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Variational Inverse Initial Value Problem for Linear Systems of Differential – Difference Equations of Neutral Type

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1 Introduction

The theory of functional differential equations has numerous applications in various fields of mechanics, physics, biology, technical and economic sciences. In relation to the rapid expansion of functional differential equations in application problems there goes a rapid development of the theory of these equations. Some of the much investigated systems of functional differential equations are the linear systems of differential difference equations (DDE), where the delay is constant. Here we notice at first the fundamental works [1]–[4] where as a main problem the initial value problem is considered, where the initial function is given in one way or another. In some cases it is necessary to investigate the questions of existence of solutions, where additional conditions for the solutions are given. Such boundary value problems have been considered in many works (see the review [5]). Recently in some applied fields of science (for example, in immunology) the problem associated with investigations of initial functions has appeared. This inverse initial value problem (i.i.v.p.) is formulated as follows: obtain the existence conditions for an absolutely continuous initial function such that the solution of the problem studied and the initial function itself satisfy some additional conditions.

In the present lecture some approaches to the investigation of the inverse initial value problem for DDE are considered.

2 Statement of the problem

Consider the following initial value problem for the differential difference equation:

$$\dot{x}(t) + c\dot{x}(t-1) = A(t)x(t-1) + B(t)x(t) + f(t),\ t \in J = [0,T]; \qquad (2.1)$$

$$x(t) = g(t),\ t \in J_0 = [-1, 0], \tag{2.2}$$

where $x(t) :\ J_0 \cup J \to \mathbb{R}^n$; $A(t), B(t) :\ J \to \mathbb{R}^{n\times n}$; $g(t) :\ J_0 \to \mathbb{R}^n$; $f(t) : J \to \mathbb{R}^n$, $c-$const.

Definition 2.1. A function $x(t) \in C^1[J]$ is said to be a *solution* of the initial value problem (2.1)–(2.2) if (2.1) is satisfied for almost all $t \in J$ and the equality (2.2) holds.

Direct initial value problem (DIVP): *Obtain the existence conditions for an absolutely continuous function $x(t)$, which should be a solution of the initial value problem* (2.1) – (2.2) *when the initial function $g(t)$ is given.*

Let the values of the functions $x(t)$ and (or) its derivatives $x^{(n)}(t),\ n = \overline{1,p}$, be given at some fixed points $t \in [-1, T]$:

$$\begin{aligned} x(t_i^0) &= x_i^0,\ \ i = \overline{1,m}; \\ \dot{x}(t_j^1) &= x_j^1,\ \ j = \overline{1,l}; \\ &\cdots\cdots\cdots\cdots \\ x^{(p)}(t_k^p) &= x_k^p,\ \ k = \overline{1,r}. \end{aligned} \tag{2.3}$$

Remark 2.1. In the following we will assume that in the sequences $\{t_i^0\}_{i=1}^m$, $\{t_j^1\}_{j=1}^l, \ldots, \{t_k^p\}_{k=1}^r$ the ordering conditions hold, *i.e.*, $t_j^i < t_{j+1}^i$, $i = \overline{0,p}$. As this takes place the cases when $t_\gamma^\alpha = t_\delta^\beta$ $(\alpha \neq \beta)$ are not ruled out.

Inverse initial value problem (IIVP): *Obtain the existence conditions for an absolutely continuous initial function $g(t)$, $t \in J_0$, such that the function $x(t)$, generated by $g(t)$, should be a solution of the initial value problem* (2.1) – (2.2) *and the conditions* (2.3) *hold.*

3 Main results

3.1 The existence theorem

We restrict ourselves intentionally to the circle of scalar IIVP in order to demonstrate clearly the idea of the method.

Taking into account (2.3) rewrite (2.1)–(2.2) in the form

$$\dot{x}(t) + c\dot{x}(t-1) = a(t)x(t-1) + b(t)x(t) + f(t),\ t \in J = [0, T]; \tag{3.1}$$

$$x(t) = g(t),\ t \in J_0 = [-1, 0], \tag{3.2}$$

$$x(t_i^0) = x_i^0,\ \ i = \overline{1,m};\ \ldots;\ x^{(p)}(t_k^p) = x_i^0,\ \ k = \overline{1,r};\ \ t_j^i \in J_0 \cup J. \tag{3.3}$$

$$T = \max\{1,\, t_m^0,\, t_l^1,\, \dots,\, t_r^p\}. \tag{3.4}$$

Represent the segment J in (3.1) as follows:

$$J = \bigcup_{n=1}^{]T[+1} J_n, \quad J_n = \begin{cases} [n-1,\ n], & n = \overline{1,]T[}; \\ [\ ]T[,\ T\ ], & n =]T[+1. \end{cases}$$

Here $]\cdot[$ is the integer part of a number.

Notation 3.1. The solution $x(t)$ of the IIVP (3.1)–(3.4) in the segment J_n will be denoted by $x_n(t)$.

Assumption 3.1. Henceforth we will consider the cases when $g(t) \in C^\sigma[J_0]$, where σ is the maximal of the upper indices i for all $t_s^i \in J_0 \cup J$ in (3.3).

Represent the function $x_0(t)$ in the form

$$x_0(t) = \sum_{u=1}^{G} \psi_0^u(t) g_u, \quad t \in J_0. \tag{3.5}$$

Here $\psi_0^u(t)$ are some known real functions, g_n are unknown constant coefficients.

Definition 3.2. The linearly independent functions $\psi_0^u(t),\ u = \overline{1, G}$, which are chosen out of the class $C^\sigma[J_0]$ are said to be the *basic functions* of the solution $x(t)$.

Considering the function $x_0(t)$ in view of (3.5) as an initial function $g(t) = x_0(t)$, we obtain the solution of (3.1)–(3.4) on $J_1 = [0, 1]$.

Rewrite (3.1)–(3.2) as follows:

$$\dot{x}(t) = b(t)x(t) + a(t)g(t-1) - c\dot{g}(t-1) + f(t), \quad x(0) = x_0(0) = x_0, \quad t \in J_1.$$

Then we have

$$x(t) = z(t)x(0) + \int_0^t c(t,s)[a(s)g(s-1) - c\dot{g}(s-1) + f(s)]\, ds, \tag{3.6}$$

where $z(t)$ is the solution of the Cauchy problem

$$\dot{z}(t) = b(t)z(t), \quad z(0) = 1$$

and $c(t,s)$ is the Cauchy function.

Taking into account (3.5), represent the formula (3.6) in the form

$$x_1(t) = x(t) = \sum_{u=1}^{G} \psi_1^u(t) g_u + f_1(t), \quad t \in J_1,$$

where

$$\psi_1^u(t) = z(t)\psi_0^u(0) + \int_0^t c(t,s)[a(s)\psi_0^u(s-1) - c\dot{\psi}_0^u(s-1)]\,ds;$$

$$f_1(t) = \int_0^t c(t,s)f(s)\,ds.$$

Continuing this process in just the same way on $J_2,\ J_3$ and so on, for $J_M = [M-1, M]$ we deduce

$$x_M(t) = \sum_{u=1}^{G} \psi_M^u(t)g_u + f_M(t),\ \ t \in J_M. \tag{3.7}$$

Here

$$\psi_M^u(t) = z(t)\psi_{M-1}^u(M-1) + \int_{M-1}^t c(t,s)[a(s)\psi_{M-1}^u(s-1) - c\dot{\psi}_0^u(s-1)]\,ds;$$

$$f_M(t) = \int_{M-1}^t c(t,s)[f_{M-1}(s-1) + f(s)]\,ds.$$

Remark 3.1. In accordance with Definition 3.2 the solution $x_M(t)$ is differentiable on J_M corresponding with the conditions (3.3) times, *i.e.*,

$$x_M^{(s)}(t) = \sum_{u=1}^{G} \frac{d^s}{dt^s}\psi_M^u(t)g_u + f_M^{(s)}(t). \tag{3.8}$$

Now we are coming to the question of finding the initial function $g(t)$ for the IIVP (3.1)–(3.4). Introduce the function

$$\delta(t) = 0 \ \text{ when }\ t \le 0;\ \ \delta(t) = \begin{cases} t & \text{if }\ t =]t[; \\ ]t[+1 & \text{if }\ t \ne]t[, \end{cases} \ \text{ when }\ t > 0. \tag{3.9}$$

Since every point $t_s^k \in J_{\delta(t_s^k)}$ by (3.8) and (3.9), from (3.3) it follows:

$$x_{\delta(t_i^0)}(t_i^0) = \sum_{n=1}^{G} \psi_{\delta(t_i^0)}^n(t_i^0)g_n + f_{\delta(t_i^0)}(t_i^0) = x_i^0,\ \ i = \overline{1,m};$$

$$\dot{x}_{\delta(t_j^1)}(t_j^1) = \sum_{n=1}^{G} \frac{d}{dt}\psi_{\delta(t_j^1)}^n(t_j^1)g_n + \dot{f}_{\delta(t_j^1)}(t_j^1) = x_j^1,\ \ j = \overline{1,l};$$

. . . .

$$x_{\delta(t_k^p)}^{(p)}(t_k^p) = \sum_{n=1}^{G} \frac{d^p}{dt^p}\psi_{\delta(t_k^p)}^n(t_k^p)g_n + f_{\delta(t_k^p)}^{(p)}(t_k^p) = x_k^p,\ \ k = \overline{1,r}. \tag{3.10}$$

Remark 3.2. In the following the function $f_0(t)$ is assumed equal to zero when $t < 0$.

Denote

$$a_{i\,s} = \psi^s_{\delta(t_i^0)}(t_i^0), \quad i = \overline{1,m}, \quad s = \overline{1,G};$$
$$a_{(m+j)s} = \frac{d}{dt}\psi^s_{\delta(t_j^1)}(t_j^1), \quad j = \overline{1,l}, \quad s = \overline{1,G};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{(m+l+\dots+k)s} = \frac{d^p}{dt^p}\psi^s_{\delta(t_k^p)}(t_k^p), \quad k = \overline{1,r}, \quad s = \overline{1,G};$$

$$b_i = x_i^0 - f_{\delta(t_i^0)}(t_i^0), \quad i = \overline{1,m};$$
$$b_{m+j} = x_i^1 - \dot{f}_{\delta(t_j^1)}(t_j^1), \quad j = \overline{1,l};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$b_{m+j+\dots+k} = x_k^p - f^{(p)}_{\delta(t_k^p)}(t_k^p), \quad k = \overline{1,r},$$

and rewrite the relation (3.10) as a linear system of algebraic equations for the unknown coefficients g_n:

$$Ag = B, \tag{3.11}$$

where $A = \{a_{ij}\}$, $i = \overline{1, m+l+\dots+r}$, $j = \overline{1,G}$, is a rectangular matrix; $g = \{g_1, g_2, \dots, g_G\}^T$ and $B = \{b_1, b_2, \dots, b_{m+l+\dots+r}\}^T$ are vectors.

Denote

$$Q = m + l + \dots + r.$$

Let the matrix

$$\bar{A} = \left|\begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1G} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2G} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{Q1} & a_{Q2} & \cdots & a_{QG} & b_Q \end{array}\right|$$

be the augmented matrix of the linear system (3.11). It is well known that

Theorem 3.1 *The linear system* (3.11) *is compatible if and only if the rank of the matrix $\bar{A}$ is equal to the rank of the matrix A, i.e.,* $\operatorname{rank}\bar{A} = \operatorname{rank} A$.

This result was proved independently by Rouché and Fontené, but it is now often called the *Rouché–Frobenius theorem*. From this theorem there follows

Corollary 3.1. *If* $\operatorname{rank}\bar{A} > \operatorname{rank}A$*, then under the basic functions* $\psi_n^0(t),\ n=\overline{1,G}$*, chosen in* (3.5) *the IIVP* (3.1) – (3.4) *is unsolvable.*

When the linear system (3.11) is compatible the following theorem is valid:

Theorem 3.2. *For the IIVP* (3.1) – (3.4) *let the following conditions hold:*

1. *the functions* $a(t)$, $b(t)$, $f(t)$ *belong to class* $C^{\sigma-1}[J]$*;*
2. *the basic functions* $\psi_n^0(t),\ \ n=\overline{1,G}$*, in* (3.5) *belong to class* $C^{\sigma}[J_0]$*;*
3. *the linear system* (3.11) *is compatible, therefore the number of the basic functions satisfies the inequality* $G \geq \operatorname{rank}A$*.*

Then the inverse initial value problem (3.1) – (3.4) *has a solution with respect to the initial function*

$$g(t)=\sum_{u=1}^{G}\psi_u^0(t)g_u,\ \ g(t)\in C^{\sigma}[J_0]. \tag{3.12}$$

Moreover, there are infinitely many solutions if $G > \operatorname{rank}A$*, and the representation of the solution in the form* (3.12) *is unique if* $G=\operatorname{rank}A$*.*

The proof of this theorem is the same as the proof of Theorem 3.2 in [6].

3.2 Variational solutions of the IIVP

Let the initial function be given as a polynomial

$$g(t)=\sum_{n=1}^{G} g_n t^{n-1}. \tag{3.13}$$

Define a functional on the set of polynomials $g(t)$:

$$\Upsilon=\left|\int_{-1}^{0} F(t,\varphi_0(t),g(t))\,dt\right|, \tag{3.14}$$

where $\varphi_0(t)$ – a given absolutely continuous on J_0 function, $F(t,\varphi_0(t),g_N(t))$ – the function, satisfying the extremum conditions of the functional.

3.2.1. Let in (3.11) $G=R$, then $g=A^{-1}B$ and $g_0(t)=\sum\limits_{n=1}^{R} g_n t^{n-1}$.

Calculate the value of the functional

$$\Upsilon_0 = \left| \int_{-1}^{0} F(t, \varphi_0(t), g_0(t))\, dt \right|. \tag{3.15}$$

3.2.2. Let in (3.11) $G = R+1$. Then $g_1(t) = \sum\limits_{n=1}^{R+1} g_n t^{n-1}$. Rewrite (3.11) in the form

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1R} \\ a_{21} & a_{22} & \cdots & a_{2R} \\ \vdots & \vdots & \ddots & \vdots \\ a_{R1} & a_{R2} & \cdots & a_{RR} \end{vmatrix} \begin{vmatrix} g_1 \\ g_2 \\ \vdots \\ g_R \end{vmatrix} = \begin{vmatrix} b_1 - a_{1(R+1)} g_{R+1} \\ b_2 - a_{2(R+1)} g_{R+1} \\ \vdots \\ b_R - a_{R(R+1)} g_{R+1} \end{vmatrix}. \tag{3.16}$$

Since rank $A = R$,

$$\begin{vmatrix} g_1 \\ g_2 \\ \vdots \\ g_R \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1R} \\ a_{21} & a_{22} & \cdots & a_{2R} \\ \vdots & \vdots & \ddots & \vdots \\ a_{R1} & a_{R2} & \cdots & a_{RR} \end{vmatrix}^{-1} \begin{vmatrix} b_1 - a_{1(R+1)} g_{R+1} \\ b_2 - a_{2(R+1)} g_{R+1} \\ \vdots \\ b_R - a_{R(R+1)} g_{R+1} \end{vmatrix},$$

or in another form

$$g_n = h_n(g_{R+1}),\ n = \overline{1, R}. \tag{3.17}$$

Then by (3.13)

$$g_1(t, g_{R+1}) = \sum_{n=1}^{R+1} h_n(g_{R+1}) t^{n-1}.$$

For the functional in (3.14) we have

$$\Upsilon_1 = \left| \int_{-1}^{0} F(t, \varphi_0(t), g_1(t, g_{R+1}))\, dt \right|.$$

By an integration we get

$$\Upsilon_1 = \Upsilon_1(g_{R+1}).$$

To obtain the minimum of the functional one takes a derivative

$$d\Upsilon_1(g_{R+1})/dg_{R+1} = 0.$$

Let g^*_{R+1} be such that

$$\min \Upsilon_1 = \Upsilon_1(g^*_{R+1}). \tag{3.18}$$

By (3.17) we obtain $g_n,\ n = \overline{1, R}$, and by (3.13) – $g_1(t)$.

Define the difference

$$\Delta\Upsilon_1 =| \min \Upsilon_1 - \Upsilon_0 \mid . \tag{3.19}$$

3.2.3. Let in (3.11) $G = R+2$, then $g_2(t) = \sum_{n=1}^{R+2} g_n t^{n-1}$ and

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1R} \\ a_{21} & a_{22} & \cdots & a_{2R} \\ \vdots & \vdots & \ddots & \vdots \\ a_{R1} & a_{R2} & \cdots & a_{RR} \end{vmatrix} \begin{vmatrix} g_1 \\ g_2 \\ \vdots \\ g_R \end{vmatrix} = \begin{vmatrix} b_1 - a_{1(R+1)}g_{R+1} - a_{1(R+2)}g_{R+2} \\ b_2 - a_{2(R+1)}g_{R+1} - a_{2(R+2)}g_{R+2} \\ \vdots \\ b_R - a_{R(R+1)}g_{R+1} - a_{R(R+2)}g_{R+2} \end{vmatrix}. \tag{3.20}$$

From here it follows that

$$\begin{vmatrix} g_1 \\ g_2 \\ \vdots \\ g_R \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1R} \\ a_{21} & a_{22} & \cdots & a_{2R} \\ \vdots & \vdots & \ddots & \vdots \\ a_{R1} & a_{R2} & \cdots & a_{RR} \end{vmatrix}^{-1} \begin{vmatrix} b_1 - a_{1(R+1)}g_{R+1} - a_{1(R+2)}g_{R+2} \\ b_2 - a_{2(R+1)}g_{R+1} - a_{2(R+2)}g_{R+2} \\ \vdots \\ b_R - a_{R(R+1)}g_{R+1} - a_{R(R+2)}g_{R+2} \end{vmatrix},$$

or in another form

$$g_n = h_n(g_{R+1}, g_{R+2}),\ n = \overline{1, R}. \tag{3.21}$$

Taking into account (3.13) and (3.14), we have

$$\Upsilon_2 = |\int_{-1}^{0} F(t, \varphi_0(t), g_2(t, g_{R+1}, g_{R+2}))\, dt|.$$

Then

$$\Upsilon_2 = \Upsilon_2(g_{R+1}, g_{R+2}).$$

Consider the system of two equations

$$\partial\Upsilon_2(g_{R+1}, g_{R+2})/\partial g_{R+1} = 0,$$

$$\partial\Upsilon_2(g_{R+1}, g_{R+2})/\partial g_{R+2} = 0. \tag{3.22}$$

Let g^*_{R+1} and g^*_{R+2} be the solutions of this system. Then

$$\min \Upsilon_2 = \Upsilon_2(g^*_{R+1}, g^*_{R+2}). \tag{3.23}$$

On the other hand, by (3.21) we obtain the coefficients $g_n,\ n = \overline{1, R}$, and the initial function $g_2(t)$.

Taking into account (3.18), define the difference

$$\Delta\Upsilon_2 =| \min \Upsilon_1 - \min \Upsilon_2 \mid . \tag{3.24}$$

In the same way for $G = R + k$ we have the system of equations

$$\partial \Upsilon_k(g_{R+1}, g_{R+2}, \cdots, g_{R+k})/\partial g_{R+i} = 0,\ i = \overline{1, k}.$$

Let $g^*_{R+1}, g^*_{R+2}, \cdots, g^*_{R+k}$ be the solutions of this system. Then we find the initial function $g_k(t) = \sum\limits_{n=1}^{R+k} g_n t^{n-1}$ and calculate the minimum of the functional

$$\min \Upsilon_k = \Upsilon_k(g^*_{R+1}, g^*_{R+2}, \cdots, g^*_{R+k}). \tag{3.25}$$

In this case

$$\Delta\Upsilon_k = \mid \min \Upsilon_{k-1} - \min \Upsilon_k \mid . \tag{3.26}$$

Definition 3.3. The initial function $g_k(t) \in C^\sigma[J_0]$ is said to be an *ε-variation solution* of IIVP (3.1)–(3.4) if there exists k such that $\Delta\Upsilon_k \leq \varepsilon$.

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