

## Considerations on the Study of the Errors in the Mathematical Model of the Lag Propagation

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### Abstract

The aim of this paper is to discuss some methods of parameter estimation, such as the direct use of the instrumental variables and an extension of Aitken's method with the least square method, applied to the multiplicative version of some model.

Let us consider the model

$$y_t = \frac{\alpha I}{I - \lambda L} x_t + u_t, \quad t = \overline{1, T}, \quad (1)$$

where  $x_t$  is an independent variable in the error process and

$$u \sim N(0, \sigma^2 I), \quad u = (u_1, u_2, \dots, u_T)^t. \quad (2)$$

We want to present a method of parameter estimation for (1), (2).

First, we shall reduce the model to

$$\begin{cases} y_t = \alpha x_t + \lambda y_{t-1} + w_t, \\ w_t = u_t - \lambda u_{t-1}, \end{cases} \quad t = \overline{2, T}. \quad (3)$$

If we deal with (3), we shall lose an observation. The simplest choice of the instrumental variable of estimation is  $x_t, x_{t-1}$ .

The instrumental equations to be solved are

$$\begin{cases} \alpha \sum_{t=2}^T x_t^2 + \lambda \sum_{t=2}^T x_t y_{t-1} = \sum_{t=2}^T x_t y_t, \\ \alpha \sum_{t=2}^T x_{t-1} x_t + \lambda \sum_{t=2}^T x_{t-1} y_{t-1} = \sum_{t=2}^T x_{t-1} y_t. \end{cases}$$

Solving this system and replacing in (3), we obtain

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\lambda} \end{pmatrix} = \begin{pmatrix} \alpha \\ \lambda \end{pmatrix} + \begin{pmatrix} \sum_{t=2}^T x_t^2 & \sum_{t=2}^T x_t y_{t-1} \\ \sum_{t=2}^T x_{t-1} x_t & \sum_{t=2}^T x_{t-1} y_{t-1} \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=2}^T x_t w_t \\ \sum_{t=2}^T x_{t-1} w_t \end{pmatrix} \iff \quad (4)$$

$$\sqrt{T} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\lambda} \end{bmatrix} - \begin{bmatrix} \alpha \\ \lambda \end{bmatrix} = \left[ \frac{1}{T} \begin{pmatrix} \sum_{t=2}^T x_t^2 & \sum_{t=2}^T x_t y_{t-1} \\ \sum_{t=2}^T x_{t-1} x_t & \sum_{t=2}^T x_{t-1} y_{t-1} \end{pmatrix} \right]^{-1} \cdot \frac{1}{\sqrt{T}} \sum_{t=2}^T \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} w_t. \tag{4'}$$

If we suppose that the explanatory variable is not random and the probability limit of the matrix that lies in the second member of (4), multiplied by  $T$ , is finite and is not stochastic, then these estimations are consistent. This is true because

$$p \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=i+1}^T x_{t-i} w_t = 0, \quad i = \overline{0, 1}.$$

In what follows we shall suppose that (2) is satisfied or  $u_t$  are independent variables, with the same finite variance  $\sigma^2$  and with finite absolute moments of the third order. We shall use the second assumption.

We shall need the next result [1]:

**Theorem 1** *Let  $(x_t)_{t \in \mathbb{N}^*}$  be a sequence of random  $m$ -dependent variables, with the expectation  $E(x_t) = 0$  and  $(\exists) K \in \mathbb{R}_+$  such that  $E(|x_t|^3) < K^3$ ,  $(\forall) t \in \mathbb{N}^*$ . Let us consider*

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n A_{i+r},$$

$$A_i = 2\text{Cov}(x_{i+j}, x_{i+m}) + \text{Var}(x_{i+m}),$$

where  $\text{Cov}(x_{i+j}, x_{i+m})$  is the covariance of  $x_{i+j}, x_{i+m}$  and  $\text{Var}(x_{i+m})$  is the variance of  $x_{i+m}$ .

If  $[\text{Cov}(x_{i+j}, x_{i+m}) + \text{Var}(x_{i+m})]$  is uniformly convergent with respect to  $i$ , then

the asymptotic distribution of  $\frac{\sum_{t=1}^T x_t}{\sqrt{T}}$  is  $N(0, \sigma^2)$ .

Let us consider the sequence of scalars

$$s_t = (a_1 x_t + a_2 x_{t-1}) w_t.$$

We have to calculate the quantity

$$R_i = 2 \sum_{i=1}^{m-1} \text{Cov}(s_{i+1}, s_i) + \text{Var}(s_{i+m})$$

that satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n R_{i+r} = R < \infty \tag{5}$$

uniformly with respect to  $r$ .

In this case  $m = 1$  and (5) holds because

$$\begin{aligned} R_i &= 2(a_1x_i + a_2x_{i-1})(a_1x_{i+1} + a_2x_i)E(w_iw_{i+1}) + (a_1x_{i+1} + a_2x_i)^2E(w_{i+1}^2) \\ &= \sigma^2(1 + \lambda^2)a_1a_2X \cdot \\ &\cdot \begin{bmatrix} x_{i+1}^2 - \frac{2\lambda}{\lambda^2+1}x_ix_{i+1} & x'x_{-1} - \frac{\lambda}{\lambda^2+1}(x'x + x'x_{-1}) \\ x_ix_{i+1} - \frac{\lambda}{\lambda^2+1}(x'x + x'x_{-2}) & x'x - \frac{2\lambda}{\lambda^2+1}x_ix_{i+1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \end{aligned}$$

It results that, asymptotically:

$$\frac{1}{\sqrt{T}} \sum_{t=2}^T \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} w_t \sim N(0, \sigma^2 B_t),$$

where

$$B_t = (1 + \lambda^2) \cdot$$

$$\cdot \lim_{t \rightarrow \infty} \frac{1}{T} \begin{bmatrix} x'x - \frac{2\lambda}{\lambda^2+1}x'x_{-1} & x_ix_{i+1} - \frac{\lambda}{\lambda^2+1}(x_i^2 + x_{i+1}x_{i-1}) \\ x'x_{i-1} - \frac{\lambda}{\lambda^2+1}(x_i^2 + x_{i+1}x_{i-1}) & x_i^2 - \frac{2\lambda}{\lambda^2+1}x'x_{-1} \end{bmatrix},$$

$$x' = (x_2, \dots, x_T)^t, \quad x_{-1} = (x_1, x_2, \dots, x_{T-1})^t, \quad x_{-2} = (0, x_1, x_2, \dots, x_{T-2})^t.$$

In order to finish we need the probability limit in (4').

It can be seen that

$$\sum_{t=2}^T x_t y_{t-1} = \sum_{t=2}^T x_t (\alpha x_{t-1}^* + u_{t-1})$$

because

$$y_{t-1} = \frac{\alpha I}{I - \lambda L} x_{t-1} + u_{t-1} = \alpha x_{t-1}^* + u_{t-1},$$

where

$$x^* = (x_1^*, \dots, x_T^*) \quad \text{and} \quad x_T^* = \sum_{i=0}^{t-1} \lambda_i x_{t-i}.$$

Since

$$p \lim_{T \rightarrow \infty} \sum_{t=2}^T x_t u_{t-1} = 0,$$

we deduce that

$$p \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=2}^T x_t u_{t-1} = \alpha \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=2}^T x_t x_{t-1}^*$$

and using

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=2}^T x_t x_{t-1}^* = \lim_{T \rightarrow \infty} \frac{1}{T} x' D x_{-1},$$

where

$$D = \begin{vmatrix} 1 & 0 & 0 & \cdots & \cdots \\ \lambda & 1 & 0 & \cdots & \cdots \\ \lambda^2 & \lambda & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda^{T-1} & \lambda^{T-2} & \lambda^{T-3} & \cdots & 1 \end{vmatrix}.$$

So, asymptotically

$$\sqrt{T} \left[ \begin{pmatrix} \tilde{\alpha} \\ \tilde{\lambda} \end{pmatrix} - \begin{pmatrix} \alpha \\ \lambda \end{pmatrix} \right] \sim N(0, C_1),$$

where

$$C_1 = \sigma^2 A_1^{-1} B_1 (A_1^T)^{-1}.$$

We discuss about the estimator proposed by Koyck.

We reduce the model and we obtain

$$y_t = \lambda y_{t-1} + \alpha x_t + (u_t - \lambda u_{t-1}). \tag{6}$$

Koyck proposed to obtain the OLS estimator for  $\alpha$  and  $\lambda$  from the previous relation and to solve the system

$$\begin{cases} \alpha \sum_{t=2}^T x_t^2 + \lambda \sum_{t=2}^T x_t y_{t-1} = \sum_{t=2}^T x_t y_t, \\ \alpha \sum_{t=2}^T x_{t-1} x_t + \lambda \sum_{t=2}^T y_{t-1}^2 = \sum_{t=2}^T y_{t-1} y_t + \frac{\lambda \tilde{w}' \tilde{w}}{1 + \lambda} \end{cases} \tag{7}$$

in order to give the final estimation for  $\alpha$  and  $\lambda$ .

We shall deduce these estimations in a different way.

The covariance matrix of the error terms from (6) is

$$\text{Cov}(w) = \sigma^2 \begin{vmatrix} 1 + \lambda^2 & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1 + \lambda^2 & -\lambda & \cdots & 0 \\ & -\lambda & 1 + \lambda^2 & \cdots & 0 \\ \cdots & & & \cdots & \cdots \\ 0 & 0 & \cdots & -\lambda & 1 + \lambda^2 \end{vmatrix} = \sigma^2 \Phi,$$

where

$$w = (w_2, w_3, \dots, w_T)^t, \quad w_t = u_t - \lambda u_{t-1}, \quad t = \overline{2, T}.$$

Minimizing with respect to  $\alpha$  and  $\lambda$  the quantity

$$Q = (y - \alpha x - \lambda y_{-1})^T \Phi (y - \alpha x - \lambda y_{-1}), \quad (8)$$

we obtain the minimum estimation,  $\chi^2$ , of the parameters from (6).

In (8) we denote

$$y = (y_2, \dots, y_T)^t, \quad x = (x_2, \dots, x_T)^t, \quad y_{-1} = (y_2, \dots, y_{T-1})^t.$$

Since the inverse is difficult to obtain and  $w_t$  are 1-dependent variables, if we neglect the dependence, then  $\text{Cov}(w)$  can be written as

$$\text{Cov}(w) = \sigma^2(1 + \lambda^2)I.$$

Replacing in (8) and minimizing with respect to  $\alpha$  and  $\lambda$ , it results:

$$Q = \frac{1}{1 + \lambda^2} (y - \alpha x - \lambda y_{-1})^T (y - \alpha x - \lambda y_{-1}), \quad (9)$$

$$\begin{cases} \alpha x'x + \lambda x'y_{-1} = x'y, \\ \alpha y_{-1}x + \lambda y'_{-1}y_{-1} = y'_{-1}y_{-1} + \frac{\lambda}{1+\lambda} w'w. \end{cases} \quad (10)$$

It can be seen that we have the same result as in (7).

Solving (10) we obtain

$$\alpha = \frac{x'y - \lambda x'_1 y_{-1}}{x'x}$$

and

$$\begin{aligned} \lambda^2 [(x'x)(y'_{-1}y_{-1}) - (x'y_{-1})(x'y)] + \lambda [(y'_{-1}y_{-1} - y'y)x'x + (x'y)^2 - (x'y_{-1})^2] \\ - [(x'x)(y'_{-1}y_{-1}) - (x'y_{-1})(x'y)] = 0. \end{aligned} \quad (11)$$

The estimation of  $\lambda$  is considered to be the smallest root in (11) and will be denoted by  $\tilde{\lambda}$ . Then the estimation of  $\alpha$  will be:

$$\alpha = \frac{x'y - \tilde{\lambda} x'_1 y_{-1}}{x'x}. \quad (12)$$

Now we prove that the solutions of (11) and (12) give consistent estimations for  $\alpha$  and  $\lambda$ .

We write (11) in standard form:

$$\lambda^2 + b\lambda - 1 = 0,$$

where

$$b = \frac{(y'_{-1}y_{-1} - y'y)x'x + (x'y)^2 - (x'y_{-1})^2}{(x'x)(y'_{-1}y_{-1}) - (x'y_{-1})(x'y)}.$$

The solutions are

$$\tilde{\lambda} = \frac{-b \pm \sqrt{b^2 + 4}}{2}$$

and

$$\begin{aligned} p \lim_{T \rightarrow \infty} \tilde{\lambda} &= \frac{-p \lim b \pm \sqrt{p \lim b^2 + 4}}{2} \Leftrightarrow \\ p \lim_{T \rightarrow \infty} \tilde{\lambda} &= p \lim_{T \rightarrow \infty} \frac{(x'y)^2 - (x'y_{-1})^2}{(x'x)(y'_{-1}y_{-1}) - (x'y_{-1})(x'y)} \end{aligned} \quad (13)$$

is the probable limit of the two roots.

The previous relation holds because

$$y'_{-1}y_{-1} - y'y = y_1^2 - y_T^2.$$

In order to determine the limit (13) we suppose that the sequence  $(x_t)_{t \in \mathbb{Z}}$  satisfies the conditions:

- there exist and are finite the limits:

$$c_\tau = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{i=1}^T x_i x_{i-\tau},$$

at least one is nonvanishing and  $c_\tau = c_{-\tau}, (\forall) \tau$ .

Let us consider

$$p \lim_{T \rightarrow \infty} \frac{1}{T} x'y = p \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{i=0}^{\infty} \lambda^i \sum_{t=2}^T x_t x_{t-i} + p \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=2}^T x_t u_t. \quad (14)$$

Since  $x_t$  and  $u_t$  are independent, the second term in the right-hand side of (14) is zero and

$$p \lim_{T \rightarrow \infty} \frac{1}{T} x'y = \alpha \sigma_0,$$

where

$$\sigma_0 = \sum_{i=0}^{\infty} \lambda^i c_i.$$

We observe that

$$\begin{aligned} x'y &= \alpha \sum_{i=0}^{\infty} \lambda^i \sum_{t=2}^T x_t x_{t-i-1} + \sum_{t=2}^T x_t x_{t-1} \Rightarrow \\ p \lim_{T \rightarrow \infty} \frac{1}{T} x'y_{-1} &= \sum_{i=0}^{\infty} \lambda^i c_{i+1} = \frac{\alpha}{\lambda} \sum_{i=0}^{\infty} \lambda^{i+1} c_{i+1} = \frac{\alpha}{\lambda} (\sigma_0 - c_0). \end{aligned}$$

But, by definition:

$$\lim_{T \rightarrow \infty} \frac{1}{T} x'x = c_0.$$

Finally, to evaluate  $y'_{-1}y$ , we introduce the notation

$$\tilde{y}_t = \frac{\alpha I}{I - \lambda L} x_t.$$

Then:

$$\begin{aligned} y_t &= \tilde{y}_t + u_t, \\ y'_{-1}y &= \sum_{t=2}^T \tilde{y}_t \tilde{y}_{t-1} + \sum_{t=2}^T \tilde{y}_t u_{t-1} + \sum_{t=2}^T \tilde{y}_{t-1} u_t + \sum_{t=2}^T u_t u_{t-1}. \end{aligned}$$

Since  $x_t, u_t$  are independent and  $\text{Cov}(u_t, u_{t-1}) = 0$ , then

$$p \lim_{T \rightarrow \infty} \frac{1}{T} y_{-1}y = \lim_{T \rightarrow \infty} \frac{1}{T} y'_{-1}y.$$

To evaluate this limit, we remark that

$$\begin{aligned} \tilde{y} &= \alpha x + \lambda \tilde{y}_{-1}, \\ y'_{-1}y &= \alpha x' \tilde{y}_{-1} + \lambda \tilde{y}'_{-1} \tilde{y}_{-1}, \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{1}{T} \tilde{y}' \tilde{y} &= \alpha^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \lambda^i \lambda^j \frac{1}{T} \sum_{t=1}^T x_{t-i} x_{t-j} \Rightarrow \\ \lim_{T \rightarrow \infty} \frac{1}{T} \tilde{y}' \tilde{y} &= \alpha^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \lambda^i \lambda^j c_{i-j}. \end{aligned}$$

Adding all the terms that contain  $\sigma_0$ , we obtain

$$\sigma_0 \sum_{i=0}^{\infty} \lambda^{2i} = \frac{\sigma_0}{1 - \lambda^2}.$$

Therefore,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \tilde{y}' \tilde{y} = \alpha^2 \left[ \frac{\sigma_0}{1 - \lambda^2} + \frac{1}{1 - \lambda^2} \sum_{j=1}^{\infty} \lambda^j c_j \right] = \frac{\alpha^2}{1 - \lambda^2} (2\sigma_0 - c_0).$$

But,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \tilde{y}'_{-1} \tilde{y}_{-1} = \lim_{T \rightarrow \infty} \frac{1}{T} \tilde{y}' \tilde{y}$$

and using (15), it can be deduced that

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \tilde{y}'_{-1} \tilde{y} &= \alpha \lim_{T \rightarrow \infty} \frac{1}{T} x' \tilde{y}'_{-1} + \lambda \lim_{T \rightarrow \infty} \frac{1}{T} \tilde{y}'_{-1} \tilde{y}_{-1} \\ &= \frac{\alpha^2}{\lambda} (\sigma_0 - c_0) + \frac{\lambda \alpha^2}{1 - \lambda^2} (2\sigma_0 - c_0) \Rightarrow \\ \lim_{T \rightarrow \infty} \frac{1}{T} \tilde{y}'_{-1} \tilde{y} &= \frac{\alpha^2}{\lambda(1 - \lambda^2)} (\sigma_0 - c_0 + \lambda^2 \sigma_0) \Rightarrow \\ p \lim_{T \rightarrow \infty} b &= \frac{1 - \lambda^2}{\lambda} \Rightarrow \\ p \lim_{T \rightarrow \infty} \tilde{\lambda} &= \frac{-(1 - \lambda^2) \pm (1 + \lambda^2)}{2\lambda} \in \left\{ \lambda, -\frac{1}{\lambda} \right\}. \end{aligned}$$

Therefore, if we choose the root with the least modulus in (11), we obtain a consistent estimation for  $\lambda$  since  $|\lambda| < 1$ .

From (12) it results that

$$p \lim_{T \rightarrow \infty} \tilde{\alpha} = \frac{\alpha \sigma_0 - \alpha (\sigma_0 - c_0)}{c_0} = \alpha,$$

*i.e.*,  $\tilde{\alpha}$  is a consistent estimation for  $\alpha$ .

**Remark 1** If  $b > 0$ , then  $\left| \tilde{\lambda}^+ \right| = \left| \frac{-b + \sqrt{b^2 + 4}}{2} \right| < 1$ .

If  $b < 0$ , then  $\left| \tilde{\lambda}^- \right| = \left| \frac{-b - \sqrt{b^2 + 4}}{2} \right| < 1$ .

If we consider  $\lambda \in (0, 1)$ , then we have also to prove that at least one root is in the interval  $(0, 1)$ .



## References

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