5–10 July 2004, Antalya, Turkey — Dynamical Systems and Applications,
Proceedings, pp. 310–322

Asymptotic Estimates for the Eigenvalues of Some Nonlinear Elliptic Problems

Ali Djellit & Nawel Benouhiba
Université de Annaba, Institut de Mathématiques
BP 12, 23000 Annaba, ALGÉRIE. Fax: 213 08 87 25 26

Abstract

This work is devoted to the study of the solutions of some semilinear elliptic problems defined in $\mathbb{R}^n$, our concern being largely the distribution of eigenvalues. We show that the number $N(\lambda)$ of eigenvalues less than λ satisfies Weyl-Courant or Wet-Mendel formulas.

AMS Subject Classification: 35J15, 35J20, 35J60.
Key words: Weighted Sobolev spaces, Counting function, Distribution of eigenvalues, Min-max principle, Semilinear problem.

1 Introduction

We consider the following eigenvalue problem

$$\begin{cases} -\Delta u + q(x)u + f(x,u) = \lambda g(x)u, & x \in \mathbb{R}^n, \\ \underset{|x|\to+\infty}{u \longrightarrow 0}, \end{cases} \tag{1.1}$$

where Δ is the Laplacian operator, λ is a real parameter, g, q are measurable functions and $f : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function.

Many authors [2, 11, 12] treated this kind of problems in bounded domains. In this work, we use the Ljusternik-Schnirelmann theory in order to establish the existence of a sequence of pairs of solutions on the differentiable manifold $M_\alpha = \left\{ u : \int_{\mathbb{R}^n} gu^2\, dx = \alpha \right\}$, $\alpha \in \mathbb{R}^*$, afterwards we determine the asymptotic behaviour of the eigenvalues and eigenfunctions of Problem (1.1). We show that the distribution of eigenvalues is given, as in the linear case, by the Weyl-Courant or Wet-Mendel formulas. To illustrate this result, we deal with the relationship between the eigenvalues of Problem (1.1) and the eigenvalues of the associated linear problem

$$\begin{cases} \text{Find } \ u \in V, \ u \not\equiv 0; \ \lambda \in \mathbb{R}, \\ \quad -\Delta u + qu = \lambda g u, \\ \quad \int\limits_{\mathbb{R}^n} g u^2 \, dx = \alpha. \end{cases} \tag{1.2}$$

So, we generalize the results in this area due to [1, 8, 9, 11, 13]. It is known (see [4]), under conditions on the mean values of q and g, that the counting function $N(\lambda)$ associated with Problem (1.2) is represented by the asymptotic formulas

$$\begin{cases} N(\lambda) \simeq (2\pi)^{-n} \omega_n \int\limits_{\mathbb{R}^n_\lambda} (\lambda g - q)^{n/2} \, dx \\ \text{if } \int\limits_{\mathbb{R}^n} g \, dx = +\infty, \end{cases} \tag{1.3}$$

$$\begin{cases} N(\lambda) \simeq (2\pi)^{-n} \omega_n \lambda^{n/2} \int\limits_{\mathbb{R}^n} g^{n/2} \, dx \\ \text{if } \int\limits_{\mathbb{R}^n} g \, dx < +\infty, \end{cases} \tag{1.4}$$

with $\mathbb{R}^n_\lambda = \{x \in \mathbb{R}^n : \ \lambda g(x) - q(x) \geq 0\}$. Considering Problem (1.1) as a nonlinear perturbation of Problem (1.2), we only need to realize the intimate connection with their eigenvalues.

2 Notations

For a given measurable function h, we denote by $h_\pm = \max\,(\pm h, 0)$ the positive and negative part of h, *i.e.*, $h = h_+ - h_-$.

We introduce the following functions in $\mathbb{R}^n$:

$$w(x) = \left(1 + |x|^2\right)^{-1/2}.$$

For a given $\tau > 0$, we denote $w_\tau(x) = w^\tau(x) \left(1 + \log \sqrt{1 + |x|^2}\right)^{-1}$ and we set

$$p_\tau(x) = w^{2\tau}(x) \ \text{ if } \ n > 2 \ \text{ and } \ p_\tau(x) = w_\tau^2(x) \ \text{ if } \ n = 2.$$

Let $L^2_{p_\tau}(\mathbb{R}^n)$ be the space $L^2(\mathbb{R}^n)$ provided with the weight p_τ, *i.e.*,

$$L^2_{p_\tau}(\mathbb{R}^n) = \left\{ u \in D'(\mathbb{R}^n) : \int\limits_{\mathbb{R}^n} p_\tau u^2 \, dx < +\infty \right\}$$

with its corresponding inner product $:= (u,v) = \int_{\mathbb{R}^n} p_\tau uv\,dx$.

We define the weighted Sobolev space

$$V = \left\{ u \in D'(\mathbb{R}^n);\;\; p_1(x)^{1/2}u \in L^2(\mathbb{R}^n),\; |\nabla u| \in L^2(\mathbb{R}^n) \right\}.$$

with its corresponding weighted norm $\|u\|_V = \left(\int_{\mathbb{R}^n} |\nabla u|^2 + p_1 u^2\,dx \right)^{1/2}$.

For $n \geq 2$, V is a separable Hilbert space. $V \hookrightarrow L^{2^*}(\mathbb{R}^n)$, the imbedding is continuous; 2^* denotes the Sobolev critical exponent of 2, *i.e.*, $2^* = \frac{2n}{n-2}$ if $n > 2$ and $(2^*)'$ is the conjugate exponent of 2^*. If $n = 2$, $V \hookrightarrow L^{\overline{2}}(\mathbb{R}^n)$ with $\overline{2} = \frac{2r}{r-2}$ and $r > 2$ (see [2]).

V^* denotes the dual space of V, and $V_\pm = \left\{ u \in V :\; \int_{\mathbb{R}^n} gu^2\,dx \gtrless 0 \right\}$.

$$B_R = \{x \in \mathbb{R}^n :\; |x| < R\},\; B'_R\; = \mathbb{R}^n \setminus B_R.$$

Definition 2.1 If Problem (1.1) has a nontrivial solution $u \in V$ ($u \not\equiv 0$) for certain λ, then λ is called an *eigenvalue*, and u is called an *eigenfunction* of Problem (1.1). The pair (λ, u) is called a *solution* of Problem (1.1).

Let X be a Hilbert space, $F \in C^1(X,\mathbb{R})$ and $G \in C^{1,1}_{Loc}(X,\mathbb{R})$. We define the manifold

$$M_\alpha = \left\{ u \in X :\; G(u) = \alpha;\; G'(u) \neq 0 \right\}.$$

A functional F satisfies the Palais-Smale condition on M_α if and only if from any sequence $(u_n) \subset M_\alpha$ satisfying

$$\begin{cases} F(u_n) \textit{ is bounded}, \\ F'_\alpha(u_n) = F'(u_n) - \frac{1}{\alpha}\left(F'(u_n, u_n\right) G'(u_n) \to 0, \end{cases}$$

we may select a convergent subsequence.

The genus of a symmetric, closed, compact set A which does not contain the origin, is given by

$$\gamma(A) = \inf\left\{ m \geq 1;\; \exists \Phi :\; A \to \mathbb{R}^m \setminus \{0\},\; \Phi \;\text{ odd and continuous} \right\}.$$

For any $n \geq 1$, we put

$$K_n(\alpha) = \left\{ A \subset M_\alpha,\; A \text{ symmetric, closed, compact and } \;\gamma(A) \geq n \right\}.$$

3 Hypotheses

We suppose that there exist

$$c_1 > 0,\ c_2 > 0 \ \text{ and } \ \eta \geq \frac{n}{2} \ \text{ such that } \ c_1 p_\eta(x) \leq |q(x)| \leq c_2 p_\eta(x). \tag{3.1}$$

In particular, $q \in L^{n/2}(\mathbb{R}^n)$.

There also exist

$$c > 0 \ \text{ and } \ \theta > \eta \ \text{ such that } \ |g(x)| \leq c p_\theta(x) \quad \text{for a.e. } \ x \in \mathbb{R}^n. \tag{3.2}$$

We assume that $f:\ \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is an odd Carathéodory function, *i.e.*,

$$f(x,-u) = -f(x,u), \tag{3.3}$$

$$|f(x,u)| \leq \sigma(x) + \rho(x)|u|^\gamma \ \text{ for all } \ x \in \mathbb{R}^n,\ u \in \mathbb{R};\ 1 < \gamma < 1 + \frac{4}{n}; \tag{3.4}$$

$$0 \leq \rho(x) \in L^{\gamma_1}(\mathbb{R}^n) \ \text{ with } \ \gamma_1 = \frac{2^*}{2^* - (\gamma+1)}; \tag{3.5}$$

$$\int_{\mathbb{R}^n} \rho^{2/\beta}(x) u^2\, dx \leq \left| \int_{\mathbb{R}^n} g(x) u^2\, dx \right| \ \text{ for all } \ u \in M_\alpha; \quad \beta = \frac{2(2^* - (\gamma+1))}{2^* - 2}; \tag{3.6}$$

$$0 \leq \sigma(x) \in L^{n/2}_{p_1^{n/2}}(\mathbb{R}^n) \cap L^{(2^*)'}(\mathbb{R}^n); \tag{3.7}$$

$$\big(f(x,y) - q^- y\big)\, y \geq 0\ \forall x \in \mathbb{R}^n,\ y \in \mathbb{R}^*. \tag{3.8}$$

In order to show the existence of solutions of Problem (1.1), we mainly use the generalization of Ljusternik–Schnirelmann theory (see [7, p. 212, Theorem 5.5]).

4 Existence of eigenvalues

We write Problem (1.1) under its variational formulation

$$\begin{cases} \text{Find } \ u \in V,\ u \not\cong 0 \ \text{ and } \ \lambda \in \mathbb{R} \ \text{ such that} \\ \int_{\mathbb{R}^n} (\nabla u \cdot \nabla v + q^+ uv)\, dx + \int_{\mathbb{R}^n} (f(x,u) - q^- u)\, v\, dx = \lambda \int_{\mathbb{R}^n} guv\, dx \ \forall\, v \in V. \end{cases} \tag{4.1}$$

Find solutions of Problem (4.1) turns out to finding nontrivial solutions of the equation

$$\big(\phi'(u), v\big) = \lambda \left(\varphi'(u), v\right) \ \forall\, v \in V. \tag{4.2}$$

where ϕ', φ' denote Gâteaux derivatives of the functionals

$$\phi(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla u|^2 + q^+ u^2\right) dx + \int_{\mathbb{R}^n} dx \int_0^u \left(f(x,s) - q^- s\right) ds$$

and $\varphi(u) = \frac{1}{2}\int_{\mathbb{R}^n} gu^2\, dx$ respectively.

The nontrivial solutions (eigenfunctions) of Problem (4.2) under the condition $\int_{\mathbb{R}^n} gu^2\, dx = \alpha$ correspond to critical points of the functional ϕ on the manifold

$$M_\alpha = \left\{u \in V :\ \varphi(u) = \frac{\alpha}{2}\right\} = \left\{u \in V :\ \int_{\mathbb{R}^n} gu^2\, dx = \alpha\right\},\ \alpha \neq 0.$$

Remark 4.1 Next we only consider the manifolds M_α with $\alpha > 0$. The case $\alpha < 0$ is studied similarly.

Lemma 4.1 (see [3])
(i) $\phi \in C^1(V, \mathbb{R}), \varphi \in C^{1,1}_{loc}(V, \mathbb{R})$, ϕ *and* φ *are even functions.*
(ii) For all $n \in \mathbb{N}, K_n(\alpha) \neq \emptyset$.
In particular, if X_n *is an* n*-dimensional subspace of* V*, then* $\gamma(M_\alpha \cap X_n) = n$.

Lemma 4.2 *The functional* ϕ *is bounded from below and satisfies the Palais-Smale condition on* M_α.

Proof. For fixed $\alpha > 0$ and for all $u \in M_\alpha$, we obtain from (3.8)

$$\begin{aligned} \phi(u) &= \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla u|^2 + q^+ u^2\right) dx + \int_{\mathbb{R}^n} dx \int_0^u \left(f(x,s) - q^- s\right) ds \\ &\geq \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla u|^2 + q^+ u^2\right) dx \\ &\geq c\,\|u\|_V^2 . \end{aligned}$$

Since $\alpha = \int_{\mathbb{R}^n} gu^2\, dx \leq c'\,\|u\|_V^2$ by virtue of the hypothesis (3.2), then $\phi(u) \geq c''\alpha$. So, the funtional ϕ is bounded below on M_α. ■

In order to verify the Palais-Smale condition, we define the following operators

$$J, G, F :\ V \longrightarrow V^*$$

by

$$(Ju, v) = \int_{\mathbb{R}^n} \left(\nabla u \cdot \nabla v + q^+ uv\right) dx,$$

$$(Gu, v) = \int_{\mathbb{R}^n} guv \, dx, \tag{4.3}$$

$$(Fu, v) = \int_{\mathbb{R}^n} \left(f(x, u) - q^- u\right) v \, dx. \tag{4.4}$$

Hence $(\phi'(u), v) = (Bu, v)$, where $B = J + F$ and $(\varphi'(u), v) = (Gu, v)$.

We denote by ϕ'_α the derivative of ϕ on M_α:

$$\begin{aligned} \phi'_\alpha(u) &= \phi'(u) - \frac{1}{\alpha}\left(\phi'(u), u\right)\varphi'(u) \\ &= Bu - \frac{1}{\alpha}(Bu, u)Gu. \end{aligned}$$

Lemma 4.3 *The operators G, F defined by* (4.3) *and* (4.4) *respectively are compact.*

Proof. Since the function g satisfies the hypothesis (3.2), the operator of multiplication G associated with g is compact.

Let $(u_n)_n$ be a weakly convergent sequence to u_0 in V, then

$$\begin{aligned} \sup_{\substack{v \in V \\ \|v\|_V = 1}} (Fu_n - Fu_0, v) \leq & \sup_{\substack{v \in V \\ \|v\|_V = 1}} \int_{B_R} |f(x, u_n) - f(x, u_0)| \, |v| \, dx \\ &+ \sup_{\substack{v \in V \\ \|v\|_V = 1}} \int_{B_R} q^- |u_n - u_0| \, |v| \, dx \\ &+ \sup_{\substack{v \in V \\ \|v\|_V = 1}} \int_{B'_R} |f(x, u_n) - f(x, u_0)| \, |v| \, dx \\ &+ \sup_{\substack{v \in V \\ \|v\|_V = 1}} \int_{B'_R} q^- |u_n - u_0| \, |v| \, dx. \end{aligned}$$

By virtue of Lemma 4.1 in [5], the quantity

$$\sup_{\substack{v\in V\\ \|v\|_V=1}} \int_{B_R} |f(x,u_n)-f(x,u_0)|\,|v|\,dx$$
$$+ \sup_{\substack{v\in V\\ \|v\|_V=1}} \int_{B_R} q^-|u_n-u_0|\,|v|\,dx \;+\; \sup_{\substack{v\in V\\ \|v\|_V=1}} \int_{B'_R} |f(x,u_n)-f(x,u_0)|\,|v|\,dx$$

converges to zero when n tends to infinity.

On the other hand, for any $v \in V$ we have

$$\int_{B'_R} q^-|u_n-u_0|\,|v|\,dx \le \left(\int_{B'_R} |u_n-u_0|^{2^*}\,dx\right)^{1/2^*} \left(\int_{B'_R} \left(q^-|v|\right)^{(2^*)'}\,dx\right)^{1/(2^*)'}$$
$$\le \left(\int_{\mathbb{R}^n} |u_n-u_0|^{2^*}\,dx\right)^{1/2^*} \left(\int_{B'_R} |v|^{2^*}\,dx\right)^{1/2^*} \left(\int_{B'_R} (q^-)^{n/2}\,dx\right)^{2/n}.$$

Since the sequence (u_n) is bounded in V, we obtain

$$\begin{aligned}\int_{B'_R} q^-|u_n-u_0|\,|v|\,dx &\le \|u_n-u_0\|_V\|v\|_V \left(\int_{B'_R} (q^-)^{n/2}\,dx\right)^{2/n}\\ &\le c\|v\|_V \left(\int_{B'_R} (q^-)^{n/2}\,dx\right)^{2/n}.\end{aligned}$$

So $\displaystyle\sup_{\substack{v\in V\\ \|v\|_V=1}} \int_{B'_R} q^-|u_n-u_0|\,|v|\,dx \le c\left(\int_{B'_R} (q^-)^{n/2}\,dx\right)^{2/n}$.

Therefore, using hypothesis (3.1), $\displaystyle\sup_{\substack{v\in V\\ \|v\|_V=1}} \int_{B'_R} q^-|u_n-u_0|\,|v|\,dx$ tends to zero when R tends to infinity. ∎

Let $(u_n)_n \subset M_\alpha$ be such that

$$\begin{cases} \phi(u_n) \ \text{ is bounded }, \\ \phi'_\alpha(u_n) = \phi'(u_n) - \frac{1}{\alpha}(\phi'(u_n), u_n)\varphi'(u_n) \to 0 \ \text{ in } \ V^*. \end{cases} \tag{4.5}$$

Then $(u_n)_n$ has a convergent subsequence.

Indeed, since ϕ is coercive, $(u_n)_n$ is bounded in V and then Gu_n converges. The numerical sequence

$$(Bu_n, u_n) = (Ju_n, u_n) + (Fu_n, u_n)$$

is bounded and thus admits a convergent subsequence.

On the other hand, from (4.5) we get

$$\phi'_\alpha(u_n) = Bu_n - \frac{1}{\alpha}(Bu_n, u_n)Gu_n \to 0,$$

then Bu_n converges. Therefore, we derive from the compactness of F that Fu_n converges, hence Ju_n also converges. At last, since the form (Ju, u) is coercive, then J has a bounded inverse, which means that $(u_n)_n$ converges. ■

Theorem 4.1 *For arbitrary $\alpha > 0$ and $n \in \mathbb{N}$, we set*

$$C_n(\alpha) = \inf_{K \in K_n(\alpha)} \sup_{u \in K} 2\phi(u).$$

Then, for all $n \in \mathbb{N}$, there exists $u_n(\alpha) \in M_\alpha$ and $\lambda_n(\alpha) \in \mathbb{R}$ such that

$$C_n(\alpha) = 2\phi(u_n(\alpha)) \quad \text{and} \quad \lambda_n(\alpha) = \frac{(\phi'(u_n(\alpha)), u_n(\alpha))}{\alpha}$$

and $(u_n(\alpha), \lambda_n(\alpha))$ is a solution of Problem (1.1).

Proof. The result is derived immediately from [7, p. 209, Theorem 5.3] and from Lemmas 4.1, 4.2 and 4.3. ■

Remark 4.2 In view of Remark 4.1, Problem (1.1) also has a sequence of solutions $(u_n(\alpha), \lambda_n(\alpha))$ for any $\alpha < 0$. For $\alpha > 0$, $u_n(\alpha) \in V_+$ and $\lambda_n(\alpha) > 0$, $\forall n \in \mathbb{N}$; similarly for any $\alpha < 0$, $u_n(\alpha) \in V_-$ and $\lambda_n(\alpha) < 0$, $\forall n \in \mathbb{N}$.

Proposition 4.1 *For arbitrary $\alpha \in \mathbb{R}^*$ we have*

$$\|u_n(\alpha)\|_V \to +\infty \quad \text{and} \quad |\lambda_n(\alpha)| \to +\infty \ \text{ when } \ n \to +\infty.$$

Proof. We have from the definition of ϕ that

$$2\phi(u) = (Ju, u) + 2(Fu, u).$$

Furthermore,

$$\begin{aligned} |(Fu,u)| &\leq \int\limits_{\mathbb{R}^n} |f(x,u)|\,|u|\,dx + \int\limits_{\mathbb{R}^n} q^-|u|^2\,dx \\ &\leq \int\limits_{\mathbb{R}^n} \sigma|u|\,dx + \int\limits_{\mathbb{R}^n} \rho|u|^{\gamma+1}\,dx + \int\limits_{\mathbb{R}^n} q^-|u|^2\,dx \\ &\leq \left(\int\limits_{\mathbb{R}^n} \sigma^{(2^*)'}\,dx\right)^{1/(2^*)'} \left(\int\limits_{\mathbb{R}^n} |u|^{2^*}\,dx\right)^{1/2^*} \\ &\quad + \left(\int\limits_{\mathbb{R}^n} \rho^{\gamma_1}\,dx\right)^{1/\gamma_1} \left(\int\limits_{\mathbb{R}^n} |u|^{2^*}\,dx\right)^{(\gamma+1)/2^*} \\ &\quad + \left(\int\limits_{\mathbb{R}^n} (q^-)^{n/2}\,dx\right)^{2/n} \left(\int\limits_{\mathbb{R}^n} |u|^{2^*}\,dx\right)^{2/2^*}. \end{aligned}$$

Hence

$$|(Fu,u)| \leq C_1 \|u\|_V + C_2 \|u\|_V^{\gamma+1} + C_3 \|u\|_V^2$$

because the imbedding is continuous from V into $L^{2^*}(\mathbb{R}^n)$.

On the other hand, we have $(Ju,u) \leq C_4 \|u\|_V^2$, consequently,

$$2\phi(u) \leq C_1 \|u\|_V + C_2 \|u\|_V^{\gamma+1} + C_5 \|u\|_V^2 .$$

Let $u_n(\alpha)$ be an eigenfunction of Problem (1.1), then

$$C_n(\alpha) = 2\phi(u_n(\alpha)) \leq C_1 \|u_n(\alpha)\|_V + C_2 \|u_n(\alpha)\|_V^{\gamma+1} + C_5 \|u_n(\alpha)\|_V^2 .$$

Letting $n \to +\infty$, we obtain $C_n(\alpha) \to +\infty$, and then $\|u_n(\alpha)\|_V \to +\infty$. Since for any $\alpha \in \mathbb{R}^*$

$$\alpha\lambda_n(\alpha) = (\phi'(u_n(\alpha)), u_n(\alpha)) \geq c\,\|u_n(\alpha)\|_V^2 \to +\infty,$$

then $|\lambda_n(\alpha)| \to +\infty$. The proof is thus complete. ■

5 Asymptotic behaviour of the eigenvalues

5.1 The case of a positive potential

We suppose in this part that the potential q is positive. We denote by λ_n^0 the eigenvalues of Problem (1.2). In order to establish a relation between $\lambda_n(\alpha)$ and λ_n^0, we need an appropriate formulation of Min-Max principle which characterize λ_n^0 (see [4]).

Lemma 5.1 (see [3]) *For any $\alpha > 0$ and $n \in \mathbb{N}$, we have*

$$\alpha\lambda_n^0 = \inf_{K \in K_n(\alpha)} \sup_{u \in K} (Ju, u)\,.$$

Proposition 5.1 *Suppose that the hypotheses* (3.1) *throughout* (3.8) *are satisfied. Then, for any $\alpha > 0$ we have*

$$\lambda_n(\alpha) = \lambda_n^0 + \circ\left(\left(\lambda_n^0\right)^{\varepsilon}\right) \quad \textit{with} \ \ 0 < \varepsilon < 1.$$

Proof. For any $u \in M_\alpha$, $2\phi(u) = (Ju, u) + 2(Fu, u)$. Furthermore,

$$\begin{aligned} |(Fu,u)| &\le c \int\limits_{\mathbb{R}^n} \left(\sigma|u| + \rho|u|^{\gamma+1}\right) dx \qquad (5.1)\\ &\le c\left\{\left(\int\limits_{\mathbb{R}^n} \sigma^{(2^*)'}\, dx\right)^{1/(2^*)'} \left(\int\limits_{\mathbb{R}^n} u^{2^*}\, dx\right)^{1/2^*} + \int\limits_{\mathbb{R}^n} \rho|u|^{\gamma+1}\, dx\right\} \end{aligned}$$

and

$$\begin{aligned} \int\limits_{\mathbb{R}^n} \rho|u|^{\gamma+1}\, dx &= \int\limits_{\mathbb{R}^n} \rho|u|^{\beta}|u|^{\gamma+1-\beta}\, dx \\ &\le \left(\int\limits_{\mathbb{R}^n} \rho^{2/\beta}|u|^2\, dx\right)^{\beta/2} \left(\int\limits_{\mathbb{R}^n} |u|^{\frac{2(\gamma+1-\beta)}{2-\beta}}\, dx\right)^{\frac{2-\beta}{2}} \end{aligned}$$

(since $\beta = \frac{2(2^* - (\gamma+1))}{2^*-2}$, then $\frac{2(\gamma+1-\beta)}{2-\beta} = 2^*$).
Hence

$$\int\limits_{\mathbb{R}^n} \rho|u|^{\gamma+1}\, dx \le c\left(\int\limits_{\mathbb{R}^n} \rho^{2/\beta}|u|^2\, dx\right)^{\beta/2} \|u\|_V^{\gamma+1-\beta}.$$

Taking into account the hypothesis (3.6), we find

$$\int_{\mathbb{R}^n} \rho|u|^{\gamma+1}\,dx \le c\alpha^{\beta/2}\|u\|_V^{\gamma+1-\beta}. \tag{5.2}$$

Substituting (5.2) in (5.1) on the right-hand side, we obtain

$$2\phi\,(u) \le (Ju,u) + c_1\,\|u\|_V + c_2\alpha^{\beta/2}\,\|u\|_V^{\gamma+1-\beta}.$$

The operator J being coercive, we obtain

$$2\phi(u) \le (Ju,u) + c_1'(Ju,u)^{1/2} + c_2'\alpha^{\beta/2}\,(Ju,u)^{\frac{\gamma+1-\beta}{2}}.$$

Let $h:\ \mathbb{R}^+ \to \mathbb{R}$ be an increasing and continuous function. It is easy to check that for all $n \in \mathbb{N}$

$$\inf_{K_n(\alpha)}\ \sup_{u\in K} h((Ju,u)) = h\left(\inf_{K_n(\alpha)}\ \sup_{u\in K}(Ju,u)\right).$$

Put $h(t) = t + c_1 t^{1/2} + c_2\alpha^{\beta/2}\,t^{\frac{\gamma+1-\beta}{2}}$, then

$$\begin{aligned} C_n(\alpha) &= \inf_{K_n(\alpha)}\ \sup_{u\in K} 2\phi(u) \\ &\le \inf_{K_n(\alpha)}\ \sup_{u\in K} h((Ju,u)) = h\left(\inf_{K_n(\alpha)}\ \sup_{u\in K}(Ju,u)\right) = h(\alpha\lambda_n^0), \end{aligned}$$

thus

$$C_n(\alpha) \le \alpha\lambda_n^0 + c_1(\alpha\lambda_n^0)^{1/2} + c_2\alpha^{\beta/2}\,(\alpha\lambda_n^0)^{\frac{\gamma+1-\beta}{2}}.$$

On the other hand,

$$2\phi(u) \ge (Ju,u) \ \text{ and then } \ C_n(\alpha) \ge \alpha\lambda_n^0;$$

$$\left|C_n(\alpha) - \alpha\lambda_n^0\right| \le c_1(\alpha\lambda_n^0)^{1/2} + c_2\alpha^{\beta/2}\,(\alpha\lambda_n^0)^{\frac{\gamma+1-\beta}{2}}. \tag{5.3}$$

It follows from the hypothesis (3.4) that $\gamma + 1 - \beta < 2$. So, from (5.3), we get

$$C_n(\alpha) \sim \alpha\lambda_n^0 \ \text{ for } \ n \ \text{ sufficiently large.}$$

Furthermore, let $u_n(\alpha)$ be the eigenfunction associated with $\lambda_n(\alpha)$, we have

$$\begin{aligned} |C_n(\alpha) - \alpha\lambda_n(\alpha)| &= \left|2\phi(u_n(\alpha)) - (\phi'(u_n(\alpha)), u_n(\alpha))\right| \\ &= \left|2\int_{\mathbb{R}^n} dx \int_0^{u_n(\alpha)} f(x,s)\,ds - \int_{\mathbb{R}^n} f(x,u_n(\alpha))u_n(\alpha)\,dx\right|. \end{aligned}$$

By a similar argument and using the hypothesis (3.4), we obtain

$$\begin{aligned} |C_n(\alpha) - \alpha\lambda_n(\alpha)| &\leq c_1(Ju_n(\alpha), u_n(\alpha))^{1/2} + c_2\alpha^{\beta/2}\,(Ju_n(\alpha), u_n(\alpha))^{\frac{\gamma+1-\beta}{2}} \\ &\leq c_1(2\phi(u_n(\alpha))^{1/2} + c_2\alpha^{\beta/2}\,(2\phi(u_n(\alpha))^{\frac{\gamma+1-\beta}{2}}. \end{aligned}$$

Hence

$$|C_n(\alpha) - \alpha\lambda_n(\alpha)| \leq c_1(C_n(\alpha))^{1/2} + c_2\alpha^{\beta/2}\,(C_n(\alpha))^{\frac{\gamma+1-\beta}{2}}. \tag{5.4}$$

Therefore, according to (5.3) and (5.4), we write

$$\begin{aligned} |\lambda_n(\alpha) - \lambda_n^0| &\leq \frac{1}{\alpha}\left(|C_n(\alpha) - \alpha\lambda_n(\alpha)| + |C_n(\alpha) - \alpha\lambda_n^0|\right) \\ &\leq C\left((\lambda_n^0)^{1/2} + (\lambda_n^0)^{\frac{\gamma+1-\beta}{2}}\right). \end{aligned}$$

Hence $\lambda_n(\alpha) \sim \lambda_n^0$ for n sufficiently large. The proof is thus complete. ∎

From Proposition 5.1 we derive the following

Theorem 5.1 *Assume that the hypotheses of Proposition 5.1 are fulfilled. Then,* $N(\lambda)$ *is given by the asymptotic formulas* (1.3) *and* (1.4).

5.2 The case of a not necessarily positive potential

In the case of a potential with nonconstant sign, the problem

$$-\Delta u + qu = \lambda gu \tag{5.5}$$

under the hypothesis

$$\exists K > 0 : \int_{\mathbb{R}^n} \left(|\nabla u|^2 + qu^2 + Kgu^2\right)\,dx > 0,\ \forall u \in V, \tag{5.6}$$

has at most a finite number of negative eigenvalues and an infinite sequence of positive eigenvalues (see [4]). The counting function $N(\lambda)$ of Problem (1.1) is given by Theorem 5.1. To prove this statement, it suffices to compare the eigenvalues of Problem (1.1) with the eigenvalues of Problem (5.5) under the condition (5.6).

Acknowledgments

Our research has been generously supported by grants from ANDRU under No. CU39904.

References

[1] Chiappinelli R., *The asymptotic distribution of eigenvalues of a nonlinear problem*, Boll. Un. Mat. Ital. B (6) **1** (1982), No. 3, 1131–1149.

[2] Chiappinelli R., *Compact embedding of some weighted Sobolev spaces on* $\mathbb{R}^n$, Math. Proc. Cambridge Philos. Soc., **117** (1995), No. 2, 333–338.

[3] Chiappinelli R., *On the eigenvalues and the spectrum for a class of semilinear elliptic operators*, Boll. Un. Mat. Ital. B (6) **4** (1985), No. 3, 867–882.

[4] Djellit A., *Valeurs propres de problèmes elliptiques indéfinis sur des ouverts non bornés de* $\mathbb{R}^n$, Thèse de Doctorat Univ. P. S. Toulouse, 1992.

[5] Djellit A. and Benouhiba N., *Existence and uniqueness of positive solution of a semilinear elliptic equation in* $\mathbb{R}^n$, Demonstratio Mathematica, **35** (2002), No. 1, 61–73.

[6] Fučik S., Nečas J., Souček J. and Souček V., *Spectral Analysis of Nonlinear Operators*, Lecture Notes in Mathematics **346**, Springer–Verlag, Berlin, 1970.

[7] Kavian O., *Introduction à la Théorie des Points Critiques et Applications aux Problèmes Elliptiques*, Springer–Verlag, Paris, 1993.

[8] Moscatelli V. F. and Thompson M., *Asymptotic distribution of Ljusternik-Schnirelmann eigenvalues for elliptic nonlinear operators*, Proceeding of the Edinburgh Mathematical Society, **33** (1990), 381–403.

[9] Rabinowitz P. H. *Variational methods for nonlinear eigenvalue problems*, in: Eigenvalues of nonlinear problems, CIME Cremonese, Rome, 1974, pp. 141–195.

[10] Shibata T., *Asymptotic properties of variational eigenvalues for semilinear elliptic operators*, Boll. Un. Mat. Ital. B (7) **2** (1988), No. 2, 411–425.

[11] Shibata T., *Asymptotic formula of eigenvalues of sublinear Sturm–Liouville problems*, Differential and Integral Equations, **8** (1995), No. 1, 183–200.

[12] Shibata T., *Deformation of domain and the limit of variational eigenvalues of semilinear elliptic operators*, Internat. J. Math. & Math. Sci., **19** (1996), No. 4, 679–688.

[13] Shibata T., *Asymptotic behaviour of variational eigenvalues of the nonlinear elliptic eigenvalue problem in a ball*, Nonlinearity, **7** (1994), No. 6, 1645–1653.