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# Approximating Fixed Points of Asymptotically Quasi-Nonexpansive Mappings by the Iterative Sequences with Errors

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**Abstract**

In this paper, by using the new analysis techniques, we prove some strong convergence theorems of the modified Ishikawa and Mann iterative sequences with errors for a class of uniformly $\phi-$continuous and asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces. Our results improve and gerneralize the recent results proved by Rhoades, Schu, Tan and Xu, Xu and Noor and many others.

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**Key words:** Uniformly convex Banach space, Asymptotically nonexpansive and quasi-nonexpansive mappings, The modified Ishikawa iterative sequence with errors, Uniformly $\phi-$continuous mapping, Uniformly Hölder continuous mapping.

Let $C$ be a nonempty subset of a real normed linear space $X$. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \geq 1$ for all $n \geq 1$ and $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

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for all $x, y \in C$ and $n \geq 1$. This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972. They proved that, if $C$ is a nonempty bounded closed convex subset of a uniformly convex Banach space $X$, then every asymptotically nonexpansive self-mapping $T$ of $C$ has a fixed point. Moreover, the set $F(T)$ of fixed points of $T$ is closed and convex. Since 1972, many authors have studied weak and strong convergence problems of the iterative sequences (with errors) for asymptotically nonexpansive mapping types in Hilbert spaces and Banach spaces.

The mapping $T$ is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists $k_n \in [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - p\| \leq k_n \|x - p\|$$

for all $x \in C,\ p \in F(T)$ and $n \geq 1$. The mapping $T$ is said to be *uniformly $\phi-$continuous* if there exists a real function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) \to 0$ as $t \to 0^+$ such that

$$\|T^n x - T^n y\| \leq \phi(\|x - y\|)$$

for all $x,\ y \in C$ and $n \geq 1$. For further details on some kinds of asymptotically nonexpansive mapping types, refer to [1].

**Remark 1** If $T : C \to C$ is uniformly Hölder continuous, *i.e.*, there exist constants $L > 0$ and $\alpha > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|^\alpha$$

for all $x,\ y \in C$ and $n \geq 1$, then it is uniformly $\phi-$continuous, but the converse is not true. In particular, if $T$ is uniformly $L-$Lipschitzian, it is uniformly Hölder continuous with constants $L > 0$ and $\alpha = 1$.

In 1991, Schu [6, 7] introduced the following iterative sequences: Let $X$ be a normed linear space, $C$ be a nonempty convex subset of $X$ and $T : C \to C$ be a given mapping. Then, for arbitrary $x_1 \in C$ the *modified Ishikawa iterative sequence* $\{x_n\}$ is defined by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, & n \geq 1, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, & n \geq 1, \end{cases} \tag{A}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0, 1]$. With $X,\ C,\ \{\alpha_n\}$ and $x_1$ as above, the *modified Mann iterative sequence* $\{x_n\}$ is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1. \tag{B}$$

Schu [6, 7] established weak and strong convergence theorems of the modified Ishikawa and Mann sequences for asymptotically nonexpansive mappings in Hilbert spaces. In particular, he proved the following strong convergence results:

**Theorem S1** ([6]) *Let $H$ be a Hilbert space and $K$ be a nonempty closed convex and bounded subset of $H$. Let $T : K \to K$ be a completely continuous and asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ for all $n \geq 1$, $k_n \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying the condition $\epsilon \leq \alpha_n \leq (1 - \epsilon)$ for some $\epsilon > 0$. Then the sequence $\{x_n\}$ defined by $(A)$ converges strongly to some fixed point of $T$.*

**Theorem S2** ([7]) *Let $H$ be a Hilbert space, $K$ a nonempty closed convex and bounded subset of $H$. Let $T : K \to K$ be a completely continuous and asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ for all $n \geq 1$, $q_n = 2k_n - 1$ satisfying the condition $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ with $\epsilon \leq \alpha_n \leq \beta_n \leq b$, $n \geq 1$, for some $\epsilon > 0$ and $b \in (0, L^{-2}\{(1+L^2)^{1/2} - 1\})$. Then the sequence $\{x_n\}$ defined by $(B)$ converges strongly to a fixed point of $T$.*

In 1994, Tan and Xu [11] extended Theorem S1 and Theorem S2 by establishing weak convergence theorems of the modified Mann and Ishikawa iteration processes for asymptotically nonexpansive mappings in uniformly convex Banach spaces. By virtue of Xu's inequality [9], Rhoades [5] extended also Theorem S1 to more general uniformly convex Banach spaces. Very recently, Liu [3] extended some results of Tan and Xu [11] to uniformly quasi-nonexpansive and uniformly Hölder continuous mappings. Namely, Liu [3] proved the following:

**Theorem L** ([3]) *Let $E$ be a nonempty compact subset of a uniformly convex Banach space $X$ and $T : E \to E$ be a uniformly Hölder continuous and asymptotically quasi-nonexpansive mapping with respect to a sequence $\{k_n\}$ with $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. For an arbitrary initial value $x_1 \in E$, define the modified Ishikawa iterative sequence by*

$$\begin{cases} y_n = \bar{a}_n x_n + \bar{b}_n T^n x_n + \bar{c}_n v_n, & n \geq 1, \\ x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n, & n \geq 1, \end{cases} \tag{C}$$

*where $\{u_n\}$, $\{v_n\}$ are bounded sequences in $E$ and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a}_n\}$, $\{\bar{b}_n\}$, $\{\bar{c}_n\}$ are sequences in $[0, 1]$ with $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1$,*

(i) $0 < \alpha \leq a_n \leq \bar{\alpha} < 1, \quad 0 < \alpha \leq \bar{a}_n,$

(ii) $0 < \bar{\beta} \leq b_n \leq \beta < 1, \quad \bar{b}_n \leq \beta < 1, \ \ n \geq 1,$

(iii) $\lim_{n\to\infty} \bar{b}_n = 0, \quad \sum_{n=1}^{\infty} c_n < \infty, \quad \sum_{n=1}^{\infty} \bar{c}_n < \infty.$

*Then the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.*

It is our purpose in this paper to extend and improve Theorem L to uniformly $\phi-$continuous and asymptotically quasi-nonexpansive mappings and, moreover, to relax the iterative parameters in our results. Also our results improve and gerneralize Theorems S1, S2 and the recent results announced by Rhoades [5], Tan and Xu [11], Xu and Noor [12] and many others. Our arguments will involve certain new ideas and techniques which are quite different from those methods that already existed in the known literature.

In order to prove the main results in this paper, we need the following lemmas:

**Lemma 1** ([1, 10]) *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n.$$

*If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. Further, if $\{a_n\}$ has a subsequence converging to 0, then $\lim_{n\to\infty} a_n = 0$.*

**Lemma 2** *Let $X$ be a normed linear space and $C$ be a nonempty convex subset of $X$. Let $T : C \to C$ be an asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{x_n\}$ be a sequence in $C$ defined by $(C)$ with the conditions $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} \bar{c}_n < \infty$. Then we have the following:*

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists for any $p \in F(T)$,*

(ii) $\lim_{n\to\infty} d(x_n, F(T))$ *exists, where $d(x, F(T))$ denotes the distance from $x$ to the set $F(T)$.*

**Proof.** Set $M = \max\{\sup\{\|u_n - p\| : n \geq 1\}, \sup\{\|v_n - p\| : n \geq 1\}\}$. Then it follows from $(C)$ that, for any given $p \in F(T)$,

$$\begin{aligned}
\|x_{n+1} - p\| &\leq a_n\|x_n - p\| + b_n k_n\|y_n - p\| + c_n\|u_n - p\| \\
&\leq (1-b_n)\|x_n - p\| + b_n k_n^2\|x_n - p\| + M(c_n + \bar{c}_n) \\
&\leq [1 + (k_n^2 - 1)]\|x_n - p\| + M(c_n + \bar{c}_n).
\end{aligned}$$

Consequently, the conclusions of the lemma follows directly from Lemma 1. This completes the proof.

**Lemma 3** ([9]) *Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space $X$ is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x-y\|)$$

*for all $x,\ y \in B_r(0) = \{x \in X : \|x\| \leq r\}$ and all $\lambda \in [0,1]$, where $w_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$.*

As an immediate consequence of Lemma 3, we have the following:

**Lemma 4** *Let $X$ be a uniformly convex Banach space and $B_r(0)$ be a closed ball of $X$. Then there exists a continuous increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x-y\|)$$

*for all $x,\ y \in B_r(0)$ and $\lambda,\ \mu,\ \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.*

**Proof.** We first observe that $\frac{\lambda}{1-\gamma}x + \frac{\mu}{1-\gamma}y \in B_r(0)$ whenever $x,\ y \in B_r(0)$ and $\lambda,\ \mu,\ \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$. It follows from Lemma 3 that

$$\begin{aligned}
&\|\lambda x + \mu y + \gamma z\|^2 \\
= \ &\|(1-\gamma)[\frac{\lambda}{1-\gamma}x + \frac{\mu}{1-\gamma}y] + \gamma z\|^2 \\
\leq \ &(1-\gamma)\|\frac{\lambda}{1-\gamma}x + \frac{\mu}{1-\gamma}y\|^2 + \gamma\|z\|^2 - w_2(\gamma) \\
\leq \ &\lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - (1-\gamma)w_2(\frac{\lambda}{1-\gamma})g(\|x-y\|) \\
\leq \ &\lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \frac{\lambda\mu}{1-\gamma}g(\|x-y\|) \\
\leq \ &\lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x-y\|)
\end{aligned}$$

since $\frac{1}{1-\gamma} \geq 1$. This completes the proof.

Now, we give the main results in this paper:

**Theorem 5** *Let $X$ be a uniformly convex Banach space, $C$ be a nonempty convex subset of $X$ and $T : C \to C$ be a uniformly $\phi-$continuous and asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{x_n\}$ be a sequence in $C$ defined by (C) with the following conditions:*

(i) $0 \le b_n \le b < 1$ *and* $b_{n+1} \le b_n$ *for all* $n \ge 1$,

(ii) $\sum_{n=1}^{\infty} b_n = \infty$,

(iii) $\lim_{n\to\infty} \bar{b}_n = 0$,

(iv) $\sum_{n=1}^{\infty} c_n < \infty$ *and* $\sum_{n=1}^{\infty} \bar{c}_n < \infty$.

*Then we have*

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0.$$

**Proof.** Since $T : C \to C$ is asymptotically quasi-nonexpansive, we have

$$\|T^n y_n - p\| \le k_n\|y_n - p\| \le k_n^2\|x_n - p\|$$

for any $p \in F(T)$. By Lemma 2, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists. Hence $\{x_n - p\}$ and $\{T^n y_n - p\}$ are all bounded sequences in $X$. Set

$$\begin{aligned} r &= \max\{\sup\{\|x_n - p\| : n \ge 1\},\ \sup\{T^n y_n - p\| : n \ge 1\}, \\ &\qquad \sup\{\|u_n - p\| : n \ge 1,\ \sup\{\|v_n - p\| : n \ge 1\}\} \end{aligned}$$

for any fixed $p \in F(T)$. Then we have $\{x_n - p\},\ \{T^n y_n - p\},\ \{u_n - p\} \subset B_r(0)$ for all $n \ge 1$. By using Lemma 4 and $(C)$, we have

$$\begin{aligned} \|y_n - p\|^2 &= \|\bar{a}_n(x_n - p) + \bar{b}_n(T^n x_n - p) + \bar{c}_n(v_n - p)\|^2 \\ &\le \bar{a}_n\|x_n - p\|^2 + \bar{b}_n\|T^n x_n - p\|^2 + \bar{c}_n\|v_n - p\|^2 \\ &\qquad -\bar{a}_n\bar{b}_n g(\|x_n - T^n x_n\|) \\ &\le (1 - \bar{b}_n - \bar{c}_n)\|x_n - p\|^2 + \bar{b}_n k_n^2\|x_n - p\|^2 + r^2\bar{c}_n \qquad (1) \\ &\le [1 + \bar{b}_n(k_n^2 - 1)]\|x_n - p\|^2 + r^2\bar{c}_n \\ &\le k_n^2\|x_n - p\|^2 + r^2\bar{c}_n. \end{aligned}$$

Again, using Lemma 4 and (C), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|a_n(x_n - p) + b_n(T^n y_n - p) + c_n(u_n - p)\|^2 \\ &\le a_n\|x_n - p\|^2 + b_n\|T^n y_n - p\|^2 + c_n\|u_n - p\|^2 \\ &\qquad -a_n b_n g(\|x_n - T^n y_n\|) \\ &\le (1 - b_n - c_n)\|x_n - p\|^2 + b_n k_n^2\|y_n - p\|^2 + r^2 c_n \qquad (2) \\ &\qquad -a_n b_n g(\|x_n - T^n y_n\|) \\ &\le (1 - b_n)\|x_n - p\|^2 + b_n k_n^2\|y_n - p\|^2 + r^2 c_n \\ &\qquad -a_n b_n g(\|x_n - T^n y_n\|). \end{aligned}$$

Substituting (1) into (2) yields

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-b_n+b_nk_n^4)\|x_n-p\|^2+r^2b_nk_n^2\bar{c_n}+r^2c_n\\ &\quad -a_nb_ng(\|x_n-T^ny_n\|)\\ &\le [1+b_n(k_n^4-1)]\|x_n-p\|^2+r^2(k_n^2\bar{c_n}+c_n)\\ &\quad -a_nb_ng(\|x_n-T^ny_n\|)\\ &\le [1+(k_n^4-1)]\|x_n-p\|^2+r^2(k_n^2\bar{c_n}+c_n)\\ &\quad -a_nb_ng(\|x_n-T^ny_n\|).\end{aligned} \tag{3}$$

Note that $\sum_{n=1}^{\infty}(k_n^2-1)<\infty$ is equivalent to $\sum_{n=1}^{\infty}(k_n^4-1)<\infty$ and so, setting $d_n=r^2(k_n^4-1)$, then $\sum_{n=1}^{\infty}d_n<\infty$. Furthermore, since $g:[0,\infty)\to[0,\infty)$ is a continuous increasing convex function and $\{x_n-T^ny_n\}$ is a bounded sequence in $X$, we assert that $g(\|x_n-T^ny_n\|)$ is bounded. Thus, by setting $e_n=c_ng(\|x_n-T^ny_n\|)$, we have $\sum_{n=1}^{\infty}e_n<\infty$. Since $\{k_n\}$ is bounded, we have also $\sum_{n=1}^{\infty}k_n^2\bar{c}_n<\infty$. Now, set

$$\gamma_n=d_n+e_n+r^2(k_n^2\bar{c}_n+c_n).$$

Then $\sum_{n=1}^{\infty}\gamma_n<\infty$. By the assumption (i), we have $(1-b_n)\ge(1-b)$. It follows from (3) that

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ \le\ &\|x_n-p\|^2-(1-b_n-c_n)b_ng(\|x_n-T^ny_n\|)\\ &\quad +d_n+r^2(k_n^2\bar{c}_n+c_n)\\ \le\ &\|x_n-p\|^2-(1-b_n)b_ng(\|x_n-T^ny_n\|)\\ &\quad +e_n+d_n+r^2(k_n^2\bar{c}_n+c_n)\\ \le\ &\|x_n-p\|^2-(1-b)b_ng(\|x_n-T^ny_n\|)+\gamma_n,\end{aligned} \tag{4}$$

which leads to

$$(1-b)b_ng(\|x_n-T^ny_n\|)\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\gamma_n \tag{5}$$

and

$$\begin{aligned}&(1-b)b_{n+1}g(\|x_{n+1}-T^{n+1}y_{n+1}\|)\\ \le\ &\|x_{n+1}-p\|^2-\|x_{n+2}-p\|^2+\gamma_{n+1}\end{aligned} \tag{6}$$

for all $n\ge1$. Adding on both sides of (5) and (6) and using the condition $b_{n+1}\le b_n$ for all $n\ge1$, then we have

$$(1-b)\sum_{n=1}^{\infty}b_{n+1}[g(\|x_{n+1}-T^{n+1}y_{n+1}\|)+g(\|x_n-T^ny_n\|)]<\infty.$$

Since $\sum_{n=1}^{\infty} b_n = \infty$ by the assumption (ii), we have

$$\liminf_{n\to\infty}[g(\|x_{n+1} - T^{n+1}y_{n+1}\|) + g(\|x_n - T^n y_n\|)] = 0.$$

By virtue of the continuity and monotonicity of function $g$, we assert that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$x_{n_j} - T^{n_j}y_{n_j} \to 0, \quad x_{n_j+1} - T^{n_j+1}y_{n_j+1} \to 0$$

as $j \to \infty$. By the assumption (iii), we see that

$$\|y_n - x_n\| \le \bar{b}_n\|x_n - T^n x_n\| + \bar{c}_n\|v_n - x_n\| \to 0$$

as $n \to \infty$ and hence $\phi(\|y_n - x_n\|) \to 0$ as $n \to \infty$. Observe that

$$\begin{aligned} \|x_n - T^n x_n\| &\le \|x_n - T^n y_n\| + \|T^n y_n - T^n x_n\| \\ &\le \|x_n - T^n y_n\| + \phi(\|y_n - x_n\|). \end{aligned}$$

It follows that $x_{n_j} - T^{n_j}x_{n_j} \to 0$ and $x_{n_j+1} - T^{n_j+1}x_{n_j+1} \to 0$ as $j \to \infty$. Since $T^{n_j}x_{n_j} - x_{n_j} \to 0$ as $j \to \infty$, we have $x_{n_j+1} - x_{n_j} = b_{n_j}(T^{n_j}x_{n_j} - x_{n_j}) \to 0$ as $j \to \infty$. Observe that

$$\begin{aligned} &\|T^{n_j}x_{n_j+1} - x_{n_j+1}\| \\ \le\ &\|T^{n_j}x_{n_j+1} - T^{n_j}x_{n_j}\| + \|T^{n_j}x_{n_j} - x_{n_j+1}\| \\ \le\ &\phi(\|x_{n_j+1} - x_{n_j}\|) + \|T^{n_j}x_{n_j} - x_{n_j}\| + \|x_{n_j+1} - x_{n_j}\| \\ &\to 0 \end{aligned} \tag{7}$$

as $j \to \infty$ and

$$\begin{aligned} &\|x_{n_j+1} - Tx_{n_j+1}\| \\ \le\ &\|x_{n_j+1} - T^{n_j+1}x_{n_j+1}\| + \|T^{n_j+1}x_{n_j+1} - Tx_{n_j+1}\| \\ \le\ &\|x_{n_j+1} - T^{n_j+1}x_{n_j+1}\| + \phi(\|T^{n_j}x_{n_j+1} - x_{n_j+1}\|). \end{aligned} \tag{8}$$

Therefore, it follows from (7) and (8) that $\|x_{n_j+1} - Tx_{n_j+1}\| \to 0$ as $j \to \infty$. This completes the proof.

Let $\{z_n\}$ be a given sequence in $C$. A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to satisfy Condition $(A)$ if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\|z_n - Tz_n\| \ge f(d(z_n, F(T)))$$

for all $n \ge 1$ (see [8]).

By using Theorem 5, we have the following:

**Theorem 6** *Let $X$ be a uniformly convex Banach space, $C$ be a nonempty closed convex subset of $X$ and $T : C \to C$ be a uniformly $\phi-$continuous and asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{x_n\}$ be a sequence in $C$ defined by $(C)$ with the following conditions:*

(i) $0 \leq b_n \leq b < 1$ *and* $b_{n+1} \leq b_n$ *for all* $n \geq 1$,

(ii) $\sum_{n=1}^{\infty} b_n = \infty$,

(iii) $\lim_{n\to\infty} \bar{b}_n = 0$,

(iv) $\sum_{n=1}^{\infty} c_n < \infty$ *and* $\sum_{n=1}^{\infty} \bar{c}_n < \infty$.

*If $T$ satisfies Condition $(A)$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p$ of $T$.*

**Proof.** It follows from Theorem 5 that

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0.$$

By Condition $(A)$, we have

$$\liminf_{n\to\infty} f(d(x_n, F(T))) = 0.$$

From the property of $f$, it follows that

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

It follows from Lemma 2 that $d(x_n, F(T)) \to 0$ as $n \to \infty$. Now, we can take an infinite subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset F(T)$ such that $\|x_{n_j} - p_j\| \leq 2^{-j}$. Set $M = \exp\{\sum_{n=1}^{\infty}(k_n^2 - 1)\}$ and write $n_{j+1} = n_j + l$ for some $l \geq 1$. Then we have

$$\begin{aligned} \|x_{n_{j+1}} - p_j\| &= \|x_{n_j+l} - p_j\| \\ &\leq [1 + (k_{n_j+l-1}^2 - 1)]\|x_{n_j+l-1} - p_j\| \\ &\leq \exp\Big\{\sum_{m=0}^{l-1}(k_{n_j+m}^2 - 1)\Big\}\|x_{n_j} - p_j\| \\ &\leq \frac{M}{2^j}. \end{aligned} \tag{9}$$

It follows from (9) that

$$\|p_{j+1} - p_j\| \leq \frac{2M+1}{2^{j+1}}.$$

Hence $\{p_j\}$ is a Cauchy sequence. Assume that $p_j \to p$ as $j \to \infty$. Then $p \in F(T)$ since $F(T)$ is closed, which implies that $x_j \to p$ as $j \to \infty$. This completes the proof.

**Remark 2** We note that, if $T : C \to C$ is completely continuous, then it must be demicompact ([7]), and, if $T$ is continuous and demicompact, it must satisfy Condition $(A)$ ([4, 8]). In view of this observation, Theorem 6 improves Theorem L in several aspects:

(i) $C$ may be not necessarily compact or bounded,

(ii) $T$ may be not uniformly Hölder continuous,

(iii) The iterative parameters in Theorem 6 are weaker than those of Theorem L.

As a corollary of Theorem 6, we have the following:

**Corollary 7** *Let $X$ be a uniformly convex Banach space, $C$ be a nonempty closed convex subset of $X$ and $T : C \to C$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{x_n\}$ be a sequence in $C$ defined by $(C)$ with the following conditions:*

(i) $0 \leq b_n \leq b < 1$ *and* $b_{n+1} \leq b_n$ *for all* $n \geq 1$*,*

(ii) $\sum_{n=1}^{\infty} b_n = \infty$*,*

(iii) $\lim_{n\to\infty} \bar{b}_n = 0$*,*

(iv) $\sum_{n=1}^{\infty} c_n < \infty$ *and* $\sum_{n=1}^{\infty} \bar{c}_n < \infty$*.*

*If $T$ satisfies Condition $(A)$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p$ of $T$.*

**Proof.** Since every asymptotically nonexpansive mapping is uniformly $\phi-$continuous, the conclusion of the corollary follows from Theorem 6.

**Remark 3** For the parameters of our theorem, we can make the following choices:

$$a_n = \bar{a}_n = 1 - \frac{1}{n+1} - \frac{1}{n^2},$$

$$b_n = \bar{b}_n = \frac{1}{n+1}, \quad c_n = \bar{c}_n = \frac{1}{n^2}$$

for all $n \geq 1$. However, Theorem S1, Theorem L and the results of Rhoades [5] and Tan and Xu [11] do not work for such chosen parameters.

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