alpha, \beta[, \\ u + \varepsilon \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{P.1}$$

where λ **is a real parameter** and ε **is a positive real number.**

We shall also use the notation $F(y, u) = \int_0^u f(y, \sigma)\, d\sigma$.

2 General results

We are now in a position to state the following result:

Proposition 1 *Let u be a solution of $(P.1)$, then for any $x \in \mathbb{R}$ and $\varepsilon > 0$,*

$$\int_\alpha^\beta \left[\lambda \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial y} \right|^2 + 2F(y, u) \right] (x, y)\, dy + \frac{1}{\varepsilon} \left[u(x, \alpha)^2 + u(x, \beta)^2 \right] = 0. \tag{2.1}$$

Proof. Let us set

$$\mathcal{K}(x) = \int_{\alpha}^{\beta} \left[\frac{\lambda}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{2} \left| \frac{\partial u}{\partial y} \right|^2 + F\ (y,u) \right] (x,y)\, dy.$$

Under the above hypothesis $\mathcal{K}$ is absolutely continuous and we have almost everywhere on $\mathbb{R}$:

$$\mathcal{K}'(x) = \int_{\alpha}^{\beta} \left[\lambda \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial^2 u}{\partial x^2} \right) + \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^2 u}{\partial x \partial y} \right) + \left(\frac{\partial u}{\partial x} \right) f(y,u) \right] (x,y)\, dy.$$

An integration by parts yields

$$\begin{aligned} \mathcal{K}'(x) &= \int_{\alpha}^{\beta} \left[\lambda \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} - \frac{d}{dy} \left(\frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial x} + f(y,u) \frac{\partial u}{\partial x} \right] (x,y)\, dy + \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial u}{\partial y} \right) \Bigg|_{y=\alpha}^{y=\beta} \\ &= \int_{\alpha}^{\beta} \left(\left(\lambda \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + f(y,u) \right) \frac{\partial u}{\partial x} \right) (x,y)\, dy + \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial u}{\partial y} \right) \Bigg|_{y=\alpha}^{y=\beta} \\ &= \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial u}{\partial y} \right) \Bigg|_{y=\alpha}^{y=\beta}. \end{aligned}$$

Or,

$$u + \varepsilon \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \Longleftrightarrow \begin{cases} \dfrac{\partial u(x,\beta)}{\partial y} + \dfrac{1}{\varepsilon} u(x,\beta) = 0, \\ \dfrac{\partial u(x,\alpha)}{\partial y} - \dfrac{1}{\varepsilon} u(x,\alpha) = 0. \end{cases}$$

If $0 < \varepsilon < +\infty$, we may write:

$$\begin{aligned} \frac{\partial u(x,\beta)}{\partial y} &= -\frac{1}{\varepsilon} u(x,\beta) \ \text{ and} \\ \frac{\partial u(x,\alpha)}{\partial y} &= \frac{1}{\varepsilon} u(x,\alpha). \end{aligned}$$

The boundary term is then equal to

$$\begin{aligned} \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial u}{\partial y} \right) \Bigg|_{y=\alpha}^{y=\beta} &= -\frac{1}{\varepsilon} \left[\left(\frac{\partial u(x,\beta)}{\partial x} \right) u(x,\beta) + \left(\frac{\partial u(x,\alpha)}{\partial x} \right) u(x,\alpha) \right] \\ &= -\frac{1}{2\varepsilon} \frac{d}{dx} \left[(u(x,\alpha))^2 + (u(x,\beta))^2 \right], \end{aligned}$$

and finally,

$$\frac{d}{dx}\left(K(x)+\frac{1}{2\varepsilon}\left[(u(x,\alpha))^2+(u(x,\beta))^2\right]\right)=0,$$

thus the expression in parentheses is constant, but

$$\int_{-\infty}^{+\infty}\left(\mathcal{K}\,(x)+\frac{1}{2\varepsilon}\left[(u(x,\alpha))^2+(u(x,\beta))^2\right]\right)\,dx<+\infty$$

implies that this constant is zero. This proves the Proposition. □

Remark 1 If $\varepsilon=0$ **(Dirichlet condition)**, $u=0$ on $\partial\Omega$ implies $\nabla u=\frac{\partial u}{\partial n}n$ and this allows us to write

$$\left(\frac{\partial u}{\partial x}\right)(x,\,y)=\left(\frac{\partial u}{\partial n}\right)n_1(x,\,y)$$

and

$$\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\Big|_{y=\alpha}^{y=\beta}$$

vanishes.

If $\varepsilon=+\infty$ **(Neumann condition)**, $\dfrac{\partial u}{\partial n}=0$ on $\partial\Omega$ becomes

$$\frac{\partial u}{\partial y}=0 \quad \text{on} \quad \partial\Omega$$

and

$$\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\Big|_{y=\alpha}^{y=\beta}$$

also vanishes.

The problem $(P.1)$ includes in fact **two types of equations** depending on whether λ is positive or negative.

2.1 Hyperbolic case

Let us present two theorems of nonexistence of nontrivial solutions.

Theorem 1 *Suppose that $u\in H^2(\Omega)\cap L^\infty(\Omega)$ is a solution of $(P.1)$, $\lambda>0$ and f satisfies*

$$F(y,\,u)\geq 0. \qquad (A)$$

Then $u\equiv 0$.

Proof. We apply formula (2.1) to obtain

$$\int_{\alpha}^{\beta} \left[\frac{\lambda}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{2} \left| \frac{\partial u}{\partial y} \right|^2 + F\ (y, u) \right] (x, y)\, dy = 0.$$

$F(y,\, u) \geq 0$ and $\lambda > 0$ yield

$$\frac{\partial u}{\partial x}(x,\, y) = \frac{\partial u}{\partial y}(x,\, y) = 0 \quad \text{in} \quad \Omega,$$

and then u is constant, but since

$$\int_{\Omega} |u(x,y)|^2\, dxdy < +\infty,$$

this constant is necessarily zero. □

Let us now see another type of nonlinearity which also provides a nonexistence result.

Theorem 2 *Let $u \in H^2(\Omega) \cap L^\infty(\Omega)$ be a solution of $(P.1)$, $\lambda > 0$ and f satisfying*

$$2F(y,u) - uf(y,u) \geq 0, \quad y \in]\alpha, \beta[. \tag{B}$$

Then the function $j(x) = \int_{\alpha}^{\beta} |u(x,y)|^2\, dy$ *is convex on* $\mathbb{R}$.

Remark 2 The convexity of $j(x)$ on $\mathbb{R}$ implies evidently the triviality of the solution u of problem $(P.1)$.

Proof. It is easy to see that almost everywhere on Ω we have

$$u\left(\frac{\partial^2 u}{\partial x^2}\right)(x,y) = \left(\frac{1}{2}\frac{\partial^2\left(u^2\right)}{\partial x^2} - \left|\frac{\partial u}{\partial x}\right|^2\right)(x,y).$$

Let us multiply equation $(P.1)$ by $\frac{1}{2}u$ and integrate over $]\alpha, \beta[$ to obtain

$$\int_{\alpha}^{\beta} \left[\frac{\lambda}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)u - \frac{1}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)u + \frac{1}{2}\left(f(y,u)\right)u \right](x,y)\, dy = 0,$$

which yields

$$\int_{\alpha}^{\beta}\left[\frac{\lambda}{2}\frac{\partial^2\left(u^2\right)}{\partial x^2}-\frac{\lambda}{2}\left|\frac{\partial u}{\partial x}\right|^2+\frac{1}{2}\left|\frac{\partial u}{\partial y}\right|^2+\frac{1}{2}f(y,u)u\right](x,y)\,dy$$
$$=\frac{1}{2}\left(u\frac{\partial u}{\partial y}\right)\Big|_{y=\alpha}^{y=\beta}=-\frac{1}{2\varepsilon}\left[(u(x,\alpha))^2+(u(x,\beta))^2\right],$$

which combined with (2.1) yields

$$\frac{\lambda}{4}\frac{d^2}{dx^2}\left[\int_{\alpha}^{\beta}|u(x,y)|^2\,dy\right]=\int_{\alpha}^{\beta}\left[\lambda\left|\frac{\partial u}{\partial x}\right|^2+F(y,u)-\frac{1}{2}uf(y,u)\right]dy.$$

The hypothesis (B) implies that

$$\frac{\lambda}{4}\frac{d^2}{dx^2}\left[\int_{\alpha}^{\beta}|u(x,y)|^2\,dy\right]\geq\lambda\int_{\alpha}^{\beta}\left|\frac{\partial u}{\partial x}\right|^2\,dy$$

and $\lambda>0$ implies the desired result. □

2.2 Elliptic equations

For the elliptic case, we have a nonexistence result stated in the following manner:

Theorem 3 *Let $u\in H^2(\Omega)\cap L^\infty(\Omega)$ be a solution of $(P.1)$, $\lambda<0$ and f satisfying*

$$2F(y,u)-uf(y,u)\leq 0,\quad y\in\,]\alpha,\beta[\,.$$

Then the function $j(x)$ defined in Theorem 2 is convex on $\mathbb{R}$.

Proof. Similar to the proof of Theorem 2. □

3 Examples

In this section, we present some examples illustrating the preceeding theorems.

Example 1 Let ρ be a function of C^1, $\rho:]\alpha,\beta[\to\mathbb{R}$, $\lambda\in\mathbb{R}$ and $f(y,\ u)\equiv\rho(y)u$. For $u\in H^2(\Omega)\cap L^\infty(\Omega)$, the problem

$$\begin{cases}\lambda\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}+\rho(y)u=0 \text{ in } \Omega,\\ u+\varepsilon\dfrac{\partial u}{\partial n}=0 \quad\text{on}\quad \partial\Omega,\end{cases}\tag{3.1}$$

does not have nontrivial solutions.

Example 2 Let us consider the **Klein-Gordon** equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} + mu - \theta_1 \left|u\right|^{p-1} u - \theta_2 \left|u\right|^{q-1} u = 0 \text{ in } \Omega, \\ u + \varepsilon \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.2}$$

where $m > 0$ is the mass of a particle, $\theta_1,\ \theta_2$ are **positive constants**, p and q are numbers greater than one. The problem (3.2) does not possess nontrivial solutions in $H^2(\Omega) \cap L^\infty(\Omega)$. It suffices to note that

$$F(y,u) - \frac{1}{2} u f(y,u) = \theta_1 \left(\frac{1}{2} - \frac{1}{p+1}\right) \left|u\right|^{p+1} + \theta_2 \left(\frac{1}{2} - \frac{1}{q+1}\right) \left|u\right|^{q+1}.$$

Example 3 Let ρ be a **nonnegative** function of class C^1, and ω a parameter, then the problem

$$\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} + \rho(y) \left(\omega u + \left|u\right|^{\tau-1} u\right) = 0 \text{ in } \Omega, \\ u + \varepsilon \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.3}$$

does not possess nontrivial solutions in $H^2(\Omega) \cap L^\infty(\Omega)$.

Remark 3 If $\Omega = \mathbb{R} \times]\alpha, +\infty[,\ \alpha \in \mathbb{R}$, we may get results on nonexistence of solutions for the problem

$$\begin{cases} \lambda \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} + f(x,u) = 0 \text{ in } \Omega, \\ u + \varepsilon \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{$P.1$}'$$

We find that

$$\int\limits_{-\infty}^{+\infty} \left[-\frac{\lambda}{2} \left|\frac{\partial u}{\partial x}\right|^2 - \frac{1}{2} \left|\frac{\partial u}{\partial y}\right|^2 + F\ (x,u) \right] (x,y)\, dx = 0.$$

Probably it would be interesting to study the problem

$$\begin{cases} \lambda \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} + f(x,y,u) = 0 \text{ in } \Omega, \\ u + \varepsilon \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{P}$$

4 Application to differential systems

In this last section we study both elliptic and hyperbolic differential systems in $\Omega = \mathbb{R} \times \mathbb{R}^+$. **Pucci** & **Serrin** [11] and **Van der Vorst** [13] have studied elliptic systems on **star-shaped** domains in $\mathbb{R}^N$. **Van der Vorst** showed that

$$\begin{cases} \Delta u = g(v) \text{ in } \Omega, \\ \Delta v = f(u) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

where f and g satisfy

$$\begin{cases} f, g \in C(\mathbb{R}), \\ f(u) > 0 \text{ if } u > 0;\ f(u) < 0 \text{ if } u < 0;\ f(0) = 0;\ NF(u) - a_1 u f(u) \leq 0,\ u \neq 0, \\ g(v) > 0 \text{ if } v > 0;\ g(v) < 0 \text{ if } v < 0;\ g(0) = 0;\ NG(v) - a_2 v g(v) \leq 0,\ v \neq 0, \\ N - a_1 - a_2 \leq 0, \end{cases}$$

does not possess nontrivial solutions in $C^2(\Omega) \cap C^1(\overline{\Omega})$.

We consider the system

$$\begin{cases} \lambda \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = g(v) \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+, \\ \lambda \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = f(u) \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+, \\ u = v = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \qquad (P.2)$$

where f and g satisfy the following hypothesis:

$$\begin{cases} f, g \in C(\mathbb{R}), \\ f(0) = g(0) = 0. \end{cases}$$

We have

Proposition 2 *Let $\lambda \in \mathbb{R}$ and $(u,\ v) \in H^2(\Omega) \cap L^\infty(\Omega) \times H^2(\Omega) \cap L^\infty(\Omega)$ be a solution of problem $(P.2)$, then, almost everywhere on $\mathbb{R}$,*

$$\int_0^{+\infty} \left[\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) - \lambda \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) + G(v) + F(u) \right](x, y)\, dy = 0, \qquad (4.1)$$

and almost everywhere on $\mathbb{R}^+$

$$\int_{-\infty}^{+\infty} \left[-\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) + \lambda \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) + G(v) + F(u) \right](x, y)\, dx = 0. \qquad (4.2)$$

Theorem 4 *Assume that f and g satisfy*

$$F(u) \cdot G(v) \geq 0. \tag{C}$$

Then the problem (P.2) does not possess nontrivial solutions (u, v) in $H^2(\Omega) \cap L^\infty(\Omega) \times H^2(\Omega) \cap L^\infty(\Omega)$.

Proof of Proposition 2. Let us set

$$\Lambda(x) = \int_0^{+\infty} \left[\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) - \lambda \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) + G(v) + F(u) \right](x, y)\, dy$$

and

$$\Gamma(y) = \int_{-\infty}^{+\infty} \left[-\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) + \lambda \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) + G(v) + F(u) \right](x, y)\, dx.$$

Under the above hypothesis $\Lambda(x)$ and $\Gamma(y)$ are absolutely continuous and we have almost everywhere on $\mathbb{R}$ and on $\mathbb{R}^+$ respectively

$$\Lambda'(x) = 0 \quad \text{and} \quad \Gamma'(y) = 0,$$

$\Lambda(x)$ and $\Gamma(y)$ are constants and as in Proposition 1, we obtain

$$\Lambda(x) \equiv 0 \quad \text{and} \quad \Gamma(y) \equiv 0.$$

The proof is complete. □

Proof of Theorem 4. From formulae (4.1) and (4.2), we obtain

$$\int_\Omega \left[\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) - \lambda \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) + G(v) + F(u) \right](x, y)\, dx\, dy = 0$$

and

$$\int_\Omega \left[-\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) + \lambda \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) + G(v) + F(u) \right](x, y)\, dx\, dy = 0.$$

Adding both formulae, we find

$$\int_\Omega \left[G(v) + F(u)\right](x, y)\, dx\, dy = 0.$$

Hypotesis (C) implies that

$$F(u) = 0 \text{ in } \Omega$$

and

$$G(v) = 0 \text{ in } \Omega.$$

As in [6, Theorem 1], the problem $(P.2)$ becomes

$$\begin{cases} \lambda \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+, \\ \lambda \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0 \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+, \\ \quad u = v = 0 \text{ on } \partial\Omega. \end{cases}$$

For any one of these equations we check that

$$\int_0^{+\infty} \left[\lambda \left| \frac{\partial u}{\partial x} \right|^2 - \left| \frac{\partial u}{\partial y} \right|^2 \right] (x, y)\, dy = 0. \tag{4.3}$$

The multiplication by u and integration over $]0, +\infty[$ yield

$$\int_0^{+\infty} \left[\frac{\lambda}{2} \frac{d^2}{dx^2} \left(|u|^2 \right) - \lambda \left| \frac{\partial u}{\partial x} \right|^2 - \left| \frac{\partial u}{\partial y} \right|^2 \right] (x, y)\, dy = 0. \tag{4.4}$$

Combining formulae (4.3) and (4.4), we get

$$\lambda \frac{d^2}{dx^2} \left[\int_0^{+\infty} |u(x, y)|^2 (x, y)\, dy \right] = 4 \int_0^{+\infty} \left| \frac{\partial u}{\partial y} \right|^2 (x, y)\, dy \geq 0.$$

If $\lambda > 0$, we conclude as in Theorem 3.

If $\lambda < 0$, (4.3) yields

$$\frac{\partial u}{\partial x} (x, y) = 0 = \frac{\partial u}{\partial y} (x, y),$$

and we conclude as in Theorem 2. □

Example 4 Let $g(v) = v$ and $f(u)$ be such that such that $F(u) \geq 0$, then the following problem

$$\begin{cases} \Delta^2 u = f(u) \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+, \\ \quad \Delta u = 0 \text{ on } \partial\Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P.2$'$}$$

does not have nontrivial solutions in $H^2(\Omega) \cap L^\infty(\Omega)$.

Proof. Let

$$\Delta u = v.$$

$(P.2)'$ reduces to

$$\begin{cases} \Delta u = v \ \text{ in } \ \Omega = \mathbb{R} \times \mathbb{R}^+, \\ \Delta v = f(u) \ \text{ in } \ \Omega = \mathbb{R} \times \mathbb{R}^+, \\ \quad u = \Delta u = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

The conclusion follows from Theorem 4. □

Example 5 Let

$$f(u) = u(u+a)(u+b) \ \text{ with } \ ab \geq \frac{2}{5}(a^2 + b^2), \quad a, b \in \mathbb{R}$$

and

$$g(v) = v.$$

The system $(P.2)$ does not possess nontrivial solutions, and it is clear that the result of **Van der Vorst** does not permit to conclude it.

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