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The Bohl Transformation and its Applications*

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Abstract

The Bohl transformation (sometimes also called the trigonometric transformation) is one of the basic tools of the oscillation theory of Sturm-Liouville second order differential equations. We present various extensions of this transformation, including the formulation of some new results and open problems associated with this transformation.

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1 Introduction

The classical Bohl transformation [5] (sometimes called the trigonometric transformation) concerns the second order Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0, \tag{1}$$

where r, c are continuous functions, $r(t) > 0$, and reads as follows:

Proposition 1 *Let x_1, x_2 be linearly independent solutions of* (1) *for which $r(x_1'x_2 - x_1x_2') = \pm 1$ and let $h = \sqrt{x_1^2 + x_2^2}$. Then the transformation $x = h(t)u$ transforms* (1) *into the equation*

$$\left(\frac{1}{q(t)}u\right)' + q(t)u = 0, \quad q := \frac{1}{rh^2}. \tag{2}$$

In particular, the solutions x_1, x_2 can be expressed in the form

$$x_1(t) = h(t)\cos\left(\int^t q(s)\,ds\right), \quad x_2(t) = h(t)\sin\left(\int^t q(s)\,ds\right).$$

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Since 1905, when the original paper of Bohl appeared, the Bohl transformation has been extended in many directions. These extensions, similarly to Proposition 1, enable to establish Sturmian-type separation theorems.

The aim of this paper is to present a survey of extensions of the Bohl transformation, to discuss applications in oscillation theory of various equations, including a formulation of some open problems. We also prove some new results concerning the Bohl transformation for symplectic dynamic systems on time scales.

The paper is organized as follows. In the next section we show the extension of the Bohl transformation to linear Hamiltonian systems. Section 3 deals with the discrete version of this transformation, applied to symplectic difference systems. The last section is devoted to the Bohl-type transformation for symplectic dynamic systems on time scales.

2 Linear Hamiltonian systems

Let $A, B, C : \mathbb{R} \to \mathbb{R}^{n\times n}$ be matrices of continuous functions such that the matrices B, C are symmetric, *i.e.*, $B^T = B$, $C^T = C$, and consider the $2n$-dimensional linear Hamiltonian differential system

$$\begin{pmatrix} x \\ u \end{pmatrix}' = \mathcal{H}(t)\begin{pmatrix} x \\ u \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}. \tag{3}$$

If $\mathcal{Z}$ is a fundamental matrix of (3) which is symplectic at some $t_0 \in \mathbb{R}$, *i.e.*,

$$\mathcal{Z}^T(t_0)\mathcal{J}\mathcal{Z}(t_0) = \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

then $\mathcal{Z}$ is symplectic everywhere. This easily follows from the identity $(\mathcal{Z}^T\mathcal{J}\mathcal{Z})' = 0$ which can be directly verified using the identity

$$\mathcal{H}^T(t)\mathcal{J} + \mathcal{J}\mathcal{H}(t) \equiv 0. \tag{4}$$

If $\mathcal{R}$ is a $2n \times 2n$ matrix of continuously differentiable functions, the transformation

$$\begin{pmatrix} x \\ u \end{pmatrix} = \mathcal{R}(t)\begin{pmatrix} y \\ z \end{pmatrix}$$

transforms (3) into the system

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \hat{\mathcal{H}}(t)\begin{pmatrix} y \\ z \end{pmatrix}, \quad \hat{\mathcal{H}} = \mathcal{R}^{-1}(-\mathcal{R}' + \mathcal{H}\mathcal{R}). \tag{5}$$

Moreover, if the matrix $\mathcal{R}$ is symplectic, the resulting system (5) is again a Hamiltonian system, *i.e.*, the matrix $\hat{\mathcal{H}}$ satisfies (4). If the transformation matrix $\mathcal{R}$ equals $\mathcal{J}$, *i.e.*, $\hat{\mathcal{H}} = \mathcal{J}^T \mathcal{H} \mathcal{J}$, system (5) is said to be *reciprocal system* to (3). This terminology is motivated by transformations of the Sturm-Liouville equation (1). This equation can be written as a 2-dimensional Hamiltonian system with $u = rx'$, $A = 0$, $B = \frac{1}{r}$, $C = -c$, and the reciprocal system is rewritten as the so-called *reciprocal* equation (if $c(t) \neq 0$)

$$\left(\frac{1}{c(t)} u' \right)' + \frac{1}{r(t)} u = 0.$$

Observe that equation (2) coincides with its reciprocal equation. Hence, the Bohl transformation can be regarded as a transformation of (1) into the *self-reciprocal* equation.

The next theorem, proved in [11], extends the Bohl transformation to (3).

Theorem 1 *Let* $\mathcal{Z} = \begin{pmatrix} X_1 & X_2 \\ U_1 & U_2 \end{pmatrix}$ *be a symplectic fundamental matrix of* (3), H *be any matrix satisfying* $HH^T = X_1 X_1^T + X_2 X_2^T$, *and* $G = (U_1 X_1^T - U_2 X_2^T) H^{T-1}$. *The transformation*

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} H & 0 \\ G & H^{T-1} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \tag{6}$$

transforms (3) *into the so-called trigonometric system*

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \tag{7}$$

where the matrix $Q = H^{-1} B H^{T-1}$ *is symmetric and the matrix* P *is antisymmetric, i.e.,* $Q = Q^T$ *and* $P = -P^T$. *Moreover, the matrix* H *can be chosen in such a way that* $P = 0$.

The concept of a trigonometric system was introduced by Barrett and Reid [4, 21] in connection with the Prüfer transformation of (3). System (7) is again a Hamiltonian system (since the transformation matrix in (6) is symplectic) and its matrix $\hat{\mathcal{H}} = \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}$ is antisymmetric, *i.e.*, the fundamental matrix of this system is (in addition to symplecticity) also orthogonal. Consequently, the trigonometric transformation from Theorem 1 can be regarded as a transformation of a system with symplectic fundamental matrix into a system with orthogonal symplectic fundamental matrix. Another point of view on the generalized Bohl transformation is that this transformation transforms general linear Hamiltonian differential system

(3) into a self-reciprocal system, since (7) coincides with its reciprocal system as can be verified by a direct computation.

Similarly to the scalar case, oscillatory properties of trigonometric systems (7) are easier to investigate than these properties of a general Hamiltonian system (3). In particular, if $\binom{S}{C}$ is a $2n \times n$ matrix solution of (7) such that the matrix $W = C + \mathrm{i}\, S$, $\mathrm{i} = \sqrt{-1}$ being the imaginary unit, is unitary for some t_0, *i.e.*, $W^*W = WW^* = I$, where W^* stands for the conjugate transpose of W, then W is unitary everywhere. Indeed, we have $W' = \mathrm{i}\, QW$ and this, together with the symmetry of Q, immediately imply that W is unitary. This means that the matrices S, C satisfy the identities

$$\begin{aligned} S^TC &= C^TS, & C^TC + S^TS &= I, \\ SC^T &= CS^T, & CC^T + SS^T &= I, \end{aligned} \tag{8}$$

and hence the matrix $V = WW^T = CC^T - SS^T + 2\mathrm{i}\, SC^T$ has symmetric real and imaginary parts. Moreover, this matrix solves the differential system

$$V' = \mathrm{i}\, [Q(t)V + VQ(t)], \tag{9}$$

and hence its determinant satisfies the Liouville-Jacobi formula

$$\det V(t) = \det V(t_0) \exp\left\{ 2\,\mathrm{i} \int_{t_0}^{t} \mathrm{Tr}\, Q(s)\, ds \right\}.$$

where Tr stands for the trace, *i.e.*, the sum of diagonal entries, of the matrix indicated.

Oscillatory properties of (3) are defined as follows. This system is said to be oscillatory if there exists a $2n \times n$ matrix solution $\binom{X}{U}$ such that the $n \times n$ matrix X^TU is symmetric, rank $\binom{X}{U} = n$ (such solution is said to be the *conjoined basis*, see [17], another terminology for such a solution, without assuming the rank conditions, is *prepared solutions*, see [16], or *isotropic solution*, see [10]), and a sequence $t_n \to \infty$ such that $\det X(t_n) = 0$, in the opposite case (3) is said to be *nonoscillatory*. Note that if (3) is supposed to be *controllable* [17] (another terminology is *identically normal* [21]), *i.e.*, the trivial solution $\binom{x}{u} \equiv \binom{0}{0}$ is the only solution for which $x(t) \equiv 0$ on an interval of positive length, then the zero points of $\det X(t)$ cannot have an accumulation point inside of the interval where the matrices A, B, C are continuous and B is nonnegative definite.

The following theorem deals with the basic oscillatory properties of (7). A part of the proof of this statement statement can be found in [3] and the rest in [14, 15].

Theorem 2 *Suppose that the matrix Q is nonnegative definite for large t, let $\binom{S}{C}$ be a conjoined basis* (7) *satisfying* (8), *and let $V = CC^T - SS^T + 2\mathrm{i}\, SC^T$.*

(i) *The number $\lambda = 1$ ($\lambda = -1$) is an eigenvalue of $V(t)$ if and only if the matrix $S(t)$ (the matrix $C(t)$) is singular. The eigenvalues of V move around the unit circle in the positive direction as t increases.*

(ii) *Let (7) be controllable and $Q(t)$ be nonnegative definite for large t. Then (7) is nonoscillatory if and only if*

$$\int^{\infty} \operatorname{Tr} Q(t)\, dt < \infty. \tag{10}$$

If $\binom{S}{C}$ is a conjoined basis of (7), then $\binom{C}{-S}$ is a conjoined basis as well and the matrix $\begin{pmatrix} C & S \\ -S & C \end{pmatrix}$ is orthogonal if and only if the matrix $C + \mathrm{i}\, S$ is unitary. By Theorem 1 the matrix $\mathcal{Z} = \begin{pmatrix} X_1 & X_2 \\ U_1 & U_2 \end{pmatrix}$ can be expressed in the form

$$\begin{pmatrix} X_1 & X_2 \\ U_1 & U_2 \end{pmatrix} = \begin{pmatrix} H & 0 \\ G & H^{T-1} \end{pmatrix} \begin{pmatrix} C & S \\ -S & C \end{pmatrix},$$

i.e., $X_1 = HC, \quad X_2 = HS$. Consequently, $\det X_1$, $\det X_2$ have the same zero points as $\det C$ and $\det S$, respectively. Tracing the movement of the eigenvalues of V around the unit circle and using Theorem 2, we get the following Sturmian-type theorem. The original proof of this statement, based of the investigation of the index of the quadratic functional associated with (3), can be found in [19].

Theorem 3 *Suppose that the matrix B is nonnegative definite and that* (3) *is identically normal on an interval $I \subset \mathbb{R}$. Let $\binom{X_1}{U_1}$, $\binom{X_2}{U_2}$ be conjoined bases of* (3) *such that the* (*constant*) *matrix $X_1^T U_2 - U_1^T X_2$ is nonsingular. If n_1, n_2 denote the number of zeros of $\det X_1$ and $\det X_2$ in an interval $I_0 \subset I$, respectively, then $|n_1 - n_2| \le n$.*

We conclude this section with some remarks related to the Bohl transformation for (1) and (3).

(i) The Bohl transformation is one of the basic tools of the transformation theory of Sturm-Liouville differential equation (1) which is elaborated in [9]. Combining the transformation described in Proposition 1 with the transformation of the independent variable, every Sturm-Liouville equation (1) can be transformed into the equation $y'' + y = 0$ considered on a suitable interval, and this equation (together with the associated interval) is taken as a canonical representative of the class of mutually transformable Sturm-Liouville equations. For details we refer to [9].

(ii) The Bohl transformation is also a basis for many oscillation criteria for (1), due to the fact that equation (2), which results from (1) upon the Bohl transformation, is oscillatory if and only if

$$\int^{\infty} q(t)\,dt = \int^{\infty} \frac{dt}{r(t)(x_1^2(t)+x_2^2(t))} = \infty, \tag{11}$$

where x_1, x_2 are linearly independent solutions of (1). Of course, the function h which transforms (1) into (2) is generally unknown, but using suitable estimates for integrands in (11), one can get explicit (non)oscillation criteria, for details see [20, 22].

(iii) Oscillation of Hamiltonian systems (3) and their Bohl-type transformation is closely related to the concept of the *argument of a symplectic matrix*, generally, to concepts of the symplectic geometry like Lagrange plane, Maslow cycle, etc., we refer to [18, 23] for details. In particular, the Lidskii [18] argument of symplectic matrix is closely related to $\det V(t)$, where V appears in (9).

3 Symplectic difference systems

Consider the symplectic difference system

$$z_{k+1} = S_k z_k, \quad k \in \mathbb{N}, \quad S = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \tag{12}$$

where $z = \binom{x}{u} \in \mathbb{R}^{2n}$, $x, u \in \mathbb{R}^n$, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} : \mathbb{N} \to \mathbb{R}^{n\times n}$, and $S : \mathbb{N} \to \mathbb{R}^{2n\times 2n}$ is a symplectic matrix for every $k \in \mathbb{N}$. Since the symplectic matrices form a subgroup of nonsingular matrices (with respect to matrix multiplication), a $2n \times 2n$ matrix solution $\mathcal{Z}$ of (12) is symplectic whenever the initial matrix Z_0 is symplectic.

Symplectic difference systems cover a large variety of difference equations and systems. Among them the Sturm-Liouville second order equation

$$\Delta(r_k \Delta x_k) + c_k x_{k+1} = 0, \quad r_k \neq 0, \tag{13}$$

the higher order self-adjoint equation

$$\sum_{\nu=0}^{n} (-1)^\nu \Delta^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k+n-\nu} \right) = 0, \quad r_k^{[n]} \neq 0, \tag{14}$$

and the linear Hamiltonian difference system

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k, \tag{15}$$

with $A, B, C \in \mathbb{R}^{n\times n}$, B, C symmetric (*i.e.*, $B = B^T$, $C = C^T$) and $I - A$ invertible. Indeed, concerning *e.g.* equation (13), this equation can be written as (15) using the substitution $u = r\Delta x$, the substitution which converts (14) into (15) can be found *e.g.* in [1]. Expanding the forward differences on the left-hand side of (15), this system can be written in the form

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{A}_k & \tilde{A}_k B_k \\ C_k \tilde{A}_k & C_k \tilde{A}_k B_k + I - A_k^T \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \tag{16}$$

where $\tilde{A} = (I - A)^{-1}$, and symplecticity of the matrix in (16) can be verified by a direct computation.

The discrete version of the Bohl transformation remained an open problem for rather long time, even in the classical setting for Sturm-Liouville equation (13). One of the reasons is that no difference equation of the form (13) is self-reciprocal. Indeed, if we denote $u = r\Delta x$ (the same substitution is used in the continuous case in computing the reciprocal equation), then u is a solution of the equation

$$\Delta\left(\frac{1}{c_k}\Delta u_k\right) + \frac{1}{r_{k+1}} u_{k+1} = 0,$$

and this shows that really no equation of the form (13) is self-reciprocal, *i.e.*, we have in disposal no Sturm-Liouville difference equations which plays the same role as equation (2) in the continuous case.

A starting point to investigate the problem of the discrete Bohl transformation from the "right point of view" was the publication of the paper [2], where the basic properties of the so-called *trigonometric difference system*

$$\begin{pmatrix} y_{k+1} \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} \mathcal{P}_k & \mathcal{Q}_k \\ -\mathcal{Q}_k & \mathcal{P}_k \end{pmatrix} \begin{pmatrix} y_k \\ z_k \end{pmatrix}, \tag{17}$$

are investigated, here $y, z \in \mathbb{R}^n$ and $\begin{pmatrix} \mathcal{P} & \mathcal{Q} \\ -\mathcal{Q} & \mathcal{P} \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$ is symplectic and orthogonal matrix, *i.e.*, its block $n \times n$ matrix entries satisfy

$$\mathcal{P}^T \mathcal{Q} = \mathcal{Q}^T \mathcal{P}, \quad \mathcal{P}^T \mathcal{P} + \mathcal{Q}^T \mathcal{Q} = I.$$

The following statement is a discrete extension of the Bohl transformation presented in Theorem 1, its proof can be found in [7].

Theorem 4 *Let* $\mathcal{Z} = \begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix}$ *be a symplectic fundamental matrix of* (12)*,* H, G *be* $n \times n$ *matrices given by* $HH^T = XX^T + \tilde{X}\tilde{X}^T$*,* $G = (UX^T + \tilde{U}\tilde{X}^T)H^{T-1}$*. The transformation* (6) *transforms* (12) *into* (17) *with the matrices* $\mathcal{P}, \mathcal{Q}$ *given by*

$$\mathcal{P}_k = H_{k+1}^{-1}(\mathcal{A}_k H_k + \mathcal{B}_k G_k), \quad \mathcal{Q} = H_{k+1}^{-1}\mathcal{B}_k H_k^{T-1}.$$

Moreover, the matrix H can be chosen in such a way that the matrix Q is symmetric and nonnegative definite.

When we apply the previous statement to (13), this equation can be transformed into the system

$$y_{k+1} = p_k y_k + q_k z_k, \quad z_{k+1} = -q_k y_k + p_k z_k, \tag{18}$$

where the sequences p, q satisfy $p_k^2 + q_k^2 = 1$ and $q_k > 0$. Then there exists a unique $\varphi_k \in (0, \pi)$ such that $p = \cos\varphi$, $q = \sin\varphi$ and

$$\begin{pmatrix} \cos\left(\sum^{k-1}\varphi_j\right) & \sin\left(\sum^{k-1}\varphi_j\right) \\ \sin\left(\sum^{k-1}\varphi_j\right) & -\cos\left(\sum^{k-1}\varphi_j\right) \end{pmatrix}$$

is a fundamental matrix of (18). Consequently, system (18) is oscillatory (*i.e.*, the first component y of any solution $\binom{y}{z}$ becomes zero or changes its sign infinitely many times in any discrete interval of the form $[N, \infty)$) if and only if

$$\sum^{\infty} \varphi_k = \sum^{\infty} \operatorname{arccot} \frac{p_k}{q_k} = \infty, \tag{19}$$

where arccot is the inverse function of the function $\cot\varphi = \frac{\cos\varphi}{\sin\varphi}$.

Recall that according to [6], system (12) is said to be *nonoscillatory* if there exists $m \in \mathbb{N}$ such that for every $N \in \mathbb{N}$, $N > m$, the $2n \times n$ matrix solution $\binom{X}{U}$ given by the initial condition $X_N = 0$, $U_N = I$ satisfies

$$Ker\, X_{k+1} \subseteq Ker\, X_k \quad \text{and} \quad X_k X_{k+1}^{\dagger} \mathcal{B}_k \geq 0,$$

where Ker, †, and $\geq$ stand for the kernel, the Moore-Penrose generalized inverse, and the nonnegative definiteness of the matrix indicated, see [6]. In the opposite case (12) is said to be *oscillatory*. In the continuous case, a necessary and sufficient condition for nonoscillation of controllable system (7) is known, it is condition (10). Concerning (17), a necessary and sufficient condition for nonoscillation of this system is an open problem and an equivalent characterization of nonoscillation is known only when the matrix $\mathcal{Q}$ is positive definite, as shows the following statement proved in [7].

Theorem 5 *Suppose that the matrix $\mathcal{Q}$ in* (17) *is symmetric and positive definite. Then this system is nonoscillatory if and only if*

$$\sum^{\infty} \operatorname{arccot}\left[\lambda^{[1]}(\mathcal{Q}_k^{-1}\mathcal{P}_k)\right] < \infty, \tag{20}$$

where $\lambda^{[1]}$ denotes the least eigenvalue of the matrix indicated.

To find an extension of this statement to a general system (17), where the matrix $\mathcal{Q}$ is allowed to be singular (which is *e.g.* the case when (17) results upon the discrete Bohl transformation applied to (12) corresponding to (14)) is a subject of the present investigation.

4 Symplectic dynamic systems on time scales

A *time scale* $\mathbb{T}$ is any closed subset of the set of real numbers $\mathbb{R}$. On a time scale $\mathbb{T}$ we define the following operators and concepts:

$$\sigma(t) := \inf\{s \in \mathbb{T},\ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T},\ s < t\}$$

are the *forward* and *backward shift operators*. A point $t \in \mathbb{T}$ is said to be *left-dense* (ld point) if $\rho(t) = t$, *right-dense* (rd point) if $\sigma(t) = t$, *left-scattered* (ls point) if $\rho(t) < t$, *right-scattered* (rs point) if $\sigma(t) > t$, and it is said to be *dense* if it is rd or ld. The *graininess* μ of a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be replaced by any Banach space) the *generalized derivative* $f^{\Delta}(t)$ is defined as follows. For every $\varepsilon > 0$ there exists a neighborhood U of t in $\mathbb{T}$ such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s| \quad \text{for all} \;\; s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$ and $f^{\Delta} = f'$ is the usual derivative. In case $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$ and $f^{\Delta} = \Delta f$ is the forward difference operator.

A function $f : [a, b] \to \mathbb{R}$ is said to be *rd-continuous* if it is continuous at each rd point and there exists a finite left limit at all ld points, and this function is said to be *rd-continuously differentiable* if its generalized derivative exists and it is rd-continuous. To every rd-continuous function f there exists its *generalized antiderivative* — a function F such that $F^{\Delta} = f$. Using the antiderivative we define $\int_a^b f(t)\Delta t := F(b) - F(a)$. A function f is said to be *regressive* if $1 + \mu(t)f(t) \neq 0$ (the mapping $x \longmapsto (\mathrm{id} + \mu(t)f(t))x$ is invertible if the range of f is a Banach space of linear operators). The initial value problem for the linear dynamic equation

$$z^{\Delta} = g(t)z, \quad z(t_0) = z_0$$

with a regressive and rd-continuous function g has the unique solution which depends continuously on the initial condition. For basic concepts of time scale theory we refer *e.g.* to [8].

A *symplectic dynamic system* on a time scale $\mathbb{T}$ is the first order linear system

$$z^{\Delta} = \mathcal{S}(t)z, \quad z = \begin{pmatrix} x \\ u \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \tag{21}$$

where $x, u : \mathbb{T} \to \mathbb{R}^n$, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} : \mathbb{T} \to \mathbb{R}^{n\times n}$ and $\mathcal{S}$ satisfies

$$\mathcal{J}\mathcal{S}(t) + \mathcal{S}^T(t)\mathcal{J} + \mu(t)\mathcal{S}^T(t)\mathcal{J}\mathcal{S}(t) = 0. \tag{22}$$

Observe that (22) reduces to (4) if $\mathbb{T} = \mathbb{R}$ and to the symplecticity of the matrix $I + S$ if $\mathbb{T} = \mathbb{Z}$. Consequently, symplectic dynamic systems (21) cover both linear Hamiltonian differential systems (3) and symplectic difference systems (12).

In [12] we have investigated oscillatory properties of the Sturm-Liouville dynamic equation

$$(r(t)x^\Delta)^\Delta + c(t)x^\sigma = 0, \quad r(t) \neq 0, \tag{23}$$

where r, c are rd-continuous functions and $x^\sigma = x \circ \sigma$, *i.e.*, $x^\sigma(t) = x(\sigma(t))$. Oscillatory properties of (23) are defined using the concept of the *generalized zero* of a solution x; we say that t is a *generalized zero* of x if $r(t)x(t)x^\sigma(t) \le 0$. In contrast to the continuous case $\mathbb{T} = \mathbb{R}$ (and in agreement with the discrete case $\mathbb{R} = \mathbb{Z}$), we generally need no sign restriction on the function r. However, to get a "reasonable" oscillation theory, we need the additional restriction

$$r(t) > 0 \text{ for rd } t \quad \text{and} \quad \lim_{s\to t-} r(t) =: l > 0 \text{ exists finite for ld } t, \tag{24}$$

see [12]. In that paper we have proved that under (24) equation (23) can be transformed (via a transformation preserving oscillatory properties of transformed equations) into an equation of the same form with $r(t) > 0$ for every $t \in \mathbb{T}$. In the next statement, which is a new result, we extend this statement to (21).

Theorem 6 *Suppose that the matrix $\mathcal{B}$ in* (21) *admits the polar decomposition $\mathcal{B}(t) = T(t)B(t)$, where B is symmetric and nonnegative definite and T is an orthogonal matrix, such that the matrix*

$$A(t) = \begin{cases} \frac{T(t)-I}{\mu(t)} & \text{if } \mu(t) > 0, \\ 0 & \text{if } \mu(t) = 0, \end{cases} \tag{25}$$

is rd-continuous. Then there exists an orthogonal matrix H such that the transformation (6) *with $G = 0$ transforms* (21) *into the system of the same form with the matrix $\mathcal{B}$ symmetric and nonnegative definite.*

Proof. First of all observe that the matrix A defined by (25) satisfies

$$A(t) + A^T(t) + \mu(t)A^T(t)A(t) = 0 \tag{26}$$

and that $I + \mu A$ is orthogonal, hence A is regressive. Let H be the solution of the matrix dynamic equation

$$H^\Delta = A(t)H, \quad H(t_0) = I, \tag{27}$$

where t_0 is a point in the time scale interval under consideration. According to (26), the matrix H is orthogonal for every $t \in I$. Indeed, we have

$$\begin{aligned}(H^T H)^\Delta &= H^T H^\Delta + (H^T)^\Delta H^\sigma = H^T A H + H^T A^T (H + \mu H^\Delta) \\ &= H^T (A + A^T + \mu A^T A) H = 0,\end{aligned}$$

and since $H(t_0) = I$, we have $H^T(t)H(t) = I$. The transformation $x = Hy$, $u = H^{T-1}z = Hz$ from our theorem transforms (12) into the system with the matrix

$$(H^\sigma)^{-1} \mathcal{B} H^{T-1} = (H^T)^\sigma \mathcal{B} H$$

instead of $\mathcal{B}$ (as can be verified by a direct computation), and

$$\begin{aligned}(H^T)^\sigma \mathcal{B} H &= [H^T + \mu (H^T)^\Delta] \mathcal{B} H = H^T (I + \mu A^T) \mathcal{B} H = H^T T^T T B H \\ &= H^T B T\end{aligned}$$

and the last matrix is symmetric and nonnegative definite. □

Remark 1 Observe that in the matrix case (in contrast to the scalar case mentioned at the beginning of this section, compare (24)), the assumption on rd-continuity of the matrix A cannot be replaced by a weaker assumption

$$\mathcal{B}(t) \text{ is positive definite at rd points and}$$

$$\lim_{s \to t-} \mathcal{B}(s) =: L \quad \text{exists finite and } L \text{ is positive definite for ld } t.$$

Indeed, let $\mathbb{T} = \left\{\frac{\pm 1}{n}\right\}_{n=1}^\infty \cup \{0\}$, and for $t \in \mathbb{T}$ let

$$\mathcal{B}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Then $\mathcal{B}(0) = I$ is positive definite and continuous at this point. However, the polar decomposition of $\mathcal{B}$ is $\mathcal{B} = \mathcal{B} \cdot I$ since $\mathcal{B}$ is orthogonal, and

$$\lim_{t \to 0+} \frac{\mathcal{B}(t) - I}{\mu(t)} = \lim_{n \to \infty} \left(\frac{1}{n-1} - \frac{1}{n} \right)^{-1} \left[\mathcal{B}\left(\frac{1}{n}\right) - I \right] \text{ does not exists (finite),}$$

i.e., this matrix is not rd-continuous, and hence condition (26) is not satisfied.

Now we turn our attention to the Bohl transformation for symplectic dynamic systems. Roughly speaking, this transformation transforms symplectic dynamic (21) into a symplectic system whose fundamental matrix is not only symplectic, but also orthogonal. The first part of the next statement was proved in [13], while the second one is a new result.

Theorem 7 *Let* $\mathcal{Z} = \begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix}$ *be a symplectic fundamental matrix of* (21), H, G *be* $n \times n$ *matrices defined in the same way as in Theorem 4. Transformation* (6) *transforms* (21) *into the system*

$$\begin{pmatrix} y \\ z \end{pmatrix}^{\Delta} = \begin{pmatrix} \mathcal{P}(t) & \mathcal{Q}(t) \\ -\mathcal{Q}(t) & \mathcal{P}(t) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \tag{28}$$

where the matrix $\hat{S} = \begin{pmatrix} \mathcal{P} & \mathcal{Q} \\ -\mathcal{Q} & \mathcal{P} \end{pmatrix}$ *satisfies the identity* $\hat{S}^T(t) + \hat{S}(t) + \mu(t)\hat{S}^T(t)\hat{S}(t) = 0$*, i.e., the fundamental matrix of* (28) *is orthogonal for* $t \in I$ *whenever it is orthogonal at one point of* I*. Moreover, if the matrix* $\mathcal{B}$ *in* (21) *is symmetric and positive definite on* I*, the matrix* H *can be taken in such a way that the matrix* $\mathcal{Q}$ *which is given by the formula* $\mathcal{Q} = (H^\sigma)^{-1}\mathcal{B}H^{T-1}$ *is also symmetric and positive definite.*

Proof. We will prove the second part of theorem only (starting with "Moreover"), the first part is proved in [13]. The matrix H is of the form $H = DG$, where G is an orthogonal matrix and $D = \left[XX^T + \tilde{X}\tilde{X}^T\right]^{1/2}$, *i.e.*, D is the unique symmetric positive definite matrix for which $D^2 = XX^T + \tilde{X}\tilde{X}^T$. Then

$$\begin{aligned} \mathcal{Q} &= (H^\sigma)^{-1}\mathcal{B}H^{T-1} = (G^T)^\sigma (D^{-1})^\sigma \mathcal{B} D^{-1} G \\ &= (G^T + \mu(G^T)^\Delta)(D^{-1} + \mu(D^{-1})^\Delta)\mathcal{B}D^{-1}G. \end{aligned}$$

Denote $\tilde{\mathcal{B}} = D^{-1}\mathcal{B}D^{-1}$, this matrix is also symmetric and positive definite. We have

$$(D^\sigma)^{-1}\mathcal{B}D^{-1} = \tilde{\mathcal{B}} + \mu(D^{-1})^\Delta \mathcal{B}D^{-1} = \tilde{\mathcal{B}} + \mu\mathcal{K},$$

where $\mathcal{K} = (D^{-1})^\Delta \mathcal{B} D^{-1}$. Now we turn our attention to the polar decomposition of the matrix $(D^\sigma)^{-1}\mathcal{B}D^{-1}$, which is of the form

$$(D^\sigma)^{-1}\mathcal{B}D^{-1} = T\hat{\mathcal{B}},$$

where T is an orthogonal matrix and $\hat{\mathcal{B}}$ is symmetric and positive definite. Using the rules for computing the matrix square root we have

$$\begin{aligned} \hat{\mathcal{B}} &= \left\{[(D^\sigma)^{-1}\mathcal{B}D^{-1}]^T[(D^\sigma)^{-1}\mathcal{B}D^{-1}]\right\}^{1/2} = \left\{(\tilde{\mathcal{B}} + \mu\mathcal{K})^T(\tilde{\mathcal{B}} + \mu\mathcal{K})\right\}^{1/2} \\ &= \left[\tilde{\mathcal{B}}^2 + \mu(\mathcal{K} + \mathcal{K}^T) + o(\mu)\right]^{1/2} = \tilde{\mathcal{B}} + \mu\tilde{\mathcal{K}} + o(\mu), \end{aligned}$$

here $\tilde{\mathcal{K}}$ is an $n \times n$ matrix (its explicit form is not important) and $o(\mu)$ is a matrix which tends to the zero matrix as $\mu \to 0+$. Now, substituting into the previous computation we obtain

$$(D^\sigma)^{-1}\mathcal{B}D^{-1} = T\hat{\mathcal{B}} = T[\tilde{\mathcal{B}} + \mu\tilde{\mathcal{K}} + o(\mu)]$$

and hence

$$(I+\mu(D^{-1})^{\Delta}D)\tilde{\mathcal{B}} = T(I+\mu\tilde{\mathcal{K}}\tilde{\mathcal{B}}^{-1}+o(\mu)]\tilde{\mathcal{B}}$$

which implies (using invertibility of $\tilde{\mathcal{B}}$)

$$\begin{aligned} T &= [I+\mu(D^{-1})^{\Delta}D][I+\mu\tilde{\mathcal{K}}\tilde{\mathcal{B}}^{-1}+o(\mu)]^{-1} \\ &= I+\mu\mathcal{N}+o(\mu), \end{aligned}$$

where $\mathcal{N}$ is an rd-continuous matrix, the explicit formula for this matrix is not important. Consequently, the matrix A defined by (25) is rd-continuous.

Now, if the matrix G is defined as the solution of (27), this solution exits, it is an orthogonal matrix and

$$\mathcal{Q} = (G^T)^{\sigma}(D^{\sigma})^{-1}\mathcal{B}D^{-1}G = G^T(I+\mu A^T)T\hat{\mathcal{B}}G = G^T\hat{\mathcal{B}}G,$$

i.e., this matrix is symmetric and positive definite what we needed to prove. □

We finish this paper with a remark concerning the oscillatory properties of trigonometric symplectic dynamic systems. First consider the scalar case $n=1$. If the function $\mathcal{Q}$ satisfies the same condition as the function r in (24), system (28) with $n=1$ is oscillatory if and only if

$$\int^{\infty}\omega(t)\Delta t = \infty, \tag{29}$$

where

$$\omega(t) = \begin{cases} \frac{1}{\mu(t)}\operatorname{arccot}\left[\frac{r(t)}{\mu(t)}(x_1(t)x_1^{\sigma}(t)+x_2(t)x_2^{\sigma}(t))\right] & \text{if } \ \mu(t)>0, \\ \frac{1}{r(t)(x_1^2(t)+x_2^2(t))} & \text{if } \ \mu(t)=0, \end{cases}$$

and x_1, x_2 are linearly independent solutions of (23), see [12]. Observe that this condition really reduces to (11) if $\mu=0$ ($\mathbb{T}=\mathbb{R}$) and to (19) if $\mu=1$ ($\mathbb{T}=\mathbb{Z}$). It is an open problem how this condition can be extended to (28). In particular, a unifying time scale condition which covers both (10) and (20) is not known yet, to find such a unifying condition is a subject of the present investigation.

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