

An Analytical Method for Some Nonlinear Difference Equations by Discrete Multiplicative Differentiation*

H. R. Khatami & M. Jahanshahi[†] & N. Aliev
Dept. of Math., Azad Islamic University of Karadj
P. O. Box 31485-313 Rajae-Shahr, Karadj, IRAN
E-mail: jahan_m15@yahoo.com
Fax: +98 (412) 4325135

Abstract

In this paper, at first we define the concept of additive and multiplicative discrete differentiations. Then, by considering these definitions of derivatives, the invariant functions with respect to additive and multiplicative derivatives are introduced respectively. Next, the differential equations with these kinds of derivatives are considered.

In the main sections, some boundary value and initial value problems including these differential equations are investigated and solved. In the second forthcoming part of this paper additive and multiplicative discrete integration are defined and considered. Finally by making use of solutions of these differential equations, the solutions of some nonlinear difference equations are found.

Key words: Discrete additive derivative, Discrete multiplicative derivative, Invariant function, Difference equation.

1 Introduction

According to Euler's theory, when we want to obtain the general solution of the n -th order ordinary linear differential equations with constant coefficients, we should use the exponential functions $e^{\lambda_i x}$, $i = 1, 2, \dots, n$, and their linear combinations because these functions are invariant with respect to ordinary differentiation [1].

According to this theory, if we want to obtain the general solution of other kinds of differential equations, we should find the respective invariant functions [5].

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[†]Author for Correspondence.

For solving differential equations with fractional and irrational order derivatives, the related invariant functions have been introduced by the authors [7].

In this paper, we consider some new kinds of ordinary differential equations with additive and multiplicative discrete derivatives.

In the second forthcoming part of this paper, additive and multiplicative discrete integration will be introduced [8].

2 Discrete additive and multiplicative differentiations

Suppose the function $f : A \subseteq \mathbb{Z} \longrightarrow \mathbb{R}$, the discrete additive derivative of f is defined in the following form:

$$f'(x) = f(x+1) - f(x), \quad x \in \mathbb{Z},$$

where $\mathbb{Z}$ is the set of integer numbers.

It is easy to see that the invariant function for this kind of differentiation will be

$$f(x) = 2^x.$$

Also, we define the discrete multiplicative differential of f in the following form:

$$f^{[1]}(x) = \frac{f(x+1)}{f(x)}, \quad x \in \mathbb{Z}.$$

The related invariant function will be

$$f(x) = C^{2^x},$$

where C is an arbitrary constant.

Remark 1 Some computational formulas have been given in [8] for elementary functions about their discrete additive and multiplicative differentiations.

Remark 2 Some computational formulas have been given in [8] for elementary functions about their discrete additive and multiplicative integration which will be given in the second part of this paper.

3 Differential equations with discrete multiplicative order

In this section, we consider a first and a second order differential equation. According to the mentioned invariant function, we propose the following function:

$$y(x) = C^{(\lambda+1)^x},$$

where C is an arbitrary constant, λ is a real number and x is an integer variable.

We consider this function is the general form of the above invariant function. In fact, we have

$$y^{[1]}(x) = \left(C^{(\lambda+1)^x}\right)^{[1]} = \left(C^{(\lambda+1)^x}\right)^\lambda.$$

We begin with the following first order equation

$$\begin{aligned} y^{[1]}(x) &= y^a(x), & x > x_0, & \quad a = \text{const}, \\ y(x_0) &= y_0. \end{aligned}$$

The general solution of this equation is

$$y(x) = C^{(a+1)^x}.$$

By imposing the initial condition we have

$$C^{(a+1)^{x_0}} = y_0 \implies C = y_0^{\frac{1}{(a+1)^{x_0}}}.$$

Therefore we have the following particular solution:

$$y(x) = y_0^{(a+1)^{x-x_0}}.$$

We now consider a second order equation:

$$y^{[11]}(x) = \left(y^{[1]}(x)\right)^a (y(x))^b, \quad x > x_0, \quad x, x_0 \in \mathbb{Z}, \quad a, b = \text{const},$$

with the following initial conditions:

$$y(x_0) = y_0, \quad y^{[1]}(x_0) = y_1.$$

By considering the invariant function

$$y(x) = C^{(\lambda+1)^x}$$

we have

$$y^{[1]}(x) = C^{(\lambda+1)^x \cdot \lambda}, \quad y^{[11]}(x) = C^{(\lambda+1)^x \cdot \lambda^2}.$$

Therefore we have the following algebraic equation:

$$\lambda^2 - \lambda a - b = 0$$

with roots

$$\lambda_k = \frac{a + (-1)^k \sqrt{a^2 + 4b}}{2}, \quad k = 1, 2.$$

The general solution is in the following form:

$$y(x) = C_1^{(\lambda_1+1)^x} \cdot C_2^{(\lambda_2+1)^x}$$

Finally, by imposing initial conditions, we have:

$$y(x) = \left(y_0^{\lambda_2} y_1^{-1}\right)^{\frac{(\lambda_1+1)^x - x_0}{\lambda_2 - \lambda_1}} \cdot \left(y_0^{-\lambda_1} y_1\right)^{\frac{(\lambda_2+1)^x - x_0}{\lambda_2 - \lambda_1}}, \quad x \in \mathbb{Z}.$$

4 Boundary value problems with discrete multiplicative derivative

In this section, we consider two boundary value problems which include a first and a second order differential equation, respectively.

Example 1 Consider the first order differential equation

$$y^{[1]}(x) = y^a(x), \quad x \in (x_0, x_1), \quad x \in \mathbb{Z}, \quad a = \text{const}$$

with the following boundary condition:

$$y(x_0) = Ay^\alpha(x_1), \quad \alpha = \text{const}.$$

We have the general solution:

$$y(x) = C^{(a+1)^x}.$$

By imposing the boundary condition, we have the following final solution:

$$y(x) = A^{\frac{(\alpha+1)^x}{(a+1)^{x_0} - (a+1)^{x_1} \cdot \alpha}} = \left(\frac{1}{A^{(a+1)^{x_0} - (a+1)^{x_1} \cdot \alpha}} \right)^{(a+1)^x}.$$

Example 2 Consider the following second order differential equation

$$y^{[11]}(x) = \left(y^{[1]}(x)\right)^a (y(x))^b, \quad x \in (x_0, x_1), \quad x_0, x_1, x \in \mathbb{Z},$$

with the following two boundary conditions:

$$\begin{aligned} y(x_0) &= A(y(x_1))^\alpha, \\ y^{[1]}(x_0) &= B\left(y^{[1]}(x_1)\right)^\beta, \end{aligned}$$

where A, B, α, β are given real constants.

We saw the general solution of this equation in the following form before:

$$y(x) = C_1^{(\lambda_1+1)^x} C_2^{(\lambda_2+1)^x},$$

where λ_1 and λ_2 are real roots of the following characteristic equation:

$$\lambda^2 - \lambda a - b = 0$$

and

$$\lambda_k = \frac{a + (-1)^k \sqrt{a^2 + 4b}}{2}, \quad k = 1, 2.$$

With differentiation of the general solution, we have

$$y^{[1]}(x) = C_1^{(\lambda_1+1)^x \cdot \lambda_1} \cdot C_2^{(\lambda_2+1)^x \cdot \lambda_2}.$$

We now impose the boundary conditions and we will have the following algebraic system with respect to C_1, C_2 :

$$\begin{cases} C_1^{(\lambda_1+1)^{x_0}} \cdot C_2^{(\lambda_2+1)^{x_0}} = A C_1^{(\lambda_1+1)^{x_1} \cdot \alpha} \cdot C_2^{(\lambda_2+1)^{x_1} \cdot \alpha}, \\ C_1^{(\lambda_1+1)^{x_0} \cdot \lambda_1} \cdot C_2^{(\lambda_2+1)^{x_0} \cdot \lambda_2} = B C_1^{(\lambda_1+1)^{x_1} \cdot \lambda_1 \beta} \cdot C_2^{(\lambda_2+1)^{x_1} \cdot \lambda_2 \beta}. \end{cases}$$

For solving this system, we can write:

$$\begin{cases} (\lambda_1 + 1)^{x_0} \ln C_1 + (\lambda_2 + 1)^{x_0} \ln C_2 = \ln A + \alpha(\lambda_1 + 1)^{x_1} \ln C_1 + \alpha(\lambda_2 + 1)^{x_1} \ln C_2, \\ \lambda_1(\lambda_1 + 1)^{x_0} \ln C_1 + \lambda_2(\lambda_2 + 1)^{x_0} \ln C_2 = \ln B + \lambda_1 \beta(\lambda_1 + 1)^{x_1} \ln C_1 \\ \quad + \lambda_2 \beta(\lambda_2 + 1)^{x_1} \ln C_2. \end{cases}$$

If we use the following substitution:

$$\begin{aligned} (\lambda_1 + 1)^{x_0} - \alpha(\lambda_1 + 1)^{x_1} &= a_{11}, & (\lambda_2 + 1)^{x_0} - \alpha(\lambda_2 + 1)^{x_1} &= a_{12}, \\ \lambda_1 [(\lambda_1 + 1)^{x_0} - \beta(\lambda_1 + 1)^{x_1}] &= a_{21}, & \lambda_2 [(\lambda_2 + 1)^{x_0} - \beta(\lambda_2 + 1)^{x_1}] &= a_{22}, \end{aligned}$$

we will have the following algebraic system:

$$\begin{aligned} a_{11} \ln C_1 + a_{12} \ln C_2 &= \ln A, \\ a_{21} \ln C_1 + a_{22} \ln C_2 &= \ln B. \end{aligned}$$

In the special case, $\alpha = \beta = 1$, we have the following determinant for this algebraic system:

$$\Delta = [(\lambda_1 + 1)^{x_0} - (\lambda_1 + 1)^{x_1}] [(\lambda_2 + 1)^{x_0} - (\lambda_2 + 1)^{x_1}] (\lambda_2 - \lambda_1)$$

which is nonzero when λ_1 and λ_2 are distinct.

5 Nonlinear difference equations

Now with the help of the concept of discrete multiplicative differential, we show that some kind of nonlinear difference equations can be derived from differential equations with discrete multiplicative orders.

For this, recall that under a discrete multiplicative differential of $y(x)$ we mean

$$y^{[1]}(x) = \frac{y(x+1)}{y(x)}, \quad y^{[11]}(x) = \frac{y(x+2)/y(x+1)}{y(x+1)/y(x)} = \frac{y(x)y(x+2)}{y^2(x+1)}.$$

Therefore the differential equation

$$y^{[11]}(x) = \left[y^{[1]}(x) \right]^a [y(x)]^b$$

can be written in the following form:

$$\frac{y(x)y(x+2)}{y^2(x+1)} = \frac{y^a(x+1)}{y^a(x)} \cdot y^b(x)$$

or

$$y(x+2) = y^{a+2}(x+1)y(x)^{b-a-1}.$$

Finally it can be written as the following difference equation:

$$y_{n+2} = y_{n+1}^{a+2} \cdot y_n^{b-a-1}.$$

At the end, we give some initial value problems about nonlinear difference equations.

Example 3 In the above obtained difference equation let $a = b = 1$ and

$$y(0) = 1, \quad y^{[1]}(0) = 2.$$

Then, we will have an initial value problem in the following form:

$$y_{n+2} = y_{n+1}^3 \cdot \frac{1}{y_n}, \quad y_0 = 1, \quad y_1 = 2.$$

The exact and analytic solution of this problem will be as follows:

$$y(x) = \left(y_0^{\lambda_2} y_1^{-1} \right)^{\frac{(\lambda_1+1)x-x_0}{\lambda_2-\lambda_1}} \cdot \left(y_0^{-\lambda_1} y_1 \right)^{\frac{(\lambda_2+1)x-x_0}{\lambda_2-\lambda_1}},$$

where λ_1 and λ_2 are real roots of the following algebraic equation:

$$\lambda^2 - \lambda - 1 = 0$$

or

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

Example 4 For the following differential equation:

$$\begin{aligned}y^{[1]}(x) &= y^a(x), \quad x > x_0, \\ y(x_0) &= y_0,\end{aligned}$$

we let

$$a = 1, \quad x_0 = 0, \quad y_0 = 3.$$

Then we will have the following initial value problem:

$$y_{n+1} = y_n^2, \quad y(0) = 3.$$

This is a nonlinear difference equation. By making use of the general solution of this differential equation, we have

$$y_n = 3^{\frac{2^n}{2^0}} = 3^{2^n}.$$

It is easy to see that the initial condition is valid.

As a last example, we put the following values in the pervious example:

$$a = 3, \quad x_0 = 1, \quad y(x_0) = 5.$$

In fact, we will solve the following initial value problem:

$$y_{n+1} = y_n^4, \quad y(1) = 5.$$

Therefore we will have the following solution:

$$y_n = (5)^{4^{n-1}}.$$

6 Conclusion

There are many physical problems as well as problems which appear in biomathematics, combinatorics and graph theory, statistics and probability theory, such that when we want to construct the corresponding mathematical models of these problems, we obtain linear or nonlinear difference equations which can be solved by these kinds of differential equations.

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