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Using Lagrange – Type $k-0$ Elements for Solving Fredholm Integral Equations of the Second Kind

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Abstract

In this paper, we describe the Petrov-Galerkin method and use Lagrange-type $k-0$ elements for solving Fredholm integral equations of the second kind on $[0,1]$ and for showing the efficiency of the method, we use numerical examples.

Key words: Integral equations, The Petrov-Galerkin method, Regular pair, Trial space, Test space.

1 Introduction

Let X be a Hilbert space with inner product and norm $\|.\|$. We assume that $K: X \to X$ is a compact operator and consider a Fredholm inegral equation of the second kind,

$$u - Ku = f, \qquad f \in X. \tag{1}$$

Numerical methods including quadrature, collocation and Galerkin and least square methods for equation (1) are used and their analysis may be found in [1, 2, 4, 5]. The Petrov-Galerkin method is established in [3] for equation (1) on $[0,1]$. One of the advantages of the Petrov-Galerkin method is that it allows us to achieve the same order of convergence as the Galerkin method with much less computational cost by choosing the test spaces to be spaces of piecewise polynomials of lower degree.

This paper is organized as follows: In Section 2, we review the Petrov-Galerkin method for equation (1). In Section 3 we describe Lagrange-type $k-0$ elements with numerical results.

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2 The Petrov-Galerkin method

In this section we follow the paper [3] with a brief review of the Petrov-Galerkin method. A similar idea has been used in solving differential equations in [6, 7].

Let X be a Banach space and X^* be its dual space of continuous linear functionals. For each positive integer n, we assume that $X_n \subset X$, $Y_n \subset X^*$ and X_n, Y_n are finite dimensional vector spaces with

$$\dim X_n = \dim Y_n, \quad n = 1, 2, \ldots \tag{2}$$

Also X_n, Y_n satisfy condition (H) : For each $x \in X$ and $y \in X^*$, there exist $x_n \in X_n$ and $y_n \in Y_n$ such that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ as $n \to \infty$.

The Petrov-Galerkin method for equation (1) is a numerical method for finding $u_n \in X_n$ such that

$$(u_n - Ku_n, y_n) = (f, y_n) \qquad \text{for all } \; y_n \in Y_n. \tag{3}$$

Define, for $x \in X$, an element $P_n x \in X_n$ called the generalized best approximation from X_n to x with respect to Y_n by the equation

$$(x - P_n x, y_n) = 0 \qquad \text{for all } \; y_n \in Y_n. \tag{4}$$

It is proved in [3] that for each $x \in X$, the generalized best approximation from X_n to x with respect to Y_n exists uniquely if and only if

$$Y_n \cap X_n^{\perp} = \{0\}. \tag{5}$$

Under this condition, P_n is a projection, *i.e.*, $P_n^2 = P_n$.

Assume that, for each n, there is a linear operator $\Pi_n : X_n \to Y_n$ with $\Pi_n X_n = Y_n$ satisfying the following two conditions

(H-1) for all $x_n \in X_n$, $\|x_n\| \leq C_1 (x_n, \Pi_n x_n)^{1/2}$,

(H-2) for all $x_n \in X_n$, $\|\Pi_n x_n\| \leq C_2 \|x_n\|$.

If a pair of sequences of spaces $\{X_n\}$ and $\{Y_n\}$ satisfies (H-1) and (H-2), we call $\{X_n, Y_n\}$ a *regular pair*. It is proved in [3] that, if a regular pair $\{X_n, Y_n\}$ satisfies $\dim X_n = \dim Y_n$ and condition (H), then the corresponding generalized projection P_n satisfies:

(1) for all $x \in X$, $\|P_n x - x\| \to 0$ as $n \to \infty$,

(2) there is a constant $C > 0$ such that $\|P_n\| < C$, $n = 1, 2, \ldots$,

(3) for some constant $C > 0$ independent of n, $\|P_n x - x\| \leq C\|Q_n x - x\|$, where $Q_n x$ is the best approximation from X_n to x.

The Petrov-Galerkin methods using regular pairs $\{X_n, Y_n\}$ of piecewise polynomial spaces are called Petrov-Galerkin elements. If we use piecewise polynomials of degree k and k' for the spaces X_n and Y_n respectively, we call the corresponding Petrov-Galerkin elements $k - k'$ elements. In Section 3, 4, we solve the equation (1) using continuous and discontinuous Lagrange-type $k - 0$ elements.

3 Lagrange-type $k - 0$ elements

We subdivide the interval $[0, 1]$ into n subintervals by a sequence of points $0 = t_0 < t_1 < \cdots < t_n = 1$. Denote $I_i = [t_{i-1}, t_i]$ and $h_i = t_i - t_{i-1}$ for $i = 1, \ldots, n$ and let X_n be the space of piecewise polynomials of degree $\leq k$ with knots at $t_i,\ i = 1, \ldots, n-1$. Let $\tau_j = \frac{2j+1}{2k+2}, j = 0, 1, \ldots, k$, and define

$$t_j^{(i)} = t_{i-1} + \tau_j h_i, \qquad j = 0, 1, \ldots, k, \quad i = 1, \ldots, n. \tag{6}$$

We define $n(k + 1)$ functions $\Phi_j^{(i)}(t)$ by letting

$$\Phi_j^{(i)}(t) = \begin{cases} \displaystyle\prod_{\substack{\ell=0 \\ \ell \neq j}}^{k} \frac{t - t_\ell^{(i)}}{t_j^{(i)} - t_\ell^{(i)}}, & t \in I_i, \\ 0, & t \notin I_i, \end{cases} \qquad \begin{array}{l} i = 1, \ldots, n, \\ j = 0, 1, \ldots, k. \end{array} \tag{7}$$

Then, for each $x_n \in X_n$, we have

$$x_n(t) = \sum_{j=0}^{k} x_n(t_j^{(i)}) \Phi_j^{(i)}(t), \quad t \in I_i, \quad i = 1, \ldots, n. \tag{8}$$

We then construct the test space Y_n by

$$\psi_j^{(i)}(t) = \begin{cases} 1, & t_{i-1} + \frac{jh_i}{k+1} \leq t \leq t_{i-1} + \frac{(j+1)h_i}{k+1}, \\ 0, & \text{otherwise}, \end{cases} \quad \begin{array}{l} j = 0, 1, \ldots, k, \\ i = 1, \ldots, n. \end{array} \tag{9}$$

Now, we define a linear operator $\Pi_n : X_n \to Y_n$ as follows:

$$\Pi_n x_n(t) = \sum_{j=0}^{k} x_n(t_j^{(i)}) \psi_j^{(i)}(t), \quad t \in I_i,\ i = 1, \ldots, n. \tag{10}$$

Then $\dim X_n = \dim Y_n = n(k+1)$ and $\Pi_n X_n = Y_n$ and in [3] it is proved that for $1 \le k \le 5$ these two space sequences form a regular pair.

Now, assume $u_n \in X_n$ and $\{b_i\}_{i=1}^n$ is a basis for X_n and $\{b_j^*\}_{j=1}^n$ is a basis for Y_n. Therefore the Petrov-Galerkin method on $[0,1]$ for equation (1) is

$$(u_n - Ku_n, b_j^*) = (f, b_j^*), \quad j = 1, \dots, n. \tag{11}$$

Let $u_n(t) = \sum\limits_{i=1}^{n} a_i b_i(t)$. Then equation (1) leads to determining $\{a_1, a_2, \dots, a_n\}$ as the solution of the linear system

$$\sum_{i=1}^{n} a_i \left\{ \int_0^1 b_i(t) b_j^*(t)\, dt - \int_0^1 \int_0^1 K(s,t) b_i(s) b_j^*(t)\, ds\, dt \right\}$$
$$= \int_0^1 f(t) b_j^*(t)\, dt, \qquad j = 1, \dots, n. \tag{12}$$

Example

$$u(t) - \int_0^1 \left(-\frac{1}{3} e^{2t-5s/3}\right) u(s)\, ds = e^{2t+1/3}, \qquad 0 \le t \le 1,$$

with exact solution $u(t) = e^{2t}$. In the following table we computed $\|u_n(t_j^{(i)}) - u(t_j^{(i)})\|_2$ for $n = 1, 2, 4, 10$ with equally spaced points and $k = 1, 2, \dots, 5$.

$k \setminus n$	1	2	4	10
1	0.0711045	0.0224796	0.0071885	0.00193661
2	0.00960519	0.000812911	0.0000710505	$2.86684 * 10^{-6}$
3	0.00287936	0.000216985	0.0000183828	$7.35154 * 10^{-7}$
4	0.000238635	$4.94556 * 10^{-6}$	$1.07522 * 10^{-7}$	$6.93185 * 10^{-10}$
5	0.0000458096	$8.48186 * 10^{-7}$	$1.78763 * 10^{-8}$	$1.14221 * 10^{-10}$

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