

## On Stochastic Response of Nonlinear Oscillators

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### Abstract

This paper is concerned with the study of some types of nonlinear oscillators for which the frequency of excitation is stochastic. The paper consists of two parts. In the first part equations of motion of weakly nonlinear oscillators are linearized. Using stochastic averaging method the differential equations for the mean and variance of the process are obtained. In the second part a number of computer simulations for strongly nonlinear motion are developed. The stochastic process is characterized by the mean and standard deviation of these realizations. Calculations were carried out for Duffing, Ueda and for forced vibrations of pendulum. The calculations showed that if attractors exist, then the deterministic vibrations (which may be chaotic) turn regular by adding noise.

### 1 Introduction

Nonlinear stochastic vibrations have been investigated by several authors. The application of numerical methods of deterministic differential equations to stochastic differential equations can lead to difficulties due to differences between deterministic and stochastic calculi [1]. Different methods such as the stochastic linearization method [2, 3], quasi-static method [4], the path-integral method [5], wavelet-based method [6] have been developed. The random excitation has been introduced using several ways, *e.g.*, additional noise [7], multiplicative noise [8] or bounded noise [9]–[11]. The basic idea of any stochastic linearization consists in the replacement of the original nonlinear equation by such a linear equation that the difference between the two systems is minimal in some probabilistic sense. Quite interesting is the relationship between chaotic and stochastic motion. The well-known fact is that the regular motion turns chaotic due to the stochastic excitation. The natural question has been raised by Szemplińska-Stupnicka: “Can chaotic motion be interpreted as nonstationary ‘free vibration’ with randomly modulated amplitude and phase?” ([12]) On the contrary, there are some papers where it is demonstrated that the additional noise may stabilize the system. Kloeden and Platen [8] considered the

Duffing–van der Pol oscillator driven by multiplicative white noise. The Milstein scheme was used starting at different initial values. The random paths remained near each other until they come close to the origin  $(0, 0)$  after which they separated and were attracted into the neighborhood of either  $(-1, 0)$  or  $(1, 0)$ . The Duffing equation with random excitation has been considered in [6, 11], [13]–[18], where the random excitation has been introduced as additional, multiplicative or bounded noise. The case where the frequency of excitation  $\varpi$  is a narrow-banded random variable has been discussed by Lepik [19]. The example has been presented where initially chaotic motion in the case of the Duffing attractor by adding noise has been turned regular. The quantity  $\varpi$  can be interpreted as the “angular velocity” of a driver and in practice it can be a random variable.

The aim of the present paper is to analyse the nonlinear oscillators with random angular velocity  $\varpi$ . In Section 2 the weakly nonlinear Duffing equation is considered whereas in Section 3 computer simulations for strongly nonlinear oscillators are applied.

## 2 Stochastic averaging

Let us consider the nonlinear differential equation

$$\ddot{x} + p \dot{x} + qx + \beta x^2 + rx^3 = s \cos \varpi t, \quad 0 \leq t \leq T, \quad (1)$$

with the boundary conditions  $x(0) = x_0$ ,  $\dot{x}(0) = \dot{x}_0$ . In (1) dots denote differentiation with respect to time  $t$  and  $p, q, \beta, r, s$  are prescribed constants. The quantity  $\varpi$  has the form

$$\varpi = \varpi_0 [1 + \alpha \xi(t)], \quad (2)$$

where  $\varpi_0$  and  $0 \leq \alpha \leq 1$  are constants;  $\xi(t)$  represents a Gaussian white noise with zero mean and standard deviation  $\sigma = 1$ . The coefficient  $\alpha$  characterizes the noise intensity (for  $\alpha = 0$  the motion is deterministic). From physical point of view the equation can model the one-mode vibration of a suspended elastic cable driven by a quasi-periodic forcing [1]. In (1)  $p$  is the measure of damping,  $\beta$  and  $r$  are nonlinearities,  $s$  is the excitation amplitude. The quadratic term may be due to the curvature of the cable whereas the cubic term may be due to the symmetric material nonlinearity. In [1] it was studied how to bring the system (1) with parametric excitation from a chaotic regime to a regular one. The aim is to integrate (1) and explore the effect of randomness to the nonlinear vibrations.

We first determine the response of the system (1) when  $\alpha = 0$ . In this case equation (1) can be written as

$$\ddot{x}_D + p \dot{x}_D + qx_D + \beta x_D^2 + rx_D^3 = s \cos \varpi_0 t, \quad 0 \leq t \leq T. \quad (3)$$

Next we introduce the noise-induced deviation  $\delta x = x - x_D$ . Combining (1) and (3) and taking into account that  $x^2 \approx x_D^2 + 2x_D\delta x$  and  $x^3 \approx x_D^3 + 3x_D^2\delta x$ , one obtains

$$\begin{aligned} \delta \dot{x} &= \delta y, \\ \delta \dot{y} &= -p\delta y - q\delta x - 2\beta x_D\delta x - 3rx_D^2\delta x + s\phi(t, \xi), \end{aligned} \tag{4}$$

where

$$\phi(t, \xi) = \cos \varpi t - \cos \varpi_0 t. \tag{5}$$

Expanding (5) into harmonic series we get

$$\begin{aligned} \phi(t, \xi) &= \cos \varpi_0 t \left[ -\frac{1}{2} (\alpha\varpi_0 t\xi)^2 + \frac{1}{24} (\alpha\varpi_0 t\xi)^4 - \dots \right] \\ &\quad - \sin \varpi_0 t \left[ \alpha\varpi_0 t\xi - \frac{1}{6} (\alpha\varpi_0 t\xi)^3 + \dots \right]. \end{aligned} \tag{6}$$

The stochastic averaging of equations (4) gives

$$\begin{aligned} E(\delta \dot{x}) &= E(\delta y), \\ E(\delta \dot{y}) &= -pE(\delta y) - (q + 2\beta x_D + 3rx_D^2)E(\delta x) + sE(\phi), \end{aligned} \tag{7}$$

where

$$E[\phi(t, \xi)] = -\frac{1}{2} (\alpha\varpi_0 t)^2 \cos \varpi_0 t \left[ 1 - \frac{1}{4} (\alpha\varpi_0 t)^2 + \dots \right]. \tag{8}$$

Introducing second order moments

$$M_x = E[(\delta x)^2], \quad M_{xy} = E(\delta x \delta y), \quad M_y = E[(\delta y)^2] \tag{9}$$

and taking into account (8), the following system of equations is obtained

$$\begin{aligned} \dot{M}_x &= 2M_{xy}, \\ \dot{M}_{xy} &= -(q + 2\beta x_D + 3rx_D^2)M_x - pM_{xy} + M_y + sE(\delta x)E(\phi), \\ \dot{M}_y &= -2(q + 2\beta x_D + 3rx_D^2)M_{xy} - 2pM_y + 2sE(\delta y)E(\phi). \end{aligned} \tag{10}$$

The equation (1) can be integrated according to the following algorithm.

STEP1. Solve (3) for boundary conditions  $x_D(0) = x_0, \quad y_D(0) = \dot{x}_0$ .

STEP2. Calculate  $E(\phi)$  from (8).

STEP3. Integrate (7) for boundary conditions  $E[\delta x(0)] = E[\delta y(0)] = 0$ .

STEP4. Integrate (10) for  $M_x(0) = M_{xy}(0) = M_y(0) = 0$ .

STEP5. Calculate

$$\begin{aligned} E(x) &= x_D + E(\delta x), \\ D(x) &= E[(x_D + \delta x - E(\delta x))^2] = E[(\delta x)^2] - [E(\delta x)]^2. \end{aligned} \tag{11}$$

Knowing the mean  $E(x)$  and variance  $D(x)$  over the time interval  $t \in [0, T]$  is usually sufficient to characterize the stochastic process (1). In the case of necessity higher moments as skewness and kurtosis can be calculated.

As an example the case  $s = 1$ ;  $p = 0.05$ ,  $q = -1$ ,  $r = 0.2$ ,  $\beta = 0.3$ ;  $x_0 = 0$ ,  $\dot{x}_0 = 1$  is considered; the results are plotted in Fig. 1.

This method can be applied only in the case of weakly nonlinear systems for which higher powers of  $\delta x$  can be neglected. Strongly nonlinear oscillators are considered in the following Sections.

### 3 Computer simulation

For numerical integration of (1) the time interval  $t \in [0, T]$  is discretized so that  $0 \leq t_1 < t_2 < \dots < t_k \leq 1$ ; here  $t_i$ ,  $i = 1, 2, \dots, k$ , are discretization points and  $k$  is the number of these points. Making use of the Gaussian pseudorandom number generator, the variable  $\xi$  is discretized at the same points; for intermediate instants the values of  $\xi$  are calculated by some appropriate interpolation method. Now the function  $\cos \varpi t$  is continuous and for integrating (1) the same techniques can be used as in the case of deterministic systems. Of course, this is an approximation of the actual stochastic process for which  $\xi$  is nondifferentiable and Itô-type equations hold. In favour of such an approach speaks the fact that in reality the forcing term  $F = s \cos \varpi t$  is continuous by physical reasons.

Integration of (1) is repeated for  $N$  independent different sequences  $\{\xi_i\}$ ; in this way  $N$  realizations of the random process are obtained. From these data the mean, variance and standard deviation are calculated with the aid of the formulae

$$\begin{aligned} E[x(t)] &= \frac{1}{N} \sum_v x^{(v)}(t), \\ D[x(t)] &= \frac{1}{N-1} \sum_v [x^{(v)}(t) - E[x(t)]]^2, \quad \sigma = \sqrt{D[x(t)]}. \end{aligned} \quad (12)$$

Here  $v$  is the number of the  $v$ -th realization.

According to this scheme computer simulations were carried out for a number of problems. The fourth order Runge-Kutta method with adapted stepsize was used. It turned out that already a small number of realizations ( $N < 10$ ) enables to estimate various statistical features of the solution.

Some results for  $\alpha = 0.2$  are plotted in Figs. 2–7. To preserve distinctness of these plots for  $N$  a small number  $N = 5$  was taken. Each plot in Figs. 2–7 consists of four parts. In parts (a) and (b) the time history and phase diagram for deterministic motion  $\alpha = 0$  are plotted. In part (c) stochastic realizations are presented; in part (d) the standard deviation as a time function is shown.

### 3.1 Duffing oscillator

Consider now equation (1) with  $\beta = 0$ :

$$\ddot{x} + p \dot{x} + qx + rx^3 = s \cos \varpi t, \quad 0 \leq t \leq T. \quad (13)$$

The unforced equation  $s = 0$  has three fixed points  $\bar{x}_1 = \bar{y}_1 = 0$  and  $\bar{x}_{2,3} = \pm\sqrt{-q/r}$ ,  $\bar{y}_{2,3} = 0$  (the notation  $y = \dot{x}$  is introduced). The eigenvalues of these fixed points are [20]

$$\lambda = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q - 3r \bar{x}_i^2} \quad (i = 1, 2, 3). \quad (14)$$

Oscillator with softening stiffness  $p > 0$ ,  $q < 0$ ,  $r > 0$ . In the case of the fixed point  $\bar{x}_1 = 0$  it follows from (14) that  $\lambda_1 < 0$ ,  $\lambda_2 > 0$  and this is a saddle point. As to  $\bar{x}_{2,3}$  then  $\lambda_1 < 0$ ,  $\lambda_2 < 0$ ; if  $p^2 + 8q > 0$ , these are stable modes, in the opposite case  $p^2 + 8q < 0$  the eigenvalues are complex numbers and the fixed points are stable focuses. So for this type of oscillator always two stable fixed points exist (two-well oscillator).

Computer simulation results for a typical case are presented in Fig. 2. Deterministic motion is chaotic, stable focuses are at  $\bar{x} = \pm 1$ . By adding noise with  $\alpha = 0.2$  the motion turns regular and terminates in the focus  $x = 1$ . The standard deviation  $\sigma$  is maximal around  $t \approx 10$  and with increasing time approaches to zero.

Calculations with other parameter values indicated that the situation, where some of the stochastic realizations are attracted by the focus  $\bar{x} = 1$  and other - by the other focus  $\bar{x} = -1$ , may exist.

Computer simulation results for  $p = 0.25$ ,  $q = -1$ ,  $r = 1$ ,  $\beta = 0.5$  are plotted in Fig. 3. It follows from this figure that stochastic realizations converge to two different solutions. The standard deviation  $\sigma$  has a decreasing tendency in time.

Assuming  $q = 0$  in (13), the Ueda equation is obtained. This equation has only one fixed point  $\bar{x}_1 = 0$ ; according to (14)  $\lambda_1 = 0$ ,  $\lambda_2 < 0$ ; consequently this is a degenerated fixed point. Computer simulation results for  $p = 0.05$ ,  $q = 0$ ,  $r = 1$  are plotted in Fig. 4. No convergence between different stochastic realizations is observed; the standard deviation  $\sigma$  also essentially differs from zero values. In view of the Hartman-Grobman theorem all this was to be expected.

### 3.2 Van der Pol-Duffing oscillator

The differential equation of this oscillator can be written in the form

$$\ddot{x} - a(1 - x^2) \dot{x} + qx + rx^3 = s \cos \varpi t \quad (a > 0). \quad (15)$$

For the conventional van der Pol equation  $r = 0$ . Since the term  $rx^3$  is characteristic to Duffing equation, then (16) is called van der Pol-Duffing equation.

The unforced equation  $s = 0$  has only one fixed point  $\bar{x} = 0$ . A linearization of (16) in the neighborhood of the fixed point gives  $\ddot{x} - a \dot{x} + qx = 0$ . This equation has the eigenvalues

$$\lambda_{1,2} = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - q}.$$

If  $a^2 > 4q$ , then  $\lambda_1 > 0$ ,  $\lambda_2 > 0$  and the fixed point is an unstable node; if  $a^2 < 4q$ , the eigenvalues are complex with a positive real part and the fixed point is an unstable focus. Hence it follows that (15) does not have any stable fixed point. But it is well known that the van der Pol equation may have a limit cycle.

Computer simulation results for  $q = 1$ ,  $r = 0$ ,  $s = 0.5$ ,  $\varpi = 1$  are plotted in Fig. 5. It can be seen from Fig. 5b that a limit cycle exists. The effect of noise to the vibrations is very small (Fig. 5c): all stochastic realizations practically coincide.

### 3.3 Vibrations of the pendulum

Consider a mathematical pendulum with mass  $m$  and length  $l$ . It is periodically driven by an external force  $F = G \cos \Omega t$ , where  $G$  and  $\Omega$  are amplitude and frequency of the excitation force. The equation of motion is

$$ml \frac{d^2 \varphi}{dt_*^2} = -\mu \frac{d\varphi}{dt_*} - (mg + G \cos \Omega t_*) \sin \varphi. \quad (16)$$

Here  $\varphi$  is the rotation angle,  $g$  – gravity constant,  $\mu$  – damping coefficient.

By the change of variables

$$t_* = t \sqrt{\frac{l}{g}}, \quad \varpi = \Omega \sqrt{\frac{l}{g}}, \quad a = \frac{G}{mg}, \quad b = \frac{\mu}{m \sqrt{lg}},$$

equation (16) can be written in the form

$$\dot{x} = y, \quad \dot{y} = -\sin x(1 + a \cos \varpi t) - by. \quad (17)$$

Here  $x = \varphi$ , dots denote differentiation with respect to nondimensional time  $t$ .

The fixed points of (17) are  $\bar{x} = k\pi$ ,  $\bar{y} = 0$ , where  $k$  is an integer. It is shown [21] that if  $k$  is an even number, the fixed points are stable focuses and saddle points if  $k$  is odd.

Depending upon the initial conditions the motion can be libration, rotation or consist of librations and rotations. As before it is assumed that  $\varpi$  is stochastic and defined by (2).

From computer simulations the results of the following two cases are presented here.

(i) The case  $a = 2$ ,  $b = 1$ ,  $\varpi_0 = 0.5\pi$ ,  $x(0) = 0$ ,  $y(0) = 1$  is plotted in Fig. 6. It follows from Fig. 6 a,b that the motion is a nonregular libration. All the stochastic realizations practically coincide and already for  $t > 10$  the motion is terminated at the fixed point  $\bar{x} = 0$ . The standard deviation is very small.

(ii) Here computations were carried out for  $a = 8$ ,  $b = 1$ ,  $\varpi_0 = 0.5\pi$ ,  $x(0) = 2$ ,  $y(0) = 0$ ; the results are plotted in Fig. 7. The deterministic motion is irregular, it consists of successive librations and rotations. The phase diagram has a rather complicated form. As to noisy motion then it is very simple: the vibrations die away very soon and the motion terminates at the focus  $\bar{x} = -2\pi$ .

## 4 Conclusions

Nonlinear vibrations for which “the angular velocity” of the driver is stochastic are investigated. Two methods of solution are suggested. For weakly nonlinearity the equations of motion are linearized. Making use of stochastic averaging mean and variance for the system variable are calculated.

In the case of strong nonlinearity computer simulation approach is used. By Runge-Kutta technique stochastic realizations of the system are computed. Divergence of these realizations is estimated by standard deviation. Calculations which were carried out for Duffing, Ueda, van der Pol attractor and for a periodically driven pendulum showed that behavior of the noisy system essentially depends upon the type of the fixed points. If the fixed points are stable nodes or focuses then the motion, which for the deterministic system could be chaotic, by adding noise turns regular and is terminated in some of the fixed points. If the system has a limit cycle then the phase portrait of the noisy motion converges to this curve.

In the case of unstable fixed points no convergence of the stochastic realizations is observed.

## 5 Acknowledgment

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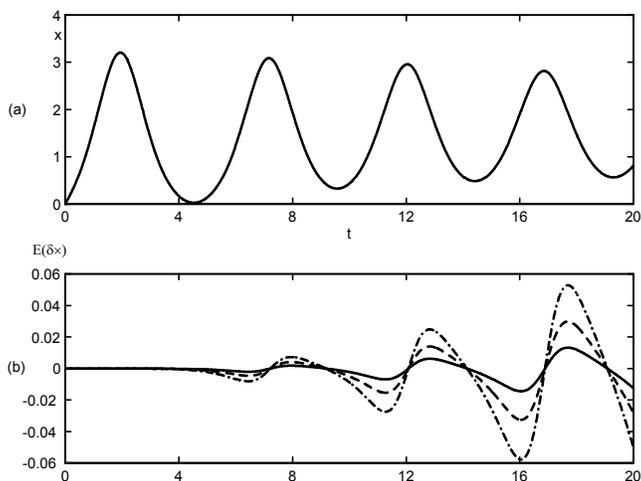


Fig. 1. Weakly nonlinear Duffing equation for  $p = 0.05$ ,  $q = -1$ ,  $r = 0.2$ ,  $\beta = 0.3$ ,  $s = 1$ ,  $\varpi_0 = 0.05$ ,  $x_0 = 0$ ,  $\dot{x}_0 = 1$ ; (a) time history of deterministic vibrations, (b) expectation of the noise induced deviation  $E(\delta x)$ : —  $\alpha = 0.1$ , - - -  $\alpha = 0.15$ , - · - · -  $\alpha = 0.2$ .

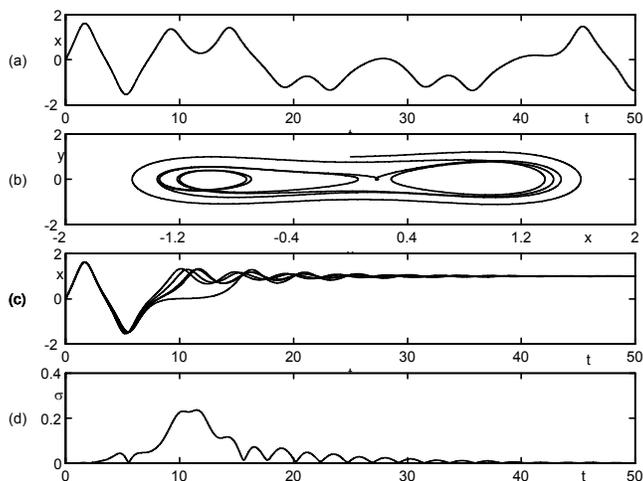


Fig. 2. Duffing equation (13) for  $p = 0.25$ ,  $q = -1$ ,  $r = 1$ ,  $s = 0.3$ ,  $\varpi_0 = 1$ ,  $x_0 = 0$ ,  $\dot{x}_0 = 1$ . In Figs. 2–8 subdiagrams (a)–(d) have the following meaning: (a) time history and (b) phase diagram in the case of deterministic motion; (c) time history and (d) standard deviator for the stochastic realizations.

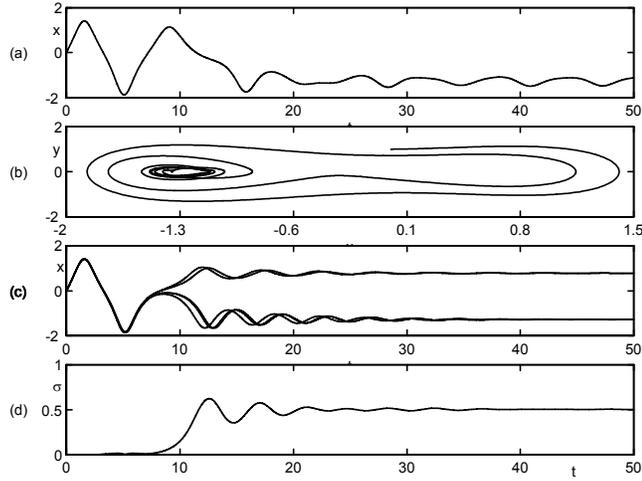


Fig. 3. Duffing equation (1) for  $p = 0.25$ ,  $q = -1$ ,  $r = 1$ ,  $s = 0.3$ ,  $\beta = 0.5$ ,  $\varpi_0 = 1$ ,  $x_0 = 0$ ,  $\dot{x}_0 = 1$ .

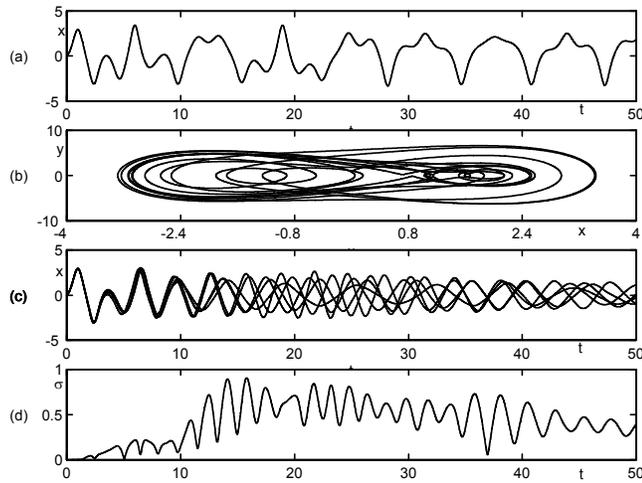


Fig. 4. Ueda oscillator; equation (1) for  $p = 0.05$ ,  $q = 0$ ,  $r = 1$ ,  $s = 7.5$ ,  $\varpi_0 = 1$ ,  $x_0 = 0$ ,  $\dot{x}_0 = 1$ .

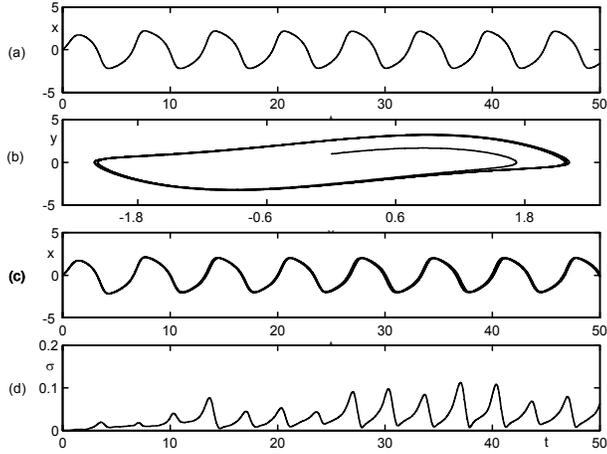


Fig. 5. Van der Pol oscillator (16) for  $a = 1$ ,  $q = 1$ ,  $r = 0$ ,  $\varpi_0 = 1$ ,  $s = 0.5$ ,  $x_0 = 0$ ,  $\dot{x}_0 = 1$ .

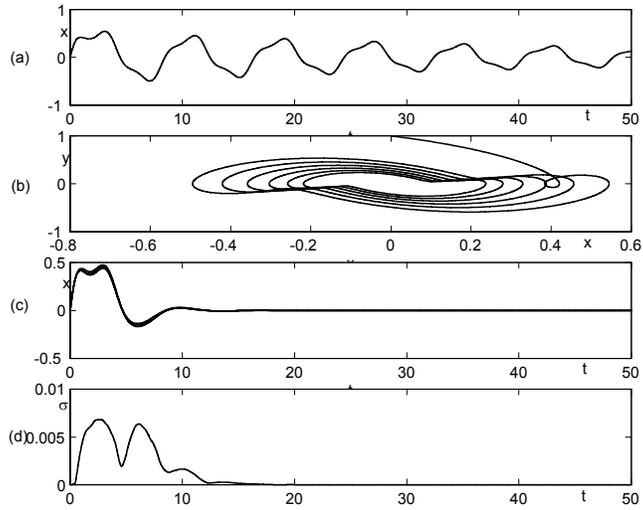


Fig. 6. Driven pendulum (17) for  $a = 2$ ,  $b = 1$ ,  $\varpi_0 = 0.5\pi$ ,  $x(0) = 0$ ,  $y(0) = 1$ .

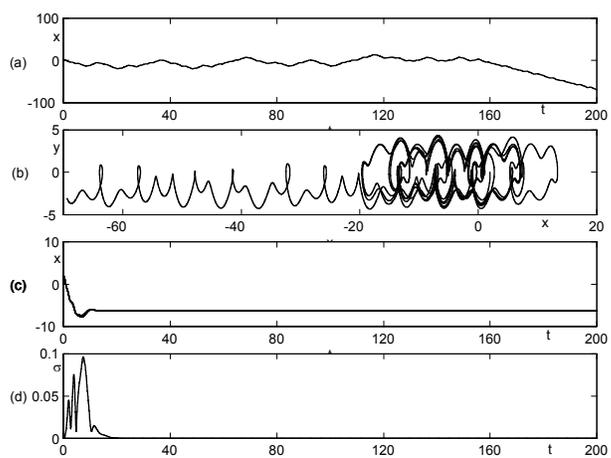


Fig. 7. Driven pendulum (17) for  $a = 8$ ,  $b = 1$ ,  $\varpi_0 = 0.5\pi$ ,  $x(0) = 2$ ,  $y(0) = 0$ .