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Periodic Solutions of the Discrete Counterpart of an Impulsive System with a Small Delay

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Abstract

A discrete analogue of an impulsive system with a small delay is considered. If the corresponding system without delay has an isolated ω-periodic solution, then in any neighbourhood of this orbit the discrete system also has a periodic solution.

1 Introduction

In the mathematical simulation of the evolution of real processes in physics, chemistry, population dynamics, radio engineering etc. which are subject to disturbances of negligible duration with respect to the total duration of the process, it is often convenient to assume that the disturbances are instantaneous, in the form of impulses. This leads to the investigation of differential equations and systems with discontinuous trajectories, or with impulse effect, called for the sake of brevity impulsive differential equations and systems.

Impulsive differential equations with delay describe models of real processes and phenomena where both dependence on the past and instantaneous disturbances are observed. For instance, the size of a given population may be normally described by a delay differential equation and, at certain instants, the number of individuals can be abruptly changed. The interaction of the impulsive perturbation and the delay makes difficult the qualitative investigation of such equations. In particular,

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the solutions are not smooth at the instants of impulse effect shifted by the delay [2].

In the present paper we derive a discrete analogue of an impulsive system with a small delay. If the corresponding system without delay has an isolated ω-periodic solution, then in any neighbourhood of this orbit the discrete system is shown to have a periodic solution.

2 Statement of the problem. Main result

Consider the system with impulses at fixed instants

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-h)), & t \neq t_j, t_j + h, \\ \Delta x(t_j) = I_j(x(t_j), x(t_j - h)), & j \in \mathbb{Z}, \\ \Delta x(t_j + h) = 0, \end{cases} \tag{1}$$

where $x \in \Omega \subset \mathbb{R}^n$, $f : \mathbb{R} \times \Omega \times \Omega \to \mathbb{R}^n$, Ω is a domain in $\mathbb{R}^n$; $\mathbb{Z}$ is the set of all integers, $\mathbb{N}$ is the set of positive integers; $\Delta x(t_j) = x(t_j + 0) - x(t_j - 0)$ are the impulses at instants t_j and $\{t_j\}_{j \in \mathbb{Z}}$ is a strictly increasing sequence such that $\lim\limits_{j \to \pm\infty} t_i = \pm\infty$; $I_j : \Omega \times \Omega \to \mathbb{R}^n$ $(j \in \mathbb{Z})$; $h > 0$ is the delay.

As usual in the theory of the impulsive differential equations [1, 4], at the points of discontinuity t_j of the solution $x(t)$ we assume that $x(t_j) \equiv x(t_j - 0)$. It is clear that, in general, the derivatives $\dot{x}(t_j)$, $j \in \mathbb{Z}$, do not exist. On the other hand, there do exist the limits $\dot{x}(t_j \pm 0)$. According to the above convention, we assume $\dot{x}(t_j) \equiv \dot{x}(t_j - 0)$.

Similarly, the derivative $\dot{x}$ does not exist at the other points of discontinuity of the right-hand side $f(t, x(t), x(t-h))$, *i.e.*, at the points $t_j + h$, $j \in \mathbb{Z}$. We require the continuity of the solution $x(t)$ at such points if they are distinct from the moments of impulse effect t_j.

For the sake of brevity we shall use the notation $\bar{x}(t) = x(t-h)$, and we will denote by $\bar{x}$ the last argument of the functions f and I_j.

In the sequel we require the fulfillment of the following assumptions:

A1. The function $f(t, x, \bar{x})$ is continuous (or piecewise continuous, with discontinuities of the first kind at the points t_j) and ω-periodic with respect to t, continuously differentiable with respect to $x, \bar{x} \in \Omega$, with locally Lipschitz continuous with respect to $x, \bar{x}$ first derivatives.

A2. The functions $I_j(x, \bar{x})$, $j \in \mathbb{Z}$, are continuously differentiable with respect to $x, \bar{x} \in \Omega$, with locally Lipschitz continuous with respect to $x, \bar{x}$ first derivatives.

A3. There exists a positive integer m such that $t_{j+m} = t_j + \omega$, $I_{j+m}(x, \bar{x}) = I_j(x, \bar{x})$ for $j \in \mathbb{Z}$ and $x, \bar{x} \in \Omega$.

Suppose, for the sake of definiteness, that

$$0 < t_1 < t_2 < \cdots < t_m < \omega.$$

Let $h_0 > 0$ be so small that for any $h \in [0, h_0]$ we should have

$$h < t_1, \ \ t_j + 2h \le t_{j+1}, \ j = \overline{1, m-1}, \ t_m + h < \omega. \tag{2}$$

If in system (1) we put $h = 0$, we obtain the so called *generating system*

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t)), & t \ne t_j, \\ \Delta x(t_j) = I_j(x(t_j), x(t_j)), & j \in \mathbb{Z}, \end{cases} \tag{3}$$

and suppose that

A4. The generating system (3) has an ω-periodic solution $\psi(t)$ such that $\psi(t) \in \Omega$ for all $t \in \mathbb{R}$.

There exists a positive number μ_0 such that Ω contains a closed μ_0-neighbourhood Ω_1 of the periodic orbit $\{x = \psi(t);\ t \in \mathbb{R}\}$. We also denote by $\Omega_{1/2}$ a closed $\mu_0/2$-neighbourhood of this orbit.

One of the most widely used techniques in the study of models involving ordinary differential equations is to approximate the system by means of a system of difference equations, whose solutions are expected to be samples of the solutions of differential equations at discrete instants of time as in the case of Euler-type methods and Runge-Kutta methods. Let us recall that convergent difference approximations for nonlinear impulsive systems of differential equations in a Banach space were obtained in [3].

We choose a uniform discretization step size h equal to the delay in system (1) and such that $N = \omega/h$ is an integer. For convenience we denote $[t/h] = i$ (the integer part of t/h), $x(ih) = x_i,\ i \in \mathbb{Z},\ [t_j/h] = n_j,\ j \in \mathbb{Z}$. By virtue of (2) and **A3** $\{n_j\}_{j\in\mathbb{Z}}$ is a strictly increasing sequence of integers such that $n_{j+1} - n_j > 1$ and $n_{j+m} = n_j + N,\ j \in \mathbb{Z}$.

Now we approximate system (1) by the discrete system

$$\begin{cases} x_{i+1} = x_i + hf(ih, x_i, x_{i-1}), & i \in \mathbb{Z} \setminus \{n_j\}_{j\in\mathbb{Z}}, \\ x_{n_j+1} = x_{n_j} + I_j(x_{n_j}, x_{n_j-1}), & j \in \mathbb{Z}. \end{cases} \tag{4}$$

By analogy with the continuous-time case we also consider the system

$$\begin{cases} x_{i+1} = x_i + hf(ih, x_i, x_i), & i \in \mathbb{Z} \setminus \{n_j\}_{j\in\mathbb{Z}}, \\ x_{n_j+1} = x_{n_j} + I_j(x_{n_j}, x_{n_j}), & j \in \mathbb{Z}, \end{cases} \tag{5}$$

which we also call generating system.

A5. For each number $N_0 > 0$ there exists an integer $N \geq N_0$ such that for $h = \omega/N$ the generating system (5) has an N-periodic solution $\{\psi_i\}_{i\in\mathbb{Z}}$ such that $\psi_i \in \Omega_{1/2}$ for all $i \in \mathbb{Z}$.

Here we may assume that $N_0 \geq \omega/h_0$.

So there exists a strictly increasing sequence of integers $\{N_k\}_{k\in\mathbb{N}}$ such that for each member of this sequence condition **A5** is valid. Henceforth N is a member of this sequence.

Now define the linearized system (also called *system in variations*) with respect to $\{\psi_i\}_{i\in\mathbb{Z}}$:

$$\begin{cases} y_{i+1} = (E + hA_i)y_i, & i \in \mathbb{Z} \setminus \{n_j\}_{j\in\mathbb{Z}}, \\ y_{n_j+1} = (E + B_j)y_{n_j}, & j \in \mathbb{Z}, \end{cases} \tag{6}$$

where E is the unit $(n \times n)$-matrix,

$$A_i = \frac{\partial}{\partial x} f(t, x, x)\bigg|_{x=\psi_i}, \quad B_j = \frac{\partial}{\partial x} I_j(x, x)\bigg|_{x=\psi_{n_j}}.$$

If $h_0 > 0$ is small enough, *i.e.*, if N_0 is large enough, then for $h \in (0, h_0]$ the matrices $E + hA_i$ are nonsingular. We also assume that

A6. The matrices $E + B_j$, $j \in \mathbb{Z}$, are nonsingular.

Now we can write down system (6) in the form

$$y_{i+1} = (E + D_i)y_i, \qquad i \in \mathbb{Z}, \tag{7}$$

where

$$D_i = \begin{cases} hA_i & \text{for} \quad i \notin \{n_j\}_{j\in\mathbb{Z}}, \\ B_j & \text{for} \quad i = n_j,\ j \in \mathbb{Z}. \end{cases} \tag{8}$$

Obviously, $D_{i+N} = D_i$ for $i \in \mathbb{Z}$. The fundamental solution $\{X_i\}_{i\in\mathbb{Z}}$ of system (7) is given by

$$X_i = \begin{cases} \prod\limits_{\nu=1}^{i} (E + D_{i-\nu}), & i > 0, \\ E, & i = 0, \\ \prod\limits_{\nu=0}^{-i-1} (E + D_{i+\nu})^{-1}, & i < 0. \end{cases}$$

Now we make the additional assumption

A7. If $\{X_i\}_{i\in\mathbb{Z}}$ is the fundamental solution of system (6), then the matrix $E - X_N$ is nonsingular.

This means that the only N-periodic solution of system (6) is $y_i = 0$ for all $i \in \mathbb{Z}$.

If conditions **A6**, **A7** are valid, then the nonhomogeneous system

$$y_{i+1} = (E + D_i)y_i + g_i, \qquad i \in \mathbb{Z}, \tag{9}$$

where D_i are given by (8) and $g_{i+N} = g_i$, has a unique N-periodic solution

$$y_i = \sum_{\nu=0}^{N-1} G(i,\nu)g_\nu, \quad i = 0, 1, \dots, N-1, \tag{10}$$

where *Green's function* $G(i,\nu)$ is defined by

$$G(i,\nu) = \begin{cases} X_i(E - X_N)^{-1}X_{\nu+1}^{-1}, & \nu = 0, 1, \dots, i-1, \\ X_{N+i}(E - X_N)^{-1}X_{\nu+1}^{-1}, & \nu = i, i+1, \dots, N-1, \end{cases}$$

and continued by periodicity for all $i \in \mathbb{Z}$.

Our result in the present paper is the following

Theorem 1 *Let conditions* **A1**–**A7** *hold. Then there exists a number* $N_1 > 0$ *so that for any integer* $N \geq N_1$ *such that for* $h = \omega/N$ *the generating system* (5) *has an* N*-periodic solution* $\{\psi_i\}_{i\in\mathbb{Z}}$*, system* (4) *has a unique* N*-periodic solution* $\{x_i\}_{i\in\mathbb{Z}}$ *such that* $\lim_{N\to\infty} \max_{0\leq i\leq N-1} |x_i - \psi_i| = 0$.

3 Proof of the main result

In system (4) we change the variables according to the formula

$$x_i = \psi_i + y_i, \qquad i \in \mathbb{Z}, \tag{11}$$

and obtain the system

$$\begin{cases} y_{i+1} &= (E + hA_i)y_i + hQ_i(y_i) + h\delta f(ih, x_i, x_{i-1}), \ i \in \mathbb{Z} \setminus \{n_j\}_{j\in\mathbb{Z}}, \\ y_{n_j+1} &= (E + B_j)y_{n_j} + J_j(y_{n_j}) + \delta I_j(x_{n_j}, x_{n_j-1}), \ j \in \mathbb{Z}, \end{cases} \tag{12}$$

where

$$\begin{aligned} Q_i(y) &\equiv f(ih, \psi_i + y, \psi_i + y) - f(ih, \psi_i, \psi_i) - A_i y, \\ J_j(y) &\equiv I_j(\psi_{n_j} + y, \psi_{n_j} + y) - I_j(\psi_{n_j}, \psi_{n_j}) - B_j y \end{aligned}$$

are nonlinearities inherent to the generating system (5), while

$$\begin{aligned} \delta f(ih, x_i, x_{i-1}) &\equiv f(ih, x_i, x_{i-1}) - f(ih, x_i, x_i), \\ \delta I_j(x_{n_j}, x_{n_j-1}) &\equiv I_j(x_{n_j}, x_{n_j-1}) - I_j(x_{n_j}, x_{n_j}) \end{aligned}$$

are increments due to the delay.

We can formally consider (12) as a nonhomogeneous system of the form (9) with nonhomogeneities

$$g_i = \begin{cases} hQ_i(y_i) + h\delta f(ih, x_i, x_{i-1}), & i \in \mathbb{Z} \setminus \{n_j\}_{j\in\mathbb{Z}}, \\ J_j(y_{n_j}) + \delta I_j(x_{n_j}, x_{n_j-1}), & j \in \mathbb{Z}. \end{cases}$$

Now let us introduce the set $\mathbb{I} = \{0, 1, \ldots, N-1\} \setminus \{n_j\}_{j=1}^m$. Clearly, $\mathbb{I}$ is a set of $N - m$ integers. Then the unique N-periodic solution $y = \{y_i\}_{i\in\mathbb{Z}}$ must satisfy the operator equation

$$y \; = \; \mathcal{U}_h y, \tag{13}$$

where

$$\begin{aligned} (\mathcal{U}_h y)_i \;\; &\equiv \;\; h\sum_{\nu\in\mathbb{I}} G(i,\nu)Q_\nu(y_\nu) + h\sum_{\nu\in\mathbb{I}} G(i,\nu)\delta f(\nu h, x_\nu, x_{\nu-1}) \\ &+ \;\; \sum_{j=1}^m G(i,n_j)J_j(y_{n_j}) + \sum_{j=1}^m G(i,n_j)\delta I_j(x_{n_j}, x_{n_j-1}) \\ &\equiv \;\; h(\mathcal{S}_1 y)_i \; + \; h(\mathcal{S}_2 y)_i \; + \; (\mathcal{S}_3 y)_i \; + \; (\mathcal{S}_4 y)_i. \end{aligned}$$

For the sake of brevity we still write x_i instead of $\psi_i + y_i$ in $\delta f(ih, x_i, x_{i-1})$ and $\delta I_j(x_{n_j}, x_{n_j-1})$ as well as in $\mathcal{S}_2 y$ and $\mathcal{S}_4 y$. Moreover, in §3.2 we will further transform the expressions $\mathcal{S}_2 y$ and $\mathcal{S}_4 y$ under the assumption that $\{x_\nu\}_{\nu\in\mathbb{Z}}$ is a solution of system (4).

An N-periodic solution $\{x_i\}_{i\in\mathbb{Z}}$ of system (4) corresponds to a fixed point $\{y_i\}_{i\in\mathbb{Z}}$ of the operator $\mathcal{U}_h$ in a suitable set of N-periodic sequences. To this end we shall prove that $\mathcal{U}_h$ maps a suitably chosen set into itself (§3.2) as a contraction (§3.3).

We first need to introduce some

3.1 Notation

For a vector $x = (x^1, x^2, \ldots, x^n)^T$ we denote $|x| = \max\limits_{1\le k\le n} |x^k|$. Denote by $\mathcal{P}_{N,n}$ the space of all sequences $w = \{w_i\}_{i\in\mathbb{Z}}$ of n-vectors w_i such that $w_{i+N} = w_i$ for all $i \in \mathbb{Z}$, equipped with the norm

$$\|w\| \; = \max_{0\le i\le N-1} |w_i|.$$

We recall that Ω_1 and $\Omega_{1/2}$ are respectively closed μ_0- and $\mu_0/2$-neighbourhoods of the orbit $\{x = \psi(t);\; t \in \mathbb{R}\}$. For $x, \bar{x} \in \Omega_1$ the functions $f(t, x, \bar{x})$ $(t \in [0, \omega])$ and

$I_j(x,\bar{x})$ $(j=\overline{1,m})$ are bounded, together with their first derivatives with respect to $x,\bar{x}$. Let us denote

$$\begin{aligned}
\mathcal{M} &= \sup\Big\{|G(i,\nu)|:\ i,\nu=\overline{0,N-1}\Big\},\\
M_0 &= \max\Big\{\sup\big\{|f(t,x,\bar{x})|:\ t\in[0,\omega],\ x,\bar{x}\in\Omega_1\big\},\\
&\qquad \sup\big\{|I_j(x,\bar{x})|:\ j=\overline{1,m},\ x,\bar{x}\in\Omega_1\big\}\Big\},
\end{aligned}$$

$$\begin{aligned}
M_1 &= \max\Big\{\sup\big\{|\partial_x f(t,x,\bar{x})|:\ t\in[0,\omega],\ x,\bar{x}\in\Omega_1\big\},\\
&\qquad \sup\big\{|\partial_{\bar{x}} f(t,x,\bar{x})|:\ t\in[0,\omega],\ x,\bar{x}\in\Omega_1\big\},\\
&\qquad \sup\big\{|\partial_x I_j(x,\bar{x})|:\ j=\overline{1,m},\ x,\bar{x}\in\Omega_1\big\},\\
&\qquad \sup\big\{|\partial_{\bar{x}} I_j(x,\bar{x})|:\ j=\overline{1,m},\ x,y\in\Omega_1\big\}\Big\}.
\end{aligned}$$

Let L be the greatest Lipschitz constant for the first derivatives of $f(t,x,\bar{x})$, $t\in[0,\omega]$, $x,\bar{x}\in\Omega_1$, and of $I_j(x,\bar{x})$, $j=\overline{1,m}$, $x,\bar{x}\in\Omega_1$, whose existence is provided by conditions **A1**, **A2** and the compactness of the set Ω_1.

For $\mu\in(0,\mu_0/2]$ define a set of sequences

$$\mathcal{T}_\mu = \{\, y\in\mathcal{P}_{N,n}:\quad \|y\|\le\mu \,\}.$$

We shall find a dependence between $N\in\{N_k\}_{k\in\mathbb{N}}$ (or $h=\omega/N$) and μ so that the operator $\mathcal{U}_h$ in (13) maps the set $\mathcal{T}_\mu$ into itself as a contraction.

3.2 Invariance of the set $\mathcal{T}_\mu$ under the action of the operator $\mathcal{U}_h$

Let $y\in\mathcal{T}_\mu$ and $N\in\{N_k\}_{k\in\mathbb{N}}$. Then $x_i=\psi_i+y_i\in\Omega_1$. We shall estimate $|(\mathcal{U}_h y)_i|$ using the representation

$$\mathcal{U}_h y = h\big(\mathcal{S}_1 y+\mathcal{S}_2 y\big)+\mathcal{S}_3 y+\mathcal{S}_4 y$$

and system (4).

First we have

$$\begin{aligned}
J_j(y_{n_j}) &= \Big\{\int_0^1 \big[\partial_x I_j(\psi_{n_j}+sy_{n_j},\psi_{n_j}+sy_{n_j}) - \partial_x I_j(\psi_{n_j},\psi_{n_j})\big]\,ds\\
&+ \int_0^1 \big[\partial_{\bar{x}} I_j(\psi_{n_j}+sy_{n_j},\psi_{n_j}+sy_{n_j}) - \partial_{\bar{x}} I_j(\psi_{n_j},\psi_{n_j})\big]\,ds\Big\}y_{n_j}.
\end{aligned}$$

Here $\psi_{n_j} \in \Omega_{1/2}$ and $\psi_{n_j} + sy_{n_j} \in \Omega_1$ for $0 \le s \le 1$, thus

$$|J_j(y_{n_j})| \le 2\int_0^1 L2s|y_{n_j}|\,ds \cdot |y_{n_j}| = 2L|y_{n_j}|^2$$

and for

$$(S_3y)_i \equiv \sum_{j=1}^{m} G(i,n_j)J_j(y_{n_j})$$

we have

$$\|\mathcal{S}_3 y\| \;\le\; 2m\mathcal{M}L\mu^2.$$

Similarly,

$$\begin{aligned} Q_\nu(y_\nu) \;&=\; \Big\{\int_0^1 \big[\partial_x f(\nu h, \psi_\nu + sy_\nu, \psi_\nu + sy_\nu) - \partial_x f(\nu h, \psi_\nu, \psi_\nu)\big]\,ds \\ &\quad + \int_0^1 \big[\partial_{\bar{x}} f(\nu h, \psi_\nu + sy_\nu, \psi_\nu + sy_\nu) - \partial_{\bar{x}} f(\nu h, \psi_\nu, \psi_\nu)\big]\,ds\Big\} y_\nu, \end{aligned}$$

thus

$$|Q_\nu(y_\nu)| \le 2\int_0^1 L2s|y_\nu|\,ds \cdot |y_\nu| = 2L|y_\nu|^2$$

and for

$$(S_1y)_i \equiv \sum_{\nu\in\mathbb{I}} G(i,\nu)Q_\nu(y_\nu)$$

we obtain

$$\|\mathcal{S}_1 y\| \;\le\; 2(N-m)\mathcal{M}L\mu^2.$$

Now

$$\begin{aligned} &\|h\mathcal{S}_1 y + \mathcal{S}_3 y\| \le (Nh - mh + m)2\mathcal{M}L\mu^2 \\ =\;\; &[\omega + m(1-h)]2\mathcal{M}L\mu^2 < 2(\omega+m)\mathcal{M}L\mu^2 \end{aligned}$$

and we can choose $\tilde{\mu}_0 \in (0, \mu_0/2]$ so that for any $\mu \in (0, \tilde{\mu}_0]$ we have

$$\|h\mathcal{S}_1 y + \mathcal{S}_3 y\| \le \mu/2. \tag{14}$$

Further on, since $n_j - n_{j-1} > 1$, we have

$$\delta I_j(x_{n_j}, x_{n_j-1}) = \int_0^1 \partial_{\bar{x}} I_j(x_{n_j}, sx_{n_j-1} + (1-s)x_{n_j})\,ds \cdot (x_{n_j-1} - x_{n_j}) \tag{15}$$

$$= -h\int_0^1 \partial_{\bar{x}} I_j(x_{n_j}, sx_{n_j-1} + (1-s)x_{n_j})\,ds \cdot f((n_j-1)h, x_{n_j-1}, x_{n_j-2}),$$

thus

$$|\delta I_j(x_{n_j}, x_{n_j-1})| \le M_0 M_1 h$$

and for

$$(\mathcal{S}_4 y)_i \equiv \sum_{j=1}^{m} G(i, n_j)\delta I_j(x_{n_j}, x_{n_j-1})$$

we have

$$\|\mathcal{S}_4 y\| \ \le \ m\mathcal{M} M_0 M_1 h. \tag{16}$$

Similarly,

$$\begin{aligned} &\delta f(\nu h, x_\nu, x_{\nu-1}) \\ = \ &\int_0^1 \partial_{\bar{x}} f(\nu h, x_\nu, s x_{\nu-1} + (1-s)x_\nu)\, ds \cdot (x_{\nu-1} - x_\nu). \end{aligned} \tag{17}$$

If $\nu - 1 \ne n_j$ for any $j = \overline{1, m}$, then

$$x_{\nu-1} - x_\nu = -h f(\nu h, x_{\nu-1}, x_{\nu-2}) \tag{18}$$

and

$$|\delta f(\nu h, x_\nu, x_{\nu-1})| \ \le \ M_0 M_1 h. \tag{19}$$

If, however, $\nu - 1 = n_j$ for some $j \in \{1, \dots, m\}$, then

$$x_{\nu-1} - x_\nu = -I_j(x_{n_j}, x_{n_j-1}) \tag{20}$$

and

$$|\delta f(\nu h, x_\nu, x_{\nu-1})| \le M_0 M_1. \tag{21}$$

Now for

$$(\mathcal{S}_2 y)_i \equiv \sum_{\nu \in \mathbb{I}} G(i, \nu)\delta f(\nu h, x_\nu, x_{\nu-1})$$

from (19) and (21) we find

$$\|\mathcal{S}_2 y\| \ \le \ \big[(N - 2m)h + m\big]\mathcal{M} M_0 M_1. \tag{22}$$

Adding together the estimates (22) and (16), we obtain

$$\|h\mathcal{S}_2 y + \mathcal{S}_4 y\| \le \mathcal{M} M_0 M_1 \big[Nh + (1-h)m\big] h < (\omega + m)\mathcal{M} M_0 M_1 h.$$

Now we can choose $N(\mu) \ge N_0$ so that for any N, $N \in \{N_k\}_{k\in\mathbb{N}}$, $N \ge N(\mu)$, and $h = \omega/N$ we have

$$\|h\mathcal{S}_2 y + \mathcal{S}_4 y\| \le \mu/2. \tag{23}$$

Finally, by virtue of the estimates (14) and (23) we obtain

$$\|\mathcal{U}_h y\| \ \le \ \mu,$$

i.e., the operator $\mathcal{U}_h$ maps the set $\mathcal{T}_\mu$ into itself for $\mu \in (0, \tilde{\mu}_0]$ and $h = \omega/N$, where $N \ge N(\mu)$ and $N \in \{N_k\}_{k\in\mathbb{N}}$.

3.3 Contraction property of the operator $\mathcal{U}_h$

Let $y', y'' \in \mathcal{T}_\mu$. Then

$$\begin{aligned}\mathcal{U}_h y' - \mathcal{U}_h y'' &= h(\mathcal{S}_1 y' - \mathcal{S}_1 y'') + h(\mathcal{S}_2 y' - \mathcal{S}_2 y'')\\ &+ (\mathcal{S}_3 y' - \mathcal{S}_3 y'') + (\mathcal{S}_4 y' - \mathcal{S}_4 y'').\end{aligned}$$

First we consider

$$(\mathcal{S}_3 y' - \mathcal{S}_3 y'')_i = \sum_{j=1}^{m} G(i, n_j)\big(J_j(y'_{n_j}) - J_j(y''_{n_j})\big).$$

We have

$$\begin{aligned}& J_j(y'_{n_j}) - J_j(y''_{n_j})\\ =\ & \big(I_j(\psi_{n_j} + y'_{n_j}, \psi_{n_j} + y'_{n_j}) - I_j(\psi_{n_j} + y''_{n_j}, \psi_{n_j} + y''_{n_j})\big) - B_j(y'_{n_j} - y''_{n_j})\\ =\ & \Big\{\int_0^1 \big[\partial_x I_j(\psi_{n_j} + s y'_{n_j} + (1-s) y''_{n_j}, \psi_{n_j} + s y'_{n_j} + (1-s) y''_{n_j})\\ & \qquad - \partial_x I_j(\psi_{n_j}, \psi_{n_j})\big]\, ds\\ +\ & \int_0^1 \big[\partial_{\bar{x}} I_j(\psi_{n_j} + s y'_{n_j} + (1-s) y''_{n_j}, \psi_{n_j} + s y'_{n_j} + (1-s) y''_{n_j})\\ & \qquad - \partial_{\bar{x}} I_j(\psi_{n_j}, \psi_{n_j})\big]\, ds\Big\} \cdot (y'_{n_j} - y''_{n_j}),\end{aligned}$$

thus

$$|J_j(y'_{n_j}) - J_j(y''_{n_j})| \le 2\int_0^1 2L\big[s|y'_{n_j}| + (1-s)|y''_{n_j}|\big]\, ds \cdot |y'_{n_j} - y''_{n_j}| \le 4L\mu\|y' - y''\|$$

and

$$\|\mathcal{S}_3 y' - \mathcal{S}_3 y''\| \le 4m\mathcal{M}L\mu\|y' - y''\|. \tag{24}$$

Next,

$$(\mathcal{S}_1 y' - \mathcal{S}_1 y'')_i = \sum_{\nu\in\mathbb{I}} G(i, \nu)\big(Q_\nu(y'_\nu) - Q_\nu(y''_\nu)\big).$$

As above we have

$$\begin{aligned}& Q_\nu(y'_\nu) - Q_\nu(y''_\nu)\\ =\ & \Big\{\int_0^1 [\partial_x f(\nu h, x_\nu(s), x_\nu(s)) - \partial_x f(\nu h, \psi_\nu, \psi_\nu)]\, ds\\ +\ & \int_0^1 [\partial_{\bar{x}} f(\nu h, x_\nu(s), x_\nu(s)) - \partial_{\bar{x}} f(\nu h, \psi_\nu, \psi_\nu)]\, ds\Big\} \cdot \big(y'_\nu - y''_\nu\big),\end{aligned}$$

where $x_\nu(s) = \psi_\nu + sy'_\nu + (1-s)y''_\nu$. Thus

$$\left|Q_\nu(y'_\nu) - Q_\nu(y''_\nu)\right| \le 2\int_0^1 2L\left[s|y'_\nu| + (1-s)|y''_\nu|\right] ds \cdot |y'_\nu - y''_\nu| \le 4L\mu\|y' - y''\|$$

and

$$\|\mathcal{S}_1 y' - \mathcal{S}_1 y''\| \le 4(N-m)\mathcal{M}L\mu\|y' - y''\|. \tag{25}$$

From (24) and (25) we obtain

$$\|h(\mathcal{S}_1 y' - \mathcal{S}_1 y'') + (\mathcal{S}_3 y' - \mathcal{S}_3 y'')\| \le 4\big[Nh + m(1-h)\big]\mathcal{M}L\mu\|y' - y''\|$$

and finally

$$\|h(\mathcal{S}_1 y' - \mathcal{S}_1 y'') + (\mathcal{S}_3 y' - \mathcal{S}_3 y'')\| \le 4(\omega + m)\mathcal{M}L\mu\|y' - y''\|. \tag{26}$$

In order to estimate $\mathcal{S}_4 y' - \mathcal{S}_4 y''$ we use the representation (15). Let $x' = \psi + y'$, $x'' = \psi + y''$. Now

$$(\mathcal{S}_4 y' - \mathcal{S}_4 y'')_i = \sum_{j=1}^m G(i, n_j)(\delta I_j(x'_{n_j}, x'_{n_j-1}) - \delta I_j(x''_{n_j}, x''_{n_j-1}))$$

and

$$\begin{aligned} & \delta I_j(x'_{n_j}, x'_{n_j-1}) - \delta I_j(x''_{n_j}, x''_{n_j-1}) \\ = \; & -h\int_0^1 \big[\partial_{\bar{x}} I_j(x'_{n_j}, sx'_{n_j-1} + (1-s)x'_{n_j}) f((n_j-1)h, x'_{n_j-1}, x'_{n_j-2}) \\ & - \partial_{\bar{x}} I_j(x''_{n_j}, sx''_{n_j-1} + (1-s)x''_{n_j}) f((n_j-1)h, x''_{n_j-1}, x''_{n_j-2})\big]\, ds. \end{aligned}$$

Further on,

$$\begin{aligned} & \big|\partial_{\bar{x}} I_j(x'_{n_j}, sx'_{n_j-1} + (1-s)x'_{n_j}) f((n_j-1)h, x'_{n_j-1}, x'_{n_j-2}) \\ & \quad - \partial_{\bar{x}} I_j(x''_{n_j}, sx''_{n_j-1} + (1-s)x''_{n_j}) f((n_j-1)h, x''_{n_j-1}, x''_{n_j-2})\big| \\ \le \; & \big|\partial_{\bar{x}} I_j(x'_{n_j}, sx'_{n_j-1} + (1-s)x'_{n_j}) - \partial_{\bar{x}} I_j(x''_{n_j}, sx''_{n_j-1} + (1-s)x''_{n_j})\big| \\ & \qquad \times \big|f((n_j-1)h, x'_{n_j-1}, x'_{n_j-2})\big| \\ + \; & \big|\partial_{\bar{x}} I_j(x''_{n_j}, sx''_{n_j-1} + (1-s)x''_{n_j})\big| \cdot \big|f((n_j-1)h, x'_{n_j-1}, x'_{n_j-2}) \\ & \qquad - f((n_j-1)h, x''_{n_j-1}, x''_{n_j-2})\big| \\ \le \; & LM_0\big(|y'_{n_j} - y''_{n_j}| + s|y'_{n_j-1} - y''_{n_j-1}| + (1-s)|y'_{n_j} - y''_{n_j}|\big) \\ + \; & M_1^2\big(|y'_{n_j-1} - y''_{n_j-1}| + |y'_{n_j-2} - y''_{n_j-2}|\big) \end{aligned}$$

and thus

$$\|\mathcal{S}_4 y' - \mathcal{S}_4 y''\| \le 2m\mathcal{M}(LM_0 + M_1^2)h\|y' - y''\|. \tag{27}$$

Similarly, in order to estimate $\mathcal{S}_2 y' - \mathcal{S}_2 y''$ we use the representation (17) with (18) or (20). If $\nu - 1 \neq n_j$ for any $j = \overline{1,m}$, then

$$|\delta f(\nu h, x'_\nu, x'_{\nu-1}) - \delta f(\nu h, x''_\nu, x''_{\nu-1})| \le 2(LM_0 + M_1^2)h\|y' - y''\|.$$

If, however, $\nu - 1 = n_j$ for some $j \in \{1, \dots, m\}$, we have

$$|\delta f(\nu h, x'_\nu, x'_{\nu-1}) - \delta f(\nu h, x''_\nu, x''_{\nu-1})| \le 2(LM_0 + M_1^2)\|y' - y''\|,$$

and as above we obtain

$$\|\mathcal{S}_2 y' - \mathcal{S}_2 y''\| \le 2[(N-2m)h + m]\mathcal{M}(LM_0 + M_1^2)\|y' - y''\|. \tag{28}$$

From (27) and (28) we obtain

$$\begin{aligned} &\big\| h\big(\mathcal{S}_2 y' - \mathcal{S}_2 y''\big) + \big(\mathcal{S}_4 y' - \mathcal{S}_4 y''\big)\big\| \\ \le\ & 2\mathcal{M}(LM_0 + M_1^2)\big[Nh + (1-h)m\big]h\|y' - y''\| \\ <\ & 2\mathcal{M}(LM_0 + M_1^2)(\omega + m)h\|y' - y''\|. \end{aligned}$$

Choose an arbitrary number $q \in (0,1)$. By virtue of (26) we can find $\mu_1 \in (0, \tilde{\mu}_0]$ so that for any $\mu \in (0, \mu_1]$ we have

$$\big\| h\big(\mathcal{S}_1 y' - \mathcal{S}_1 y''\big) + \big(\mathcal{S}_3 y' - \mathcal{S}_3 y''\big)\big\| \le \frac{q}{2}\|y' - y''\|.$$

Next we find $N_1 \ge N(\mu_1)$ so that for any $N \in \{N_k\}_{k\in\mathbb{N}},\ N \ge N_1$ and $h = \omega/N$ we have

$$\big\| h\big(\mathcal{S}_2 y' - \mathcal{S}_2 y''\big) + \big(\mathcal{S}_4 y' - \mathcal{S}_4 y''\big)\big\| \le \frac{q}{2}\|y' - y''\|.$$

Then for any $\mu \in (0, \mu_1]$ and $N \in \{N_k\}_{k\in\mathbb{N}},\ N \ge N_1$ and $h = \omega/N$, the estimate

$$\|\mathcal{U}_h y' - \mathcal{U}_h y''\| \le q\|y' - y''\|, \qquad q \in (0,1),$$

is valid for any $y', y'' \in \mathcal{T}_\mu$.

Thus the operator $\mathcal{U}_h$ has a unique fixed point in $\mathcal{T}_\mu$, which is an N-periodic solution $\{y_i\}_{i\in\mathbb{Z}},\ y_i \equiv y_i(h)$, of system (12). Now $\{x_i\}_{i\in\mathbb{Z}},\ x_i = x_i(h) = \psi_i + y_i(h)$, is the unique N-periodic solution of system (4). For any $\mu > 0$ we can choose N so large that for $h = \omega/N,\ i = \overline{0, N-1}$, we have $|x_i - \psi_i| = |y_i| \le \mu$, *i.e.*, $\lim\limits_{N\to\infty} \max\limits_{0\le i\le N-1} |x_i - \psi_i| = 0$. This completes the proof of Theorem 1.

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References

[1] Bainov D. D. and Covachev V., *Impulsive Differential Equations with a Small Parameter*, Series on Advances in Mathematics for Applied Sciences, **24**, World Scientific, Singapore, 1994.

[2] Bainov D., Covachev V. and Stamova I., *Stability under persistent disturbances of impulsive differential–difference equations of neutral type*, J. Math. Anal. Appl., **187** (1994), 799–808.

[3] Covachev V., Akça H. and Yeniçerioğlu F., *Difference approximations for impulsive differential equations*, Applied Mathematics and Computation, **121** (2001), 383–390.

[4] Lakshmikantham V., Bainov D. D. and Simeonov P. S., *Theory of Impulsive Differential Equations*, Series in Modern Applied Mathematics, **6**, World Scientific, Singapore, 1989.