

An Analytic-Numerical Method for Solving Difference Equations with Variable Coefficients by Discrete Multiplicative Integration*

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Abstract

There are some physical phenomena and engineering problems whose mathematical models appear as difference equations with variable coefficients. In this paper, at first we define the concept of discrete multiplicative derivative and discrete multiplicative integration, then the invariant function with respect to this derivative is introduced.

Next differential equations with this type of derivative are considered. In the final section, we consider some initial and boundary value problems which include difference equations with variable coefficients. At the end, by making use of linear algebra and numerical differentiation and discrete multiplicative integration, we present an analytic-numerical method for solving these difference equations.

Key words: Difference equation, Invariant function, Discrete multiplicative differentiation, Discrete multiplicative integration.

1 Introduction

To get an idea and detailed information about additive and multiplicative discrete differentiation we could refer to many books and papers like [1, 2] and [3].

The main problems of additive discrete differentiation are to discretize the linear differential equation and use the discretization to find an approximate solution of the above mentioned differential equation [1] and [4]. In addition, there are certain models in physics and engineering and especially in mathematics such that when we make their mathematical method they become discrete problems and systems, such as to find the general term of an arithmetic sequence [5] and the reproduction

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of the rabbits which is given by the Fibonacci sequence [7] and recursive formula for evaluating the n -th order determinants [8].

Except for some problems and examples of nonlinear discrete multiplicative analysis, the above mentioned problem has not been discussed as a system of equations. Some discussion about continuous multiplicative differentiation and integration and their properties in [9] has been done by Gantmacher. The aim of this paper is discussing the form of a system of equations. Some properties of discrete additive and multiplicative differentiation are discussed in [10]. The simplest example for discrete multiplicative differentiation is the geometric progression which will be in the form of a Cauchy problem that contains first order, we could use discrete additive differentiation to find the general term of the arithmetic progression.

Finally in the first part of the paper [10], after introducing the concept and properties of discrete multiplicative and additive differentiation, some kinds of nonlinear difference equations are solved by using multiplicative differentiation.

In this paper we also introduce a numerical and approximate solution of ordinary linear differential equations with variable coefficients by the help of discrete multiplicative analysis.

2 Discretization process

First with the help of classical analysis, we consider the following n -th order linear differential equation

$$\begin{aligned} & y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + a_2(x)y^{(n-2)}(x) + \dots \\ & + a_{n-1}(x)y'(x) + a_n(x)y(x) = 0, \quad x > 0, \end{aligned} \quad (1)$$

with the following initial conditions:

$$y^{(k)}(0) = \alpha_k, \quad k = 0, 1, 2, \dots, n-1. \quad (2)$$

First with the following change of variables we transfer this problem to a system of first order differential equations:

$$\begin{aligned} y(x) &= y_1(x), \\ y'(x) &= y_1'(x) = y_2(x), \\ y''(x) &= y_1''(x) = y_2'(x) = y_3(x), \\ &\vdots \\ y^{(n-1)}(x) &= y_1^{(n-1)}(x) = y_2^{(n-2)}(x) = \dots = y_{n-2}''(x) = y_{n-1}'(x) = y_n(x). \end{aligned}$$

We have in this case

$$y^{(n)}(x) = y_1^{(n)}(x) = y_2^{(n-1)}(x) = \cdots = y_{n-1}''(x) = y_n'(x),$$

and finally problem (1) and (2) will be in the form of the following system of equations:

$$\begin{cases} y_j'(x) - y_{j+1}(x) = 0, & j = 1, \dots, n-1, \\ y_n'(x) + a_1(x)y_n(x) + a_2(x) + y_{n-1}(x) + \cdots + a_n(x)y_1(x) = 0, & x > 0. \end{cases} \quad (3)$$

Then we have initial conditions as follows:

$$\begin{cases} y_1(0) = \alpha_0, \\ y_2(0) = \alpha_1, \\ \vdots \\ y_n(0) = \alpha_{n-1}. \end{cases} \quad (4)$$

The above Cauchy problem with the help of the theory of matrices can be written compactly as follows. We put

$$Z(x) = (y_1(x), y_2(x), \dots, y_n(x))^T.$$

With the help of the matrix

$$a(x) = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ & & \cdots & & & \\ 0 & 0 & \cdots & 0 & 0 & -1 \\ a_n(x) & a_{n-1}(x) & \cdots & a_3(x) & a_2(x) & a_1(x) \end{pmatrix}$$

the Cauchy problem (3) and (4) will be compactly written as follows:

$$Z'(x) + a(x)Z(x) = 0, \quad x > 0, \quad (5)$$

$$Z(0) = \alpha, \quad (6)$$

where

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})^T.$$

We must note that both coefficients and unknowns are of the same type, it means that in problem (3), (4) they are functions and in problem (5) and (6) they are not,

but the coefficients of the equations are in the form of a functional matrix and its unknown is a column functional vector. To eliminate this difficulty we put

$$\bar{\alpha} = (\alpha, \alpha, \dots, \alpha)$$

in which $\bar{\alpha}$ is an n -th degree matrix and problem (5), (6) will be changed as follows:

$$\begin{aligned}\bar{Z}'(x) + a(x)\bar{Z}(x) &= 0, \\ \bar{Z}(0) &= \bar{\alpha}\end{aligned}\tag{7}$$

in which $\bar{Z}(x)$ is an n -th order matrix, and is the unknown of the Cauchy problem (7).

Note that in problem (7) the coefficient, the right-hand side of the initial condition and the unknown are of the same type. It means that every one of them is an n -th order square matrix. To solve the above problem, we use a method similar to the case of a function.

It is clear that there are many ways to discretize the differentiation and the more the points (nods) are, the more accurate the approximation is. To use discrete differentiation analysis we replace ordinary differentiation by numerical differentiation with two points, such that the selection of points of network in the form of $x = hk$, in which $h \in \mathbb{R}^+$ and $k \in \mathbb{N} \cap \{0\}$ and constant, and the differentiation $\bar{Z}(x)$ is as follows:

$$\bar{Z}'(x) \approx \frac{\bar{Z}((k+1)h) - \bar{Z}(kh)}{h}, \quad k \geq 0.$$

In this case if we put

$$\begin{aligned}\bar{Z}(kh) &= Z_k, & k \geq 0, \\ a(kh) &= a_k, & k \geq 0,\end{aligned}$$

then problem (7) will be in the following discrete form

$$\begin{cases} Z_{k+1} = [I - ha_k] Z_k, & k \geq 0, \\ Z_0 = \bar{\alpha}, \end{cases}\tag{8}$$

in which

$$a_0 = \lim_{x \rightarrow 0} a(x).$$

The equation (8) can be written as follows:

$$Z_{k+1} \cdot Z_k^{-1} = I - ha_k$$

and $Z_{k+1}Z_k^{-1}$ is the right discrete multiplicative differentiation of the matrix Z_k , and problem (8) can be considered as a system of equations with discrete multiplicative differentiation such that its solution can be found with the help of discrete multiplicative integration as follows

$$Z_k = \left[\int_0^k (I - ha_j) \right] \bar{\alpha}, \quad k \geq 0, \quad (9)$$

in which the symbol $\int$ denotes the discrete multiplicative integration and the lower arrow limit represents the direction of increasing indices j from 0 to k and the matrices will be written from right to left.

Remark 1 In [11], the symbol $\int$ is used to show discrete additive integration. When the lower and upper limits of discrete additive integration are equal, the result will be zero. But if the lower and upper limits of discrete multiplicative integral are equal, the result will be one.

Remark 2 Whenever in the compactly written problem (3) we replace $Z(x)$ with the row vector $(y_1, y_2, \dots, y_n)$ and $b(x)$ with the matrix

$$b(x) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_n(x) \\ -1 & 0 & 0 & 0 & \cdots & a_{n-1}(x) \\ & & \cdots & & & \\ 0 & 0 & \cdots & 0 & 0 & a_2(x) \\ 0 & 0 & \cdots & 0 & -1 & a_1(x) \end{pmatrix},$$

instead of system (5) we obtain

$$\begin{cases} Z'(x) + Z(x)b(x) = 0, & x > 0, \\ Z(0) = \beta, \end{cases}$$

in which

$$\beta = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$$

is a row vector. In this case the discretized problem (8) will be as follows:

$$\begin{cases} Z_{k+1} = Z_k(I - hb_k), & x > 0, \\ Z_0 = \bar{\beta}, \end{cases}$$

in which $\bar{\beta}$ is a square matrix of order n with n times repeating of the first row. The solution of the above discrete problem similar to (9) will be as follows:

$$Z_k = \bar{\beta} \left[\int_0^k (I - hb_j) \right]$$

in which the arrow in the above limit is related to an increase of the indices, j from zero to k , and the multiplication of matrices is from left to right.

Remark 3 If in the initial value problem (1) and (2) the existing differentiation is discrete additive differentiation and its variable x does not change over real numbers, then in problem (8) which results from problem (7) the relation (9) which is the solution of (8) is real and is the exact solution of problem (1) and (2) without any error.

Remark 4 Under the conditions of Remark 3, problem (1) and (2) is in fact a difference variable coefficient equation whose real solution is given by relation (9).

3 Discrete multiplicative integration

Consider the Cauchy problem for the second order linear differential equation

$$\begin{cases} y''(x) + a(x)y(x) = 0; & x > 0, \\ y(0) = \alpha_0, \quad y'(0) = \alpha_1. \end{cases}$$

We put the first differential coefficient zero (reducing order method). To change the above problem to a system of first order differential equations we put

$$y(x) = y_0(x) \implies y' = y'_0 = y_1, \quad y'' = y''_0 = y'_1.$$

In this case we have

$$\begin{cases} y'_0(x) - y_1(x) = 0, \\ y'_1(x) + a(x)y_0(x) = 0, \end{cases}$$

in which the initial conditions are as follows

$$\begin{cases} y_0(0) = \alpha_0, \\ y_1(0) = \alpha_1. \end{cases}$$

The following symbols are used for this purpose:

$$Z(x) = \begin{pmatrix} y_0(x) \\ y_1(x) \end{pmatrix}, \quad a(x) = \begin{pmatrix} 0 & -1 \\ a(x) & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}.$$

In this case the problem will be written as follows:

$$\begin{cases} Z'(x) + a(x)Z(x) = 0, & x > 0, \\ Z(0) = \alpha. \end{cases}$$

Now we discretize the above problem. For this purpose we divide the real axis by step h , we have

$$x = kh, \quad k \geq 0, \quad h \in \mathbb{R}^+$$

and consider the first differential of $Z(x)$ by two points:

$$Z'(x) \approx \frac{Z((k+1)h) - Z(kh)}{h}$$

and if we put $Z(hk) = Z_k$, then we will have

$$\begin{cases} Z_{k+1} = (I - ha_k)Z_k, & k \geq 0, \\ Z_0 = \bar{\alpha}, \end{cases}$$

in which Z_k is a square matrix of order 2, $\bar{\alpha}$ is a matrix of the same type whose every column is equal to the vector α . Also,

$$a_0 = \lim_{x \rightarrow 0} a(x).$$

Now, if we put $k = 0$, we have

$$Z_1 = (I - ha_0)Z_0 = (I - ha_0)\bar{\alpha}$$

and if $k = 1$, we have:

$$\begin{aligned} Z_2 &= (I - ha_1)Z_1 \\ &= (I - ha_1)(I - ha_0)\bar{\alpha}. \end{aligned}$$

If we continue this process, we will have:

$$\begin{aligned} Z_k &= (I - ha_{k-1})(I - ha_{k-2}) \cdots (I - ha_1)(I - ha_0)\bar{\alpha} \\ &= \left[\prod_{j=0}^{k-1} (I - ha_j) \right] \bar{\alpha} \end{aligned}$$

in which the above arrow of multiplication will show an increase of j from right to left. Then, if we use the discrete multiplicative integration symbol, we have

$$Z_k = \left[\int_0^k (I - ha_j) \right] \bar{\alpha}.$$

To find the approximate solution of the Cauchy problem, we calculate the product of the above functional matrix and the vector $\alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$. The first component of the resulting vector is the approximate solution of the problem.

We calculate the fundamental matrix for some values of k :

$$\begin{aligned} Z_1 &= \begin{pmatrix} 1 & h \\ -ha(0) & 1 \end{pmatrix} \bar{\alpha}, \quad k = 1, \\ Z_2 &= \begin{pmatrix} 1 & h \\ -ha(1) & 1 \end{pmatrix} \begin{pmatrix} 1 & h \\ -ha(0) & 1 \end{pmatrix} \bar{\alpha} = \begin{pmatrix} 1 - h^2a(0) & 2h \\ -h(a(1) + a(0)) & -h^2a(1) + 1 \end{pmatrix} \bar{\alpha}, \\ Z_3 &= \begin{pmatrix} 1 - h^2(2a(0) + a(1)) & h(3 - h^2a(1)) \\ -h(a(2) + a(1) + a(0)) + h^3a(2)a(0) & 1 - h^2(2a(2) + a(1)) \end{pmatrix} \bar{\alpha}. \end{aligned}$$

Considering the above mentioned matrix value for the unknown function in the 3rd step is as follows:

$$y(3h) = [1 - h^2(2a(0) + a(1))] \alpha_0 + h(3 - h^2a(1)) \alpha_1.$$

If we want to calculate $y'(3h)$, we use the 2nd row of Z_3 :

$$y'(3h) = [-h(a(2) + a(1) + a(0)) + h^3a(2)a(0)] \alpha_0 + [1 - h^2(2a(2) + a(1))] \alpha_1.$$

4 Difference equation

According to Remark 4, we consider the Cauchy problem containing a second order difference equation with variable coefficients and we get its analytic solution according to the discrete multiplicative integration:

$$\begin{cases} y_{n+2} + a_n y_{n+1} + (a_n - 1)y_n = 0, & n \geq 0, \\ y_0 = \alpha_0, & y_1 = \alpha_1. \end{cases}$$

With the help of the definition of discrete additive differentiation

$$\begin{aligned} y'_n &= y_{n+1} - y_n, \\ y''_n &= y_{n+2} - 2y_{n+1} + y_n, \end{aligned}$$

the Cauchy problem will be in the following form:

$$\begin{cases} y''_n + (a_n + 2)y'_n + 2a_n y_n = 0, & n \geq 0, \\ y_0 = \alpha_0, & y'_0 = \alpha_1 - \alpha_0. \end{cases}$$

With the change of variable $y'_n = Z_n$ we have the following system of equations:

$$\begin{cases} y'_n - Z_n = 0, & y_0 = \alpha_0, \\ Z'_n + (a_n + 2)Z_n + 2a_n y_n = 0, & Z_0 = \alpha_1 - \alpha_0, \end{cases}$$

and with the substitution

$$W_n = \begin{pmatrix} y_n \\ Z_n \end{pmatrix}, \quad A_n = \begin{pmatrix} 0 & -1 \\ 2a_n & a_n + 2 \end{pmatrix}$$

the problem will be compactly written as follows:

$$\begin{aligned} W'_n + A_n W_n &= 0, \\ W_0 &= \alpha, \end{aligned}$$

in which $\alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 - \alpha_0 \end{pmatrix}$.

Now with calculating the eigenvalues and eigenvectors of the matrix A_n , we will have its canonical form as follows:

$$\begin{aligned} \lambda_1(n) &= 2, & \lambda_2(n) &= a_n, \\ v_1 &= \begin{pmatrix} 1 \\ -2 \end{pmatrix}, & v_2 &= \begin{pmatrix} 1 \\ -a_n \end{pmatrix}. \end{aligned}$$

In this case

$$\Lambda_n = \begin{pmatrix} 2 & 0 \\ 0 & a_n \end{pmatrix}, \quad m_n = \begin{pmatrix} 1 & 1 \\ -2 & -a_n \end{pmatrix}.$$

Assume that

$$\det m_n = 2 - a_n \neq 0.$$

In this case with the help of the substitution

$$W_n = m_n X_n$$

we will have the following solution for X_n :

$$X_n = \begin{pmatrix} (-1)^n C_0 + \int_0^n (-1)^{n+1+j} b_j \\ \frac{2-a_0}{2-a_n} v_0 \int_0^n (1-a_j) \end{pmatrix}$$

in which

$$b_j = -\frac{a'_j}{2-a_j} \cdot \frac{2-a_0}{2-a_{j+1}} v_0 \int_0^{n+1} (1-a_j).$$

If we put X_n in W_n and find the arbitrary constants C_0 and v_0 from the initial condition $W_0 = \alpha$, we have

$$W_n = \begin{pmatrix} y_n \\ z_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -a_n \end{pmatrix} \left((-1)^n \frac{(1-a_0)\alpha_0 - \alpha_1}{2-a_0} + \int_0^n (-1)^{n+1+j} \frac{-a'_j}{2-a_j} \cdot \frac{\alpha_0 + \alpha_1}{2-a_{j+1}} \int_0^{j+1} (1-a_k) \right. \\ \left. + \frac{\alpha_0 - \alpha_1}{2-a_n} \int_0^n (1-a_j) \right).$$

From this we have the final analytic solution of the Cauchy problem of Example 2:

$$y_n = (-1)^n \frac{(1-a_0)\alpha_0 - \alpha_1}{2-a_0} + \int_0^n (-1)^{n+j} \frac{a'_j(\alpha_0 + \alpha_1)}{(2-a_j)(2-a_{j+1})} \int_0^{j+1} (1-a_k) + \frac{\alpha_0 + \alpha_1}{2-a_n} \int_0^n (1-a_j).$$

We can easily see that the above analytic solution satisfies both the difference equation with variable coefficients and the given initial conditions.

Remark 5 As we said [11] the discrete additive integration and discrete multiplicative integration are represented as follows:

$$\int_0^n f_k = \sum_{k=0}^{n-1} f_k, \quad \int_0^n f_k = \prod_{k=0}^{n-1} f_k.$$

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