5–10 July 2004, Antalya, Turkey — Dynamical Systems and Applications, Proceedings, pp. 361–370

Oscillatory Solutions of Third Order Nonlinear Difference Equations

Zuzana Došlá, Aleš Kobza
Department of Mathematics, Masaryk University Brno
Janáčkovo nám. 2a, 602 00 Brno, Czech Republic
E-mail: dosla@math.muni.cz, akob@math.muni.cz

Abstract

We consider the third order nonlinear difference equations

$$\Delta(p_n\Delta(r_n\Delta x_n)) - q_n f(x_{n+2}) = 0,$$

where (p_n), (r_n) and (q_n) are sequences of positive real numbers and the function f satisfies $f(u)u > 0$ for $u \neq 0$. We study generalized zeros of solutions and asymptotic properties of oscillatory solutions in terms of a certain energy function.

Key words: Nonlinear third order difference equation, Oscillatory solution, Generalized zeros, Asymptotic behavior.

1 Introduction

The aim of the paper is to study oscillatory solutions of the third order nonlinear difference equations

$$\Delta\left(p_n\Delta\left(r_n\Delta x_n\right)\right) - q_n f\left(x_{n+2}\right) = 0, \tag{1}$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$. Throughout the paper we assume that (p_n), (r_n) and (q_n) are sequences of positive real numbers for $n \in \mathbb{N}$ such that

$$\Delta\left(\frac{p_n}{r_{n+1}}\right) \leq 0 \quad \text{for } n \in \mathbb{N} \tag{2}$$

and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $f(u)u > 0$ for $u \neq 0$.

As usual, a solution x of (1) is a real sequence (x_n) defined for all $n \in \mathbb{N}$ and satisfying (1) for all $n \in \mathbb{N}$. A solution of (1) is called *nontrivial* if for any $n_0 \geq 1$ there exists $n > n_0$ such that $x_n \neq 0$. Otherwise, the solution is called *trivial.* A nontrivial solution x of (1) is said to be *oscillatory* if for any $n_0 \geq 1$ there exists $n > n_0$ such that $x_{n+1}x_n \leq 0$. Otherwise, the nontrivial solution is said to be *nonoscillatory.*

Denote quasidifferences $x^{[i]}$, $i = 0, 1, 2$, of a solution x of (1) as follows:

$$x_n^{[0]} = x_n, \quad x_n^{[1]} = r_n \Delta x_n, \quad x_n^{[2]} = p_n \Delta x_n^{[1]}, \quad x_n^{[3]} = \Delta x_n^{[2]}.$$

Asymptotic and oscillatory properties of solutions of (1) have been recently investigated in the literature. For instance, the linear equation with $p_n = r_n \equiv 1$ has been studied in [7, 8, 9] and the nonlinear equation (1) with $r_n \equiv 1$ in [6]. A similar problem for the equation

$$\Delta \left(p_n \Delta \left(r_n \Delta x_n\right)\right) + q_n f \left(x_{n+1}\right) = 0, \tag{3}$$

where $q_n > 0$, has been investigated in [3] and in the case $p_n = r_n = 1$ in [1, Problem 6.24.45]. Nonoscillatory solutions of nonlinear equations

$$\Delta(p_n \Delta(r_n \Delta x_n)) + q_n f(x_{n+\sigma}) = 0, \tag{4}$$

where $\sigma \in \{0, 1, 2\}$, have been studied in [3, 5] (case $q_n > 0$) and in [4] (case $q_n < 0$).

In this paper, we will introduce a certain energy function which enables us to describe generalized zeros of all solutions of (1) and asymptotic properties of oscillatory solutions of (1). In addition, we describe nonoscillatory solutions with regard to the energy function and we give examples illustrating our results.

2 Energy function

We start with the following result which will be used later.

Lemma 1 *Let x be a solution of* (1)*.*

(i) *If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = x_{n_0+2} = 0$, then $x_n = 0$ for all $n \in \mathbb{N}$.*

(ii) *If there exists $a \in \mathbb{N}$ such that $x_a \neq 0$, then x is nontrivial solution.*

Proof. See [4, Proposition 2.1]. □

In what follows, an important role will be played by the following energy function.

Lemma 2 *Let x be a solution of* (1) *and*

$$G_n = \frac{p_n}{r_{n+1}} \left(x_n^{[1]}\right)^2 - 2x_{n+1}x_n^{[2]}. \tag{5}$$

Then

$$\Delta G_n = \Delta\left(\frac{p_n}{r_{n+1}}\right)\left(x^{[1]}_{n+1}\right)^2 - 2q_n x_{n+2} f(x_{n+2}) - \frac{1}{p_n r_{n+1}}\left(x^{[2]}_n\right)^2 \le 0, \qquad (6)$$

that is, $G = (G_n)$ is a nonincreasing sequence for $n \ge 1$.

In addition, if x is a nontrivial solution of (1), then there does not exist $a \in \mathbb{N}$ such that $G_a = G_{a+1} = G_{a+2}$.

Proof. By (2) we have

$$\begin{aligned}
\Delta G_n &= \Delta\left(\frac{p_n}{r_{n+1}}\right)\left(x^{[1]}_{n+1}\right)^2 + \frac{p_n}{r_{n+1}}\left(x^{[1]}_{n+1}\Delta x^{[1]}_n + x^{[1]}_n \Delta x^{[1]}_n\right) \\
&\quad -2x^{[2]}_n \Delta x_{n+1} - 2x_{n+2}\Delta x^{[2]}_n \\
&= \Delta\left(\frac{p_n}{r_{n+1}}\right)\left(x^{[1]}_{n+1}\right)^2 + \frac{p_n}{r_{n+1}}\Delta x^{[1]}_n\left(x^{[1]}_{n+1} + x^{[1]}_n\right) \\
&\quad -2p_n \Delta x^{[1]}_n \frac{x^{[1]}_{n+1}}{r_{n+1}} - 2q_n x_{n+2} f(x_{n+2}) \\
&= \Delta\left(\frac{p_n}{r_{n+1}}\right)\left(x^{[1]}_{n+1}\right)^2 - 2q_n x_{n+2} f(x_{n+2}) - \frac{p_n}{r_{n+1}}\Delta x^{[1]}_n\left(x^{[1]}_{n+1} - x^{[1]}_n\right) \\
&= \Delta\left(\frac{p_n}{r_{n+1}}\right)\left(x^{[1]}_{n+1}\right)^2 - 2q_n x_{n+2} f(x_{n+2}) - \frac{p_n}{r_{n+1}}\left(\Delta x^{[1]}_n\right)^2 \\
&= \Delta\left(\frac{p_n}{r_{n+1}}\right)\left(x^{[1]}_{n+1}\right)^2 - 2q_n x_{n+2} f(x_{n+2}) - \frac{1}{p_n r_{n+1}}\left(x^{[2]}_n\right)^2 \le 0.
\end{aligned}$$

Assume $G_a = G_{a+1} = G_{a+2}$. Then $\Delta G_a = 0$, which implies $x_{a+2} = 0$ and $x^{[2]}_a = 0$. Similarly, from $\Delta G_{a+1} = 0$ it follows $x_{a+3} = 0$ and $x^{[2]}_{a+1} = 0$. Taking into account $\Delta x^{[1]}_{a+1} = 0$ and using $\Delta x_{a+2} = 0$, we have $\Delta x_{a+1} = 0$ and so $x_{a+1} = 0$. Now the conclusion follows from Lemma 1. □

The function G determines the dichotomy for all nontrivial solutions of (1). Solutions of (1) for which $G_n > 0$, $n \ge 1$ are said to be G_+ *solutions*, and solutions of (1) for which $G_n < 0$ eventually are said to be G_- *solutions*.

There always exist G_- solutions, because any nontrivial solution with a double zero or any solution for which there exists $a \in \mathbb{N}$ such that $x_a = x_{a+1} = x_{a+2} \ne 0$ or any nontrivial solution with the initial condition $G_1 \le 0$ is a G_- solution.

Theorem 1 *If x is a solution of (1) such that $\lim_{n\to\infty} G_n > -\infty$, then*

$$\sum_{n=1}^{\infty} \Delta\left(\frac{p_n}{r_{n+1}}\right)\left(x_{n+1}^{[1]}\right)^2 > -\infty, \tag{7}$$

$$\sum_{n=1}^{\infty} q_n x_{n+2} f(x_{n+2}) < \infty, \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{1}{p_n r_{n+1}}\left(x_n^{[2]}\right)^2 < \infty. \tag{9}$$

Proof. Clearly,

$$G_{m+1} - G_1 = \sum_{n=1}^{m} \Delta G_n.$$

Using (6) we have

$$G_{m+1} - G_1 = \\ = \sum_{n=1}^{m} \Delta\left(\frac{p_n}{r_{n+1}}\right)\left(x_{n+1}^{[1]}\right)^2 - 2\sum_{n=1}^{m} q_n x_{n+2} f(x_{n+2}) - \sum_{n=1}^{m} \frac{1}{p_n r_{n+1}}\left(x_n^{[2]}\right)^2.$$

Letting $m \to \infty$ we get the conclusion. □

Example 1 Consider equation (1) where $p_n = r_n = 1$ and its solution x given by the initial conditions $x_1 = 2$, $x_2 = 1$, $x_3 = 0$. By Lemma 1, such solution is nontrivial and satisfies $G_1 = G_2 = 1$. This illustrates that G need not be decreasing.

3 Generalized zeros of solutions

For a nontrivial solution $x = (x_n)$ of (1), we say $k \in \mathbb{N}$ is a *zero* if $x_k = 0$, $k \in \mathbb{N}$ is a *node* if $x_k x_{k+1} < 0$ and $a \in \mathbb{N}$ is a *double zero* if $x_a = x_{a+1} = 0$. Zeros and nodes are called *generalized zeros.*

Theorem 2 *Any nontrivial solution x of (1) has the following properties:*

(i) *there exists no $a \in \mathbb{N}$ such that $x_a = x_{a+1} = x_{a+2} = x_{a+3} \neq 0$;*

(ii) *there exists at most one $a \in \mathbb{N}$ such that $x_a = x_{a+1} = x_{a+2} \neq 0$;*

(iii) *x has at most one double zero;*

(iv) *if there exists a double zero then there exists no $a \in \mathbb{N}$ such that $x_a = x_{a+1} = x_{a+2}$ and vice versa, if there exists $a \in \mathbb{N}$ such that $x_a = x_{a+1} = x_{a+2}$, then there exists no double zero.*

Proof. Claim i). Assume by contradiction that there exists $a \in \mathbb{N}$ such that $x_a = x_{a+1} = x_{a+2} = x_{a+3} \neq 0$. From $x_a = x_{a+1} = x_{a+2}$ we have $x_a^{[1]} = 0$, $x_{a+1}^{[1]} = 0$, which implies $x_a^{[2]} = 0$, and by (5) we have $G_a = 0$. Similarly from $x_{a+1} = x_{a+2} = x_{a+3}$ it follows $G_{a+1} = 0$ and hence $\Delta G_a = 0$. According to (6) this means that $x_{a+2} = 0$ and so $x_i = 0$ for $i = a, a+1, a+2, a+3$. By Lemma 1 x is trivial solution, which yields a contradiction.

Claim ii). Assume that there exist $a, b \in \mathbb{N}$ such that $x_a = x_{a+1} = x_{a+2} \neq 0$ and $x_b = x_{b+1} = x_{b+2} \neq 0$, which implies that $G_a = G_b = 0$. Without loss of generality we can suppose that $a \leq b$. If $a < b$ then in view of claim i) we have $b \geq a+3$. This gives a contradiction because by Lemma 2 it holds $G_n < 0$ for $n \geq a+2$. Thus $a = b$.

Claim iii). Assume that x has a double zero at a and b and let $a \leq b$. From (5) we have $G_a = G_b = 0$. Suppose $a < b$. By Lemma 1 we have $x_{a+2} \neq 0$, so $b \neq a+1$ and $b \neq a+2$. By the same argument as in claim ii) we prove that $b \geq a+3$ gives a contradiction with the fact $G_n < 0$ for $n \geq a+2$. Thus $a = b$.

Claim iv). Assume by contradiction that x has a double zero at a and there exists $b \in \mathbb{N}$ such that $x_b = x_{b+1} = x_{b+2} \neq 0$. Clerly $a \notin \{b-1, b, b+1, b+2\}$ and $G_a = G_b = 0$. If $a \leq b-2$ [$a \geq b+3$], then by Lemma 2 $G_b < 0$ [$G_a < 0$], a contradiction. □

Remark 1 There always exists a nontrivial solution of (1) such that $x_a = x_{a+1} = x_{a+2}$ at some $a \in \mathbb{N}$, *e.g.*, a solution of (1) with the initial value $x_1 = x_2 = x_3 = 1$ has such property. Similarly, there always exist nontrivial solutions of (1) with a double zero, *e.g.*, a solution of (1) with the initial value $x_1 = x_2 = 0$, $x_3 = 1$.

Remark 2 Let x be a solution of (1). If there exists $a \in \mathbb{N}$ such that $G_a \neq 0$, then x is a nontrivial solution. Indeed, if such solution is trivial, then by Lemma 1 $x_n = 0$ for $n \in \mathbb{N}$, and by (5) $G_n = 0$ for $n \in \mathbb{N}$, a contradiction.

Lemma 3 *Let a solution x of* (1) *have two succesive nodes at n and m, $m \geq n+1$, i.e.,*

$$x_n x_{n+1} < 0,\ x_{n+1}x_{n+2} > 0, \ldots, x_{m-1}x_m > 0,\ x_m x_{m+1} < 0.$$

Then $x_{n+1}x_n^{[2]} < 0$. In addition, if $x_n x_{n+1} < 0$ and $x_{n+1}x_{n+2} < 0$, then $x_{n+2}x_{n+3} > 0$.

Proof. Without loss of generality we can assume $x_n < 0$, $x_{n+1} > 0$,. . .,$x_m > 0$ and $x_{m+1} < 0$.

Let $m = n+1$. Then $\Delta x_n > 0$, $\Delta x_{n+1} < 0$ and so $x_n^{[1]} > 0$, $x_{n+1}^{[1]} < 0$. Hence $\Delta x_n^{[1]} < 0$, *i.e.*, $x_n^{[2]} < 0$. Therefore $x_{n+1}x_n^{[2]} < 0$. Since $x_{n+2} < 0$, we have from (1) that $\Delta x_n^{[2]} < 0$, *i.e.*, $x_{n+1}^{[2]} < x_n^{[2]} < 0$. If $x_{n+3} \geq 0$, then $x_{n+2}^{[1]} > 0$ and $x_{n+1}^{[2]} > 0$ which would be a contradiction. Therefore, $x_{n+2}x_{n+3} > 0$.

Let $m > n+1$. Since $x_k > 0$ for $k = n+1, \ldots, m$, we get from (1) $\Delta x_k^{[2]} > 0$ for $k = n-1, \ldots, m-2$ and because of $x_{m+1} < 0$, we have $\Delta x_m < 0$, *i.e.*, $x_m^{[1]} < 0$. If $x_n^{[2]} \geq 0$, then $x_k^{[2]} \geq 0$, *i.e.*, $\Delta x_k^{[1]} \geq 0$ for $k = n, \ldots, m-1$ and because of $x_n^{[1]} > 0$ we would have $x_m^{[1]} > 0$, which would be a contradiction. Hence $x_n^{[2]} < 0$ and so $x_{n+1}x_n^{[2]} < 0$.

The opposite sign case can be treated in the same manner. □

4 Oscillatory solutions

Our main result is as follows.

Theorem 3 *Every oscillatory solution x of* (1) *has the following properties:*

(i) *x is a G_+ solution;*

(ii) *x has no double zero and there exists no $a \in \mathbb{N}$ such that $x_a = x_{a+1} = x_{a+2}$;*

(iii) *x satisfies* (7) – (9).

Proof. Claim i). Assume by contradiction that there exists $a \in \mathbb{N}$ such that $G_a < 0$, which implies according to Lemma 2 that $G_n < 0$ for all $n \geq a$. If there exists a zero m, $m > a$, then from (5) it follows that $G_{m-1} \geq 0$, which would be a contradiction. Therefore any generalized zero of x is a node for $n > a$. Consider two succesive nodes at n and m, such that $a < n < m$. According to Lemma 3 we get $x_{n+1}x_n^{[2]} < 0$, which implies by (5) that $G_n \geq 0$, a contradiction.

Claim ii). By claim i) any oscillatory solution x of (1) is a G_+ solution and one can verify that any nontrivial solution with a double zero or any nontrivial solution for which $x_a = x_{a+1} = x_{a+2}$ is a G_- solution. So the conclusion holds.

Claim iii). The statement follows from claim i) and Theorem 1. □

From Theorem 3 we have the similar property which holds for the corresponding differential equation.

Corollary 1 *Every solution with a double zero is nonoscillatory.*

Remark 3 From Lemma 3 it follows that no solution of (1) has three nodes at succesive indices, say at n, $n+1$, $n+2$. In particular, the sequence $\{x_n\}$, where $x_n = (-1)^n$, is never a solution of (1). Similarly, this sequence is never a solution of equation (3), see Remark 2 in [3].

This is not true for other types of third order difference equations (4) as the following example shows.

Example 2 The sequence $\{x_n\}$, where $x_n = (-1)^n$, is a solution of the following third order equations

$$\Delta^3 x_n + 8x_n = 0,$$
$$\Delta^3 x_n - 8x_{n+1} = 0,$$
$$\Delta^3 x_n + 8x_{n+2} = 0.$$

Similarly, equations with quasidifferences

$$\begin{aligned} \Delta\left\{\left(\frac{3}{4}\right)^n \Delta\left[\left(\frac{2}{3}\right)^n \Delta x_n\right]\right\} + \frac{5}{2^{n+1}}\, x_n &= 0, \\ \Delta\left\{\left(\frac{3}{4}\right)^n \Delta\left[\left(\frac{2}{3}\right)^n \Delta x_n\right]\right\} - \frac{5}{2^{n}}\, x_{n+1} &= 0, \\ \Delta\left\{\left(\frac{3}{4}\right)^n \Delta\left[\left(\frac{2}{3}\right)^n \Delta x_n\right]\right\} + \frac{10}{2^{n}}\, x_{n+2} &= 0 \end{aligned}$$

have the oscillatory solution $x_n = (-\frac{1}{2})^n$.

Example 3 Consider the linear equation

$$\Delta^3 x_n - q_n x_{n+2} = 0 \tag{10}$$

where

$$\liminf_{n\to\infty} q_n > 0. \tag{11}$$

According to Theorem 3–iii) every oscillatory solution satisfies (8). From here and (11) it follows that oscillatory solutions of (10) satisfy

$$\sum_{n=1}^{\infty} x_n^2 < \infty,$$

i.e., oscillatory solutions belong to the space ℓ_2.

5 Nonoscillatory solutions

Nonoscillatory solutions have been investigated in [4] where all nonoscillatory solutions x of (1) have been classified to the following classes:

$$\begin{aligned} N_0 &= \{x : \exists n_x \text{ such that } x_n x_n^{[1]} < 0,\ x_n x_n^{[2]} > 0 \text{ for all } n \ge n_x\}, \\ N_1 &= \{x : \exists n_x \text{ such that } x_n x_n^{[1]} > 0,\ x_n x_n^{[2]} < 0 \text{ for all } n \ge n_x\}, \\ N_2 &= \{x : \exists n_x \text{ such that } x_n x_n^{[1]} > 0,\ x_n x_n^{[2]} > 0 \text{ for all } n \ge n_x\}, \\ N_3 &= \{x : \exists n_x \text{ such that } x_n x_n^{[1]} < 0,\ x_n x_n^{[2]} < 0 \text{ for all } n \ge n_x\}. \end{aligned}$$

Solutions from the class N_2 are sometimes called *strongly monotone solutions* and they always exist, see [3, Theorem 3.2].

By Theorem 3 and the definition of G we have the next result.

Corollary 2 *Any $x \in N_1 \cup N_3$ and any oscillatory solution of* (1) *is a G_+ solution. If x is a G_- solution, then $x \in N_0 \cup N_2$.*

Theorem 4 *Let x be a strongly monotone solution of* (1). *Assume* (11),

$$\liminf_{|u|\to\infty} |f(u)| > 0, \tag{12}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{r_{k+2}} \sum_{j=1}^{k} \frac{1}{p_{j+1}} \sum_{i=1}^{j} q_i = \infty. \tag{13}$$

Then x is a G_- solution satisfying $\lim_{n\to\infty} G_n = -\infty$.

Proof. By [3, Theorem 3.3] we have $\lim_{n\to\infty} |x_n| = \infty$. In view of (2), (6), (11) and (12) we obtain $\lim_{n\to\infty} \Delta G_n = -\infty$. From here it follows that $\lim_{n\to\infty} G_n = -\infty$. □

Corollary 3 *Assume* (11), (12) *and*

$$\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty, \quad \sum_{j=1}^{\infty} \frac{1}{r_j} = \infty. \tag{14}$$

Then G_+ solutions coincide with oscillatory solutions and G_- solutions coincide with strongly monotone solutions.

Proof. By Theorems 2.3 and 2.4 in [3], the classes N_0, N_1, N_3 are empty. If $x \in N_2$, then by Theorem 4 we have that x is a G_- solution. Now the conclusion follows from Corollary 2. □

Corollary 4 *Assume* (11), (12) *and* (14). *Then, any nonoscillatory solution x satisfies*

$$\lim_{n\to\infty} G_n = -\infty, \quad \lim_{n\to\infty} |x_n^{[i]}| = \infty \quad \textit{for} \quad i = 0, 1, 2. \tag{15}$$

In addition, if (p_n), (r_n) are bounded, then any oscillatory solution x satisfies

$$\lim_{n\to\infty} G_n = 0, \quad \lim_{n\to\infty} x_n^{[i]} = 0 \quad \textit{for} \quad i = 0, 1, 2. \tag{16}$$

Proof. Let x be a nonoscillatory solution. By Corollary 3 x is a strongly monotone solution. Now the conclusion follows from Theorem 4 and [3, Theorem 3.3].

Let x be an oscillatory solution. By Theorem 3-iii) it satisfies (8). From here and (11) it follows that

$$\lim_{n\to\infty} x_{n+2} f(x_{n+2}) = 0.$$

This, together with the continuity and the sign condition of f implies $\lim_{n\to\infty} x_n = 0$. Since (p_n) and (r_n) are bounded, it holds $\lim_{n\to\infty} x_n^{[i]} = 0$ for $i = 1, 2$ and by Lemma 2 we have $\lim_{n\to\infty} G_n = 0$. □

Example 4 Consider the equation

$$\Delta(\frac{1}{n+1}(\Delta\frac{1}{n}\Delta x_n))) - n\,|x_{n+2}|^{\lambda}\,\mathrm{sgn}\, x_{n+2} = 0,$$

where $\lambda > 0$. Applying Corollaries 3 and 4, we get that nonoscillatory solutions coincide with G_- solutions and satisfy (15), while oscillatory solutions coincide with G_+ solutions and satisfy (16).

Acknowledgment. This research was supported by the grant 201/04/0580 of the Czech Grant Agency.

References

[1] Agarwal R. P., *Difference Equations and Inequalities*, 2nd Edition, Pure Appl. Math., **228**, Marcel Dekker, New York, 2000.

[2] Bartušek M. and J. Osička J, *Asymptotic behaviour of solutions of a third-order nonlinear differential equation*, Nonlinear Analysis. T., M. & Appl., **34** (1998), 653–664.

[3] Došlá Z. and Kobza A., *Global asymptotic properties of third order difference equations,* Comput. Math. Appl., **48** (2004), 191–200.

[4] Došlá Z. and Kobza A., *On nonoscillatory solutions of third order difference equations*, in: Proceedings of the 8th International Conference of Difference Eqs. Appl. held in Brno 2003. (to appear)

[5] Kobza A., *Property A for third order difference equations*, Studies of the University of Žilina, Mathematical Series, **17** (2003), 109–114.

[6] Migda M., Schmeidel E. and Drozdowicz A., *Nonoscillation results for some third order nonlinear difference equations,* Folia FSN Universitatis Masarykianae Brunensis, Mathematica, **13**, Colloquium on Differential and Difference Equations, Brno, (2002), pp. 185–192.

[7] Popenda J. and Schmeidel E., *Nonoscillatory solutions of third order difference equations,* Portugaliae Mathematica, **49** (1992), 233–239.

[8] Smith B., *Oscillatory and asymptotic behavior in certain third order difference equations,* Rocky Mountain J. Math., **17** (1987), 597–606.

[9] Smith B., *Oscillation and nonoscillation theorems for third order quasi-adjoint difference equations,* Portugaliae Mathematica, **45** (1988), 229–243.