

Bifurcations and Attractors in Bogdanov Map

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Abstract

In this paper, we study the bifurcation space and the phase plane of the Bogdanov map. Specific bifurcation structures can be observed in the parameter space, related for instance to embedded boxes structure and configurations of bifurcation curves of periodic points near the cusp. This model is a diffeomorphism. The dynamics is extremely rich, involving periodicity, quasiperiodicity and chaos. The method of the study is a numerical iteration to an attractor in which the guesses are inspired by the theory.

Bifurcation diagrams obtained in different parameter planes are given and a sketch showing the cusp bifurcations, for the versal unfolding of Bogdanov map, is given. The phase plane is also studied, different attractors are shown, their evolution giving rise to chaotic attractors is explained. Basins of attraction are considered and fuzzy boundaries of basins are put in evidence. The study of such kind of diffeomorphisms can give an interesting contribution to nonlinear systems.

Keywords: Bifurcation, Basin, Bogdanov map, Invertible, Chaos.

1 Introduction

We study a diffeomorphism. The dynamics involves various transitions by bifurcations. A planar quadratic map is considered which is conjugate to the Hénon map in its conservative limit (see [10]). Perturbation theory here reveals periodicity and quasiperiodicity, parametrically organized by embedded boxes structure. Outside these parameter regimes the perturbation becomes larger and bifurcations complicate the dynamics, often involving chaos.

On the one hand, this concerns local bifurcations, *e.g.*, cascades of flips and cusp bifurcations. On the other hand, we have to deal with homoclinic bifurcations. It also undergoes a Hopf bifurcation which creates a stable closed curve growing out, at its destruction appears a saddle connection. Here the dynamical features are explored by numerical methods. Also, we often iterate to a periodic attractor. In several cases we find interesting dynamical objects predicted by the theory and global phenomena in the parameter plane. This kind of scanning has been made for giving a first idea about bifurcation organization. That is, the dynamic behaviors

may be, as expected, regular (equilibrium, stable cycles and stable invariant curves) or chaotic. The object of the present study is to investigate the main routes to complex behaviors in attracting sets and their basins of attraction associated with the dynamics of invertible nonlinear systems which are still of great interest. The Bogdanov map provides a good approximation to the dynamics of the Poincaré map of periodically forced oscillators, first considered by Bogdanov (see [4]), Takens and Arnold [1] in their study of the double zero eigenvalue singularity. They proved the existence of a codimension-two fixed point at the origin, called a Bogdanov-Takens cusp with a nonzero Jordan canonical form but the best contribution comes from Arrowsmith *et al.* [2, 3].

We emphasize that our aim is to study simply the map without emulating the Bogdanov vector field. This diffeomorphism can be considered as a model giving rise to a complete set of bifurcations with a fractal structure, and with birth of cyclic invariant curves (impossible to obtain with Hénon map). From this point of view it gives the fundamental mechanisms occurring in the nonautonomous differential equation with quasi periodic solutions, or more complex solutions.

The diffeomorphism considered is the following one, called Bogdanov map:

$$T : \begin{cases} x_{n+1} = x_n + y_{n+1} , \\ y_{n+1} = y_n + \varepsilon y_n + h x_n (x_n - 1) + \mu x_n \cdot y_n , \end{cases} \quad (1)$$

where x, y are real variables, ε, μ are real parameters. The parameter h is the step length in the discretization, it takes small values. T has a nonconstant Jacobian determinant $J = 1 + \varepsilon + \mu x$, then it results from this situation that a cycle can undergo the Hopf bifurcation and for $\varepsilon = \mu = 0$, we observe the Hamiltonian nature and we have a globally area preserving map. Eq. (1) has only two fixed points $(0, 0)$ and $(1, 0)$. Both these fixed points can undergo identical sets of bifurcations in the parameter space, this is due to the symmetry in the map (see Fig. 1(a, b, c)) under the transformation $(\bar{x}, \bar{y}) = (1 - x, -y)$ which switches the fixed points.

We study the bifurcation diagram which shows bifurcation curves in the parameter (ε, h) , (ε, μ) planes together with representative phase portraits. The Figures 1(a, b, c) give the parameter value for which at least one fixed point is attractive (blue domain corresponding to the value 1). More generally, the Figures 1(a, b, c) give the regions of parameter (ε, h) or (ε, μ) plane for which at least one periodic orbit of order k exists ($k = 1, 2, \dots, 14$). Each coloured part given by the upper coloured squares corresponds to the existence of a periodic orbit of order (period) k . The black regions ($k = 15$) correspond to the existence of bounded iterated sequences.

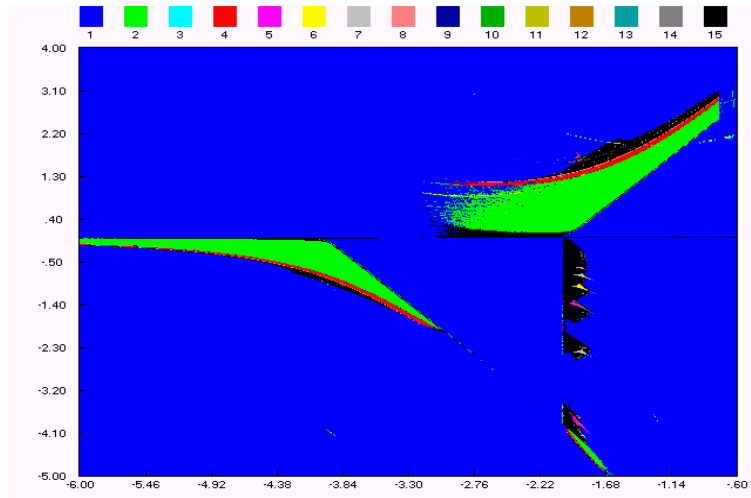


Figure (1.a) : bifurcation structure in (ε, h) -plane, $\mu = 2$.

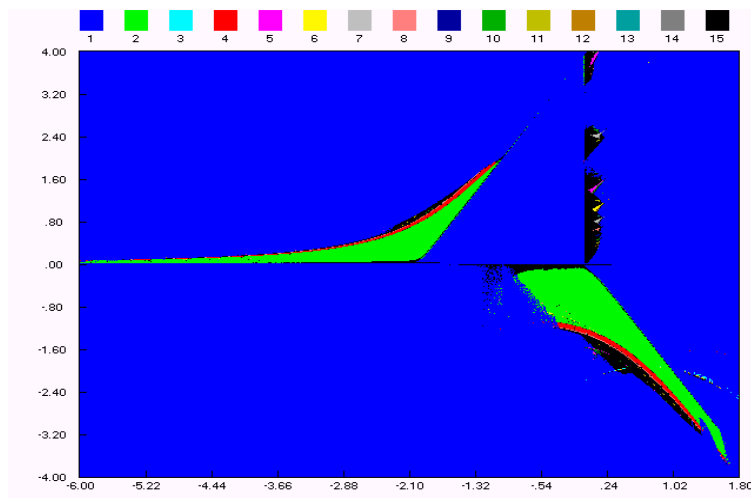


Figure (1.b) : bifurcation structure in (ε, h) -plane, $\mu = -2$.

These figures are typical of maps with dominating quadratic nonlinear terms, analogous bifurcation diagrams have already been obtained for systems with nonlinearity given by hyperbolic tangency [14, 16, 17]. We can recognize on the diagram period doubling bifurcation, saddle area and spring area [14]. The bifurcation structure is a quadratic “box-within-a-box” type (see [13, 7]), as is well known infinitely many periodic windows are opened by fold bifurcations and are closed by homoclinic bifurcations by the intriguing “box-within-a-box” bifurcation structure. There is a fold bifurcation on the line $h = 0$, a flip bifurcation on the line $\varepsilon = h/2 - 2$, and a Hopf bifurcation on the line $\varepsilon = 0$ for $0 < h < 4$. The eigenvalues of the fixed point at the origin depend on the parameters (ε, h) , in the interior of the curve $\varepsilon^2 + h^2 - 2\varepsilon h - 4h = 0$ we have complex eigenvalues.

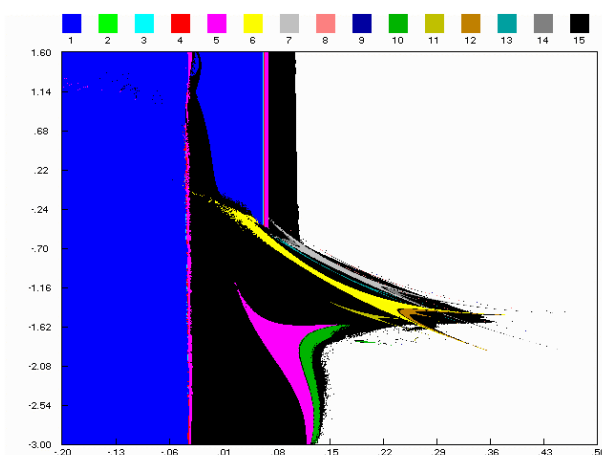


Figure (1.c) : bifurcation structure in (ε, μ) -plane, $h = 1.44$.

2 Definitions and some properties

This section essentially concerns a two-dimensional map T , considered here as a discrete dynamical system in \mathbb{R}^2 :

$$(x_{n+1}, y_{n+1}) = T(x_n, y_n) = (f(x_n, y_n; \lambda), g(x_n, y_n; \lambda)),$$

where $f(x, y, \lambda)$ and $g(x, y, \lambda)$ are continuous and differentiable functions with respect to the real variables x, y , and continuous with respect to the real parameter λ .

It is recalled that a closed and invariant set A is called an attracting set if some neighborhood U of A may exist such that $T(U) \subset U$, and $T^n(x, y) \rightarrow A$ as $n \rightarrow \infty$, $\forall (x, y) \in U$. An attracting set A may contain one or several attractors coexisting with sets of repulsive points (strange repulsors) to either chaotic transients towards these attractors, or fuzzy boundaries of their basin [15]. The set

$D = \cup_{n \geq 0} T^{-n}(U)$ is called the total basin (or influence domain) of A . That is, D is the open set of points x whose forward trajectories (set of images of x) converge towards A . D is invariant under backward iteration T^{-1} of T , but not necessarily invariant by T :

$$T^{-1}(D) = D, \quad T(D) \subseteq D.$$

The relations above hold also for the closure \bar{D} of D . The boundary of D is denoted by ∂D , is defined by the geometrical equality $\partial D = \bar{D} \cap \overline{C(D)}$, where $C(D)$ denotes the complementary set of D . This boundary satisfies:

$$T^{-1}(\partial D) = \partial D, \quad T(\partial D) \subseteq \partial D.$$

The basin D may be connected or nonconnected. A connected basin may be simply connected or multiply connected (which means connected with holes).

An attracting set is called *periodic of period k* if it is made up of k disjoint sets, $A = \cup_{i=1}^k A_i$, where each A_i is an attracting set of the map T^k .

When A is an attracting set of period $k = 1$, then its total basin is given by $D = D_0$ if it is connected, and by $D = \cup_{n \geq 0} T^{-n}(D_0)$ if it is nonconnected. When A is an attracting set of period $k > 1$, the immediate basin D_0 of A is the open set $D_0 = \cup_{i=1}^n D_{0,i}$, $D_{0,i}$ being open disjoint basins of A_i . If A is a connected attractor, the immediate basin D_0 of A is defined as the widest connected component of D containing A . When A is the widest attracting set of a map T , its basin D is the total basin of bounded iterates. That is, the open set D contains A such that \bar{D} is the locus of points of the plane having bounded trajectories.

Definition 2.1 A chaotic area A is an invariant absorbing area ($T(A) = A$), the points of which give rise to iterated sequences having the property of sensitivity to initial conditions.

About chaotic areas, it is important to emphasize that the study of such area has only the purpose to obtain properties giving rise to fractal basin boundaries.

Definition 2.2 We say that $\lambda = \lambda^*$ is a bifurcation of contact of A , if a contact between the boundary of A and the boundary of its basin of attraction takes place.

Proposition 2.1 *When a bifurcation of contact of a chaotic area A arises for a value $\lambda = \lambda^*$, the crossing of this value leads to the destruction of A , either to a qualitative modification of the properties of A (i.e., a sudden and important modification of the size of such an area or its basin of attraction).*

The destruction of A , after the crossing of the bifurcation value has been demonstrated by Gumowski & Mira [9]. The qualitative change of properties of A has been described by Mira et al. [15].

Definition 2.3 Let S be a saddle fixed point of T ; a point q is called homocline to S if $q \in W^s(S) \cap W^u(S)$ and $q \neq S$. q is a transversal homocline point if $W^s(S)$ intersects transversally $W^u(S)$.

We say the basin boundary is fractal if it contains a transversal homoclinic point.

Definition 2.4 One calls homoclinic orbit $O_o(q)$ associated with q , q belonging to a neighborhood $U(S)$ of S , a set constituted of successive iterates of q , and its infinite sequence of preimages obtained by application of the local inverse map T^{-1} of T in $U(S)$.

Definition 2.5 Let T be a diffeomorphism of \mathbb{R}^2 depending on a parameter λ and let S be a saddle point of T . A homoclinic bifurcation takes place if for a value $\lambda = \lambda^*$, there is an apparition (or disappearance) of an infinity of homoclinic orbits.

Overlapping of global stable and unstable manifolds usually leads to nonlocal bifurcations. The existence of a transversal point of a planar map leads to very complicated behavior of orbits nearby. Such dynamical complexity is often dubbed as chaos, the onset of chaos typically occurs at the parameters values for which the stable and unstable manifolds of a saddle point come into contact tangentially.

3 Basins and attractors

In this section we give a sketch on bifurcations which give rise to fractalization of the attractor and its basin boundary. The main purpose of this section is to give some results on basin structure of invertible maps and their bifurcations. Especially, we explain basin bifurcations. These basic bifurcations result from *the contact of a basin boundary with a chaotic attractor*, such a bifurcation leads either to a destruction of the chaotic area, or a sudden and important modification of this area [9, 12]. These bifurcations are called *contact bifurcations*.

Different types of fractalization of basin boundary occur, a part of them resulting from a phenomenon of islands aggregation for nonconnected basins [14]. It is worth noting that the results presented in this section were essentially obtained by a numerical method and using the bifurcation theory tool and the terminology of [14]. Unfortunately, taking into account the complexity of the matter and its nature, it seems difficult to carry out the study with success by another method.

The following figures show the corresponding basin structure. For fixed parameter values, it is possible to define in the phase plane the basin of an attractor, this means the set of initial conditions leading to iterated sequences converging towards the considered attractor. Figures 2 ($a(1, 2)$, $b(1, 2)$, $c(1, 2)$, $d(1, 2)$) represent the existing attractors (order k cycles, invariant curves or chaotic attractors) and their basins.

$$\varepsilon = 0.15, \mu := -1.70, h := 0.30$$

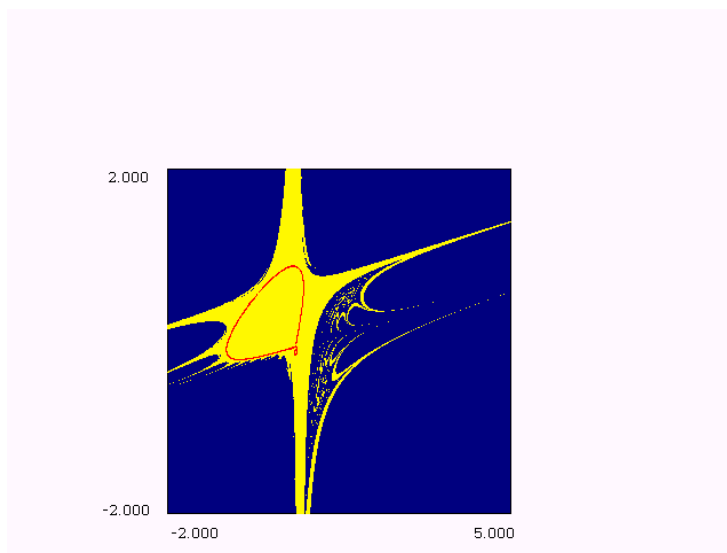


Figure 2 (a.1) : A closed curve and its basin.

$$\varepsilon = 0.15, \mu := -1.70, h := 0.30$$

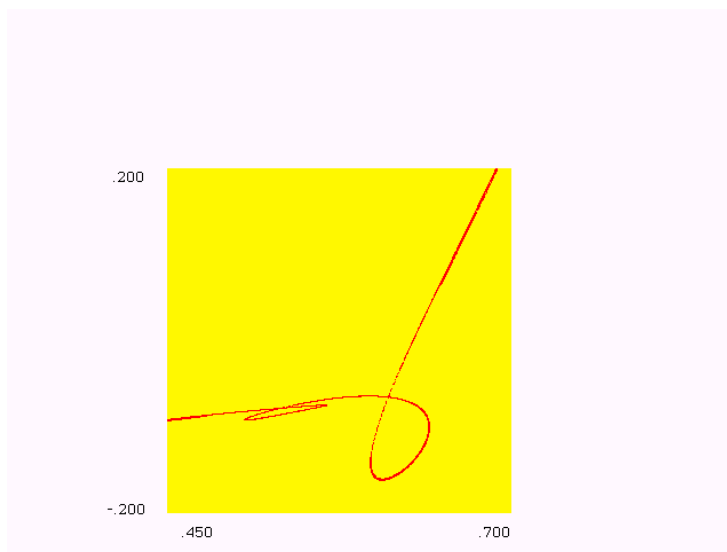


Figure 2 (a.2) : Enlargement of a part of Fig. 2 (a.1).

$$\varepsilon = 0.15, \mu := -1.70, h := 0.285$$

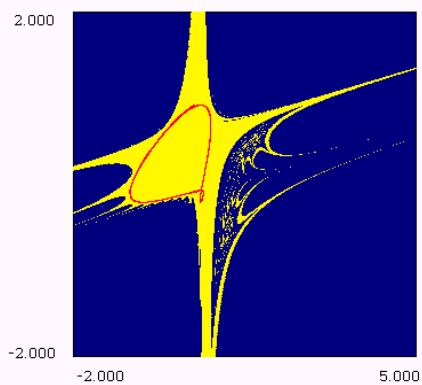


Figure 2 (b.1) : Chaotic attractor, tongues appear on basin boundary partially fractal.

$$\varepsilon = 0.15, \mu := -1.70, h := 0.285$$

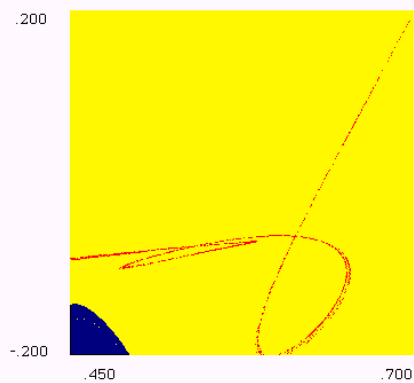


Figure 2 (b.2) : Enlargement of a part of Fig 2 (b.1).

$$\varepsilon = 0.15, \mu := -1.70, h := 0.275$$

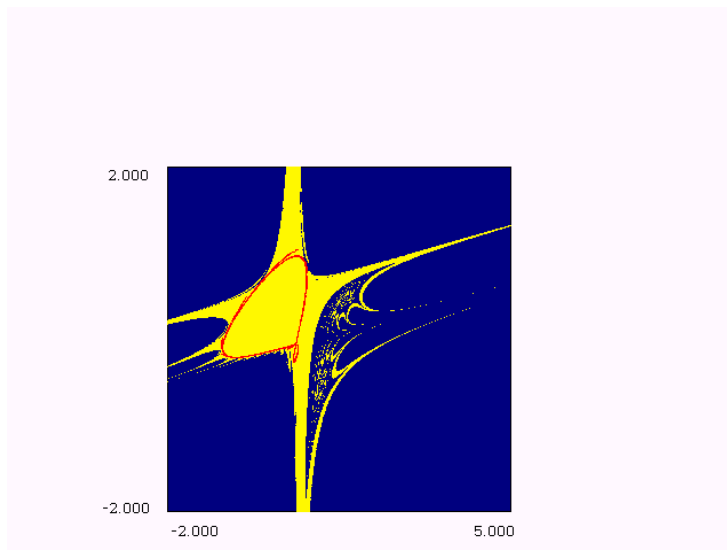


Figure 2 (c.1) : Contact bifurcation of the chaotic attractor with its basin boundary.

$$\varepsilon = 0.15, \mu := -1.70, h := 0.275$$

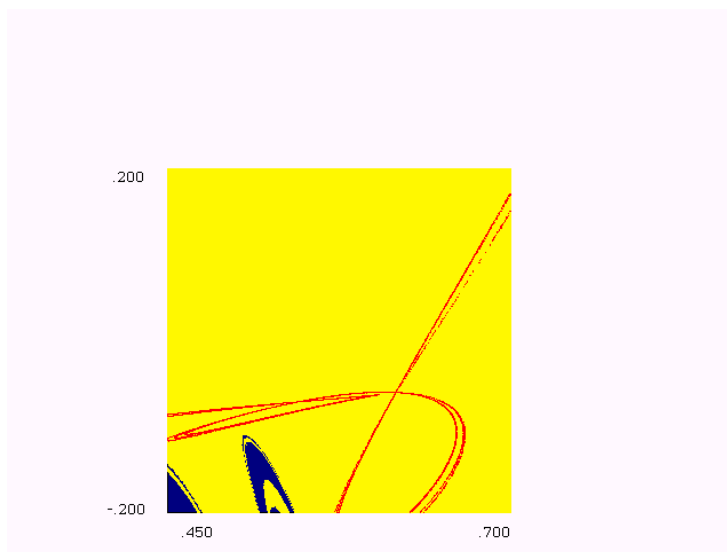


Figure 2 (c.2) : Enlargement of a part of Fig 2 (c.1).

$$\varepsilon = 0.15, \mu := -1.70, h := 0.273$$

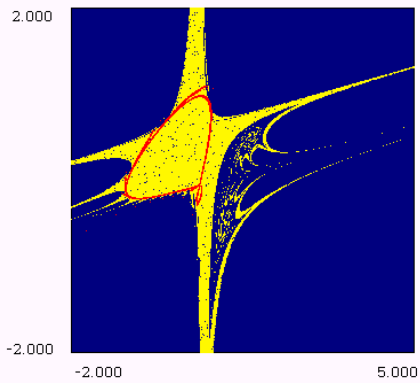


Figure 2 (d.1) : More and more loops are created on the boundary of the chaotic attractor, granulated basin.

$$\varepsilon = 0.15, \mu := -1.70, h := 0.27$$

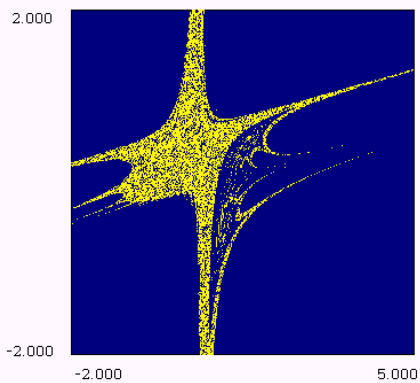


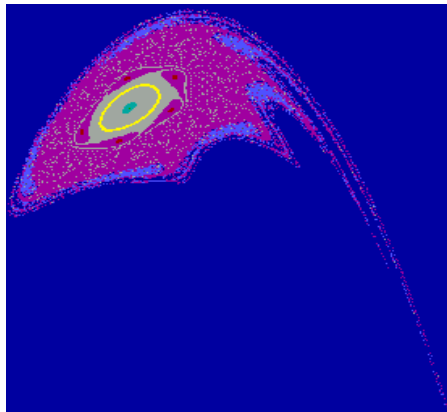
Figure 2 (d.2) : Situation just after the disappearance of the chaotic attractor and before the disappearance of the basin.

The evolution of attractors and their basins is given directly in the figures, the parameters (ε, μ) have been chosen constant and h varies. There exist several ways in which a dynamical system can become chaotic, of which the period-doubling route to chaos is the best known. Another scenario leading to chaos is the destruction of a closed invariant curve, identified here in this model, the closed curve can lose its smoothness and turn into an irregular invariant set near a homoclinic structure formed by the intersection of the stable and unstable manifolds of the saddle point. When h decreases, a process gives rise to new appearance of tongues on the basin boundary, and there is appearance of oscillations of the closed invariant curve and then we see a cyclic chaotic behavior. We can also see that the bifurcation which is put in evidence, corresponds to a limit contact between the basin boundary ∂D and the chaotic attractor. A further decrease of the parameter h causes a contact between these two boundaries which marks the destruction of the basin of attraction. As argued in [9], we can see the ghost of this area destroyed at the contact.

4 Multistability and Hopf bifurcation

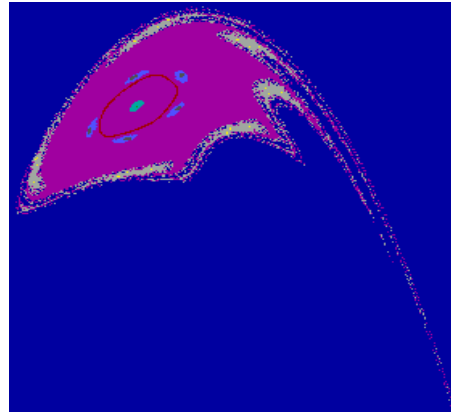
This diffeomorphism can be considered as a model giving rise to a complete and interesting set of bifurcations with birth of cyclic invariant curves (impossible to obtain with Hénon map) in relation with quasi periodic solutions, or more complex solutions occurring in the nonautonomous differential equation. Invariant closed curves are issued from Hopf bifurcations (see Fig. 3 (a, b, c, d, e, f, g, h, i, j, k, l)), these complicated attracting sets exist for very small parameter ranges. Arrow-smith [2] has explained the relationship between the tongues in the Hamiltonian structure and the invariant closed curves (circles) and clarified various points. Eventual destruction of the Hopf circles occurs as they grow outwards and their basin boundaries jump inwards in a series of steps, accumulating on the stable manifolds of subharmonic saddles.

$\varepsilon = 0.0005, \mu := -0.1000, h := 1.4400$

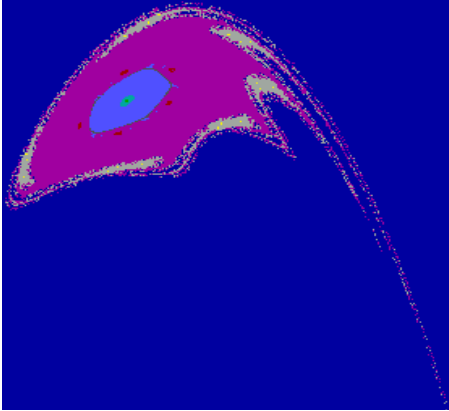


(a)

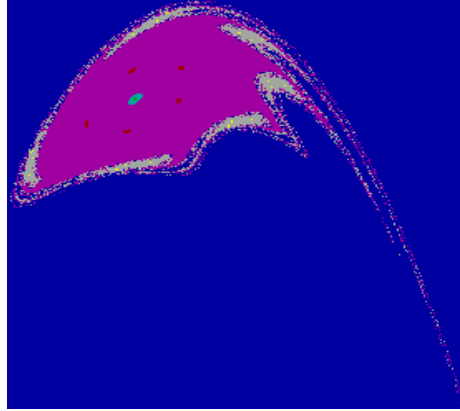
$\varepsilon = 0.001, \mu := -0.1000, h := 1.4400$



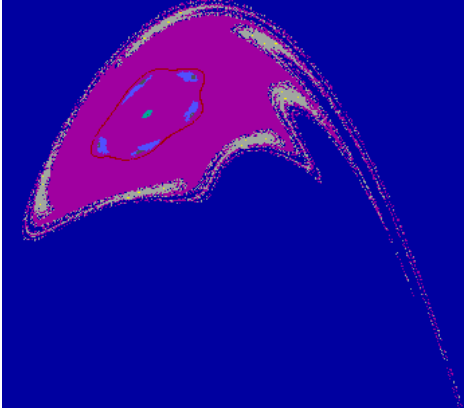
(b)

$\varepsilon = 0.0011, \mu := -0.1000, h := 1.4400$ 

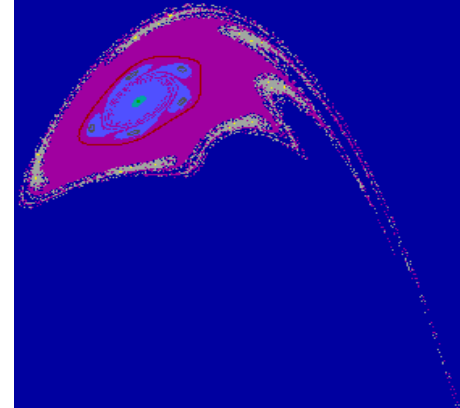
(c)

 $\varepsilon = 0.0012, \mu := -0.1000, h := 1.4400$ 

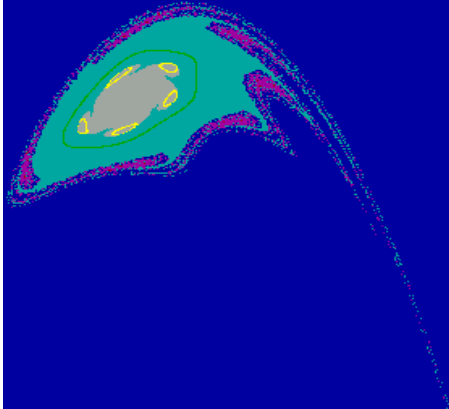
(d)

 $\varepsilon = 0.0019, \mu := -0.1000, h := 1.4400$ 

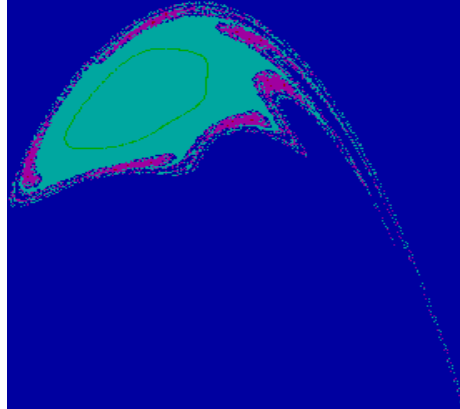
(e)

 $\varepsilon = 0.0025, \mu := -0.1000, h := 1.4400$ 

(f)

 $\varepsilon = 0.0029, \mu := -0.1000, h := 1.4400$ 

(g)

 $\varepsilon = 0.0032, \mu := -0.1000, h := 1.4400$ 

(h)

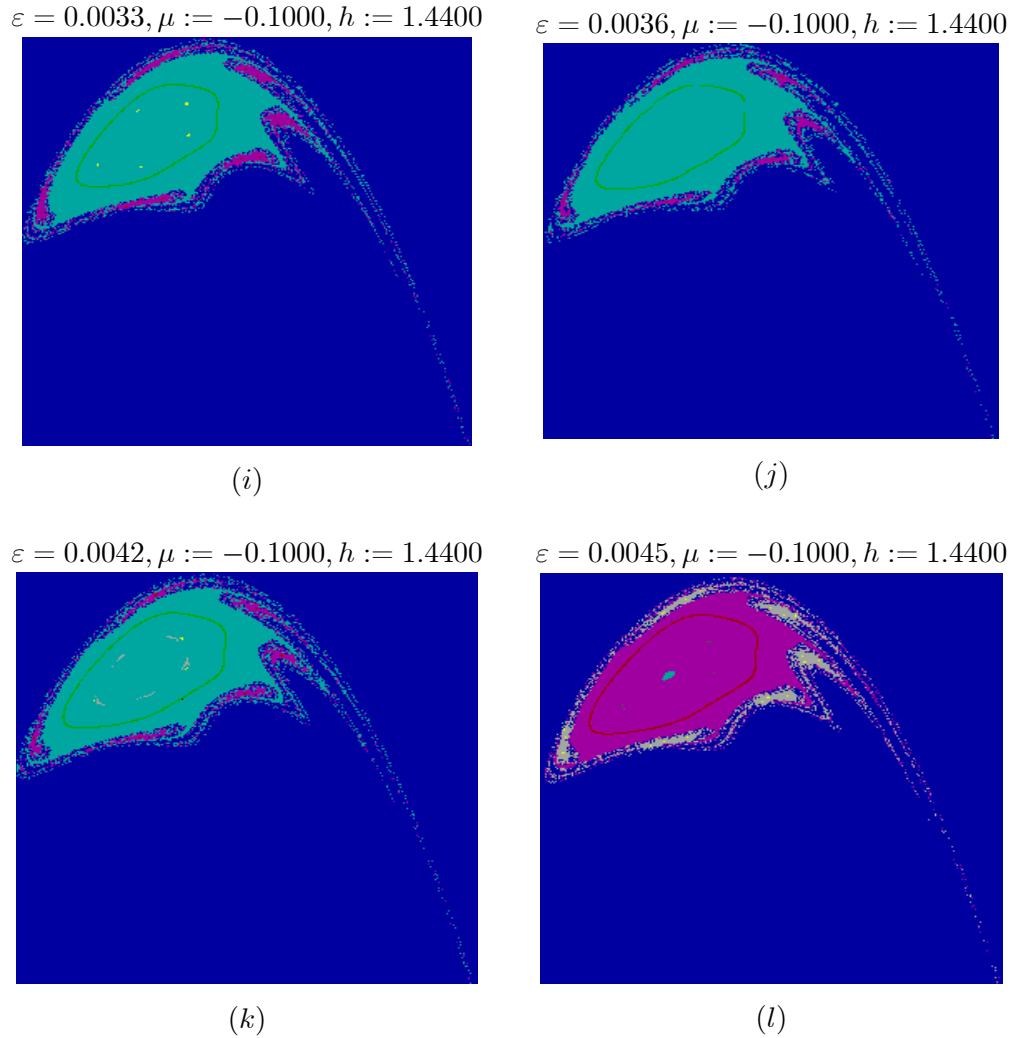


Figure 3: A sequence of phase portraits, we can see the basin of period 5, that of period 6 and invariant circles formed at different Hopf bifurcations.

5 Properties of $W^s(S)$ and $W^u(S)$

This section is devoted to the study of the evolution of attractors when parameters vary. This evolution can involve relations with the basin of the attractor and homoclinic and heteroclinic bifurcations.

In the case of a diffeomorphism, the stable invariant manifold $W^s(S)$ of a saddle point S constitutes in general the boundary of the basin of attraction of an associated attractor A . When $W^s(S)$ becomes nonconnected, each of its connected components again approaches itself to constitute the boundary of an island. It transforms the connected basin of attraction into a nonconnected basin. Holes appear inside the basin, which becomes multiply connected. All these phenomena

generate very big sensitivity to initial conditions, due principally to multistability, interconnection between basins and fractalization of basins.

5.1 Some properties of invariant curves

For fixed parameter values, we plot the attraction basin of an attractor. When there exist several attracting sets, it is possible to define a global basin, that means the set of initial conditions giving rise to bounded iterated sequences, independently of the fact that they converge to one attractor or another. We shall see that particular homoclinic bifurcations which occur are related with the structure of the foliation of the invertible map.

Other phenomena can also occur and generate sensitivity to initial conditions. This is the case where there exist two or more different attractors and where their own basins are considered. In this case, one basin constitutes holes in the other. This situation is shown in Fig. 5, where three different cycles coexist, two basins give rise to holes inside the third one.

Proposition 5.1 *Consider the invertible map T depending on a parameter λ . If the number of connected components of D changes when λ crosses through a bifurcation value λ^* , then the basin D may undergo a qualitative change of one of the following types:*

- (a) *simply connected basin \leftrightarrow multiply connected basin (when the number of connected components of D changes).*
- (b) *modification of the number of holes (lakes) in D , or new arborescent sequence of holes.*

The numeric studies give to the following results, for values: $\varepsilon = 0.0$, $\mu := -0.10$, $h := 1.44$.

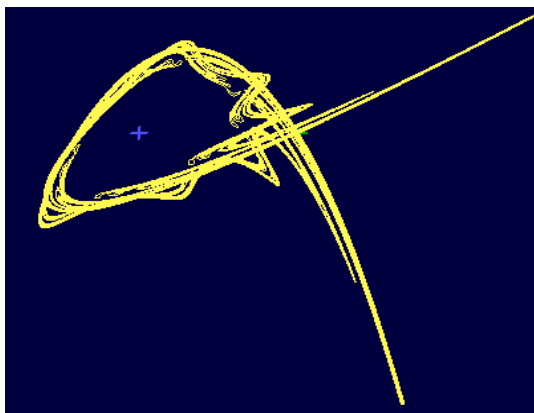


Figure 4: Plot of invariant manifolds of the saddle point $(1, 0)$.

Numerical plot of stable and unstable manifolds of the saddle point at $(1, 0)$ shows that the orbit sweeps around $(0, 0)$ twice, we can see homoclinic tangle.

There are four attractors, the first is a stable focus $f(0,0)$ of which the basin of attraction has for external boundary the stable invariant manifold of the saddle point $S(1,0)$. The second is the stable focus cycle of *order* 5 (represented in five small pink domains in the Figure 5).

These attractors coexist with a cycle of *order* 6 represented by grey domains in the Figure 5, and an invariant stable circle (in yellow color).

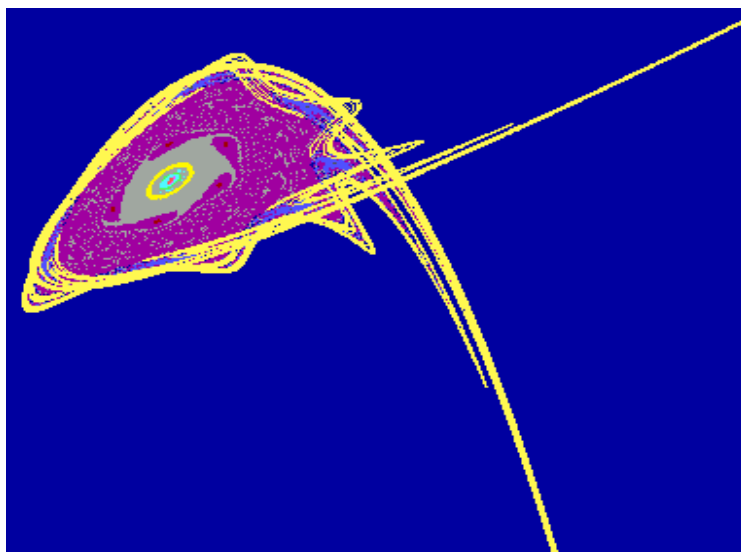


Figure 5 : f coexists with an attractive focus of order 5, its basin is multiply connected.

6 Conclusion

In this paper, specific bifurcations arising in invertible maps of quadratic type have been studied. These bifurcations concern the evolution of chaotic attractors, which appear or disappear by contact bifurcations, with their own basin boundary. In such maps, chaotic attractors can be issued of Hopf bifurcation giving rise to invariant closed curves which evolve into chaotic attractor or of a succession of period-doubling bifurcations [8, 6, 5]. As usual in many nonlinear systems, important changes can occur with small changes of parameter values.

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