5–10 July 2004, Antalya, Turkey — Dynamical Systems and Applications,
Proceedings, pp. 228–252

Smooth Solutions of Iterative Functional Differential Equations

Sui Sun Cheng
Department of Mathematics
Tsing Hua University
Hsinchu, Taiwan 30043, R. O. C.

Abstract

Iterative functional differential equations are equations involving derivatives and iterates of the unknown function. Over the past fifty years, they have attracted the attention of many people and qualitative properties of such equations have been reported. In particular, there are now a number of existence results for smooth solutions of these equations. In this note, we present some of these results and explain the techniques that are involved in their derivation.

1 Introduction

Let $x^{[0]}(t) = t,\ x^{[1]}(t) = x(t),\ x^{[2]}(t) = x(x(t)),\ x^{[3]}(t) = x(x(x(t))))$, etc. be the iterates of the function $x(t)$. When the derivatives and the iterates of an unknown function appear in a well defined functional relation, we are then dealing with an iterative functional differential equation. As early as 1815, Babbage investigated the problem of finding a function such that its n-th iterate $x^{[n]}$ is equal to a given function. Later Cooke [7] and Eder [16] pointed out that equations of the form

$$x^{(n)}(t) = H\left(t, x^{[1]}(t), \ldots, x^{[m]}(t)\right)$$

arise in problems related to infection models and also to motions of charged particles with retarded interactions.

Since then similar equations such as

$$G\left(t, x'(t), \ldots, x^{(n)}, x^{[1]}(t), \ldots, x^{[m]}(t)\right) = 0$$

have attracted the attention of a number of authors in the recent years [1–86].

The theory of iterative functional differential equations is not fully developed at this point. Before any general theory is elaborated, a stock of significant examples would thus be helpful.

Although there are many such examples, we note that iterative functional differential equations are quite different from the usual differential equations, therefore the standard existence and uniqueness theorems cannot be applied and it is therefore of interest to present existence results for smooth solutions of iterative functional differential equations.

We will present some examples and explain some of the techniques that are involved in deriving such results. We also hope that the specific results in this review would lead to more general ones in the future.

2 Closed form solutions

For the equation

$$x'(z) = x^{[m]}(z), \tag{1}$$

we may consider solutions of the form

$$x(z) = \beta z^{\gamma},$$

which leads to

$$\beta\gamma z^{\gamma-1} = \beta\left(\cdots\beta\left(\beta z^{\gamma}\right)^{\gamma}\cdots\right)^{\gamma} = \beta^{\gamma^{m-1}+\cdots+\gamma+1} z^{\gamma^m}.$$

Comparing coefficients, we obtain

$$\gamma^m = \gamma - 1,$$

$$\beta^{\gamma^{m-1}+\cdots+\gamma} = \gamma.$$

Thus for each solution (β_*, γ_*) of the above system, we obtain a corresponding solution of $x'(z) = x^{[m]}(z)$.

It is of interest to note that each root γ_* of $\gamma^m - \gamma + 1 = 0$ is a fixed point of the solution $x(z) = \beta_* z^{\gamma_*}$. Indeed,

$$x(\gamma_*) = \beta_* {\gamma_*}^{\gamma_*} = \gamma_*^{(\gamma_*-1)/(\gamma_*^m-\gamma_*)} \gamma_*^{\gamma_*} = \gamma_*^{-(\gamma_*-1)} \gamma_*^{\gamma_*} = \gamma_*. \tag{2}$$

In particular, when $m = 2$, the corresponding pair of equations is

$$\gamma^2 - \gamma + 1 = 0,$$

$$\beta^{\gamma} = \gamma,$$

which yields

$$\gamma_\pm = \frac{1 \pm \sqrt{3}i}{2}, \quad \beta_- = \gamma_-^{1/\gamma_-} \approx 2.145 - 1.238i, \quad \beta_+ = \gamma_+^{1/\gamma_+} \approx 2.145 + 1.238i.$$

Note that $|\gamma_\pm| = 1$ and $\gamma_\pm^6 = 1$, thus both of them are roots of unity.

When $m = 3$, the equation

$$\gamma^3 - \gamma + 1 = 0,$$

has roots $\gamma_1 \approx -1.3, \gamma_\pm = 0.6624 \pm 0.5625i$. Note that $|\gamma_1| > 1$. When $m = 4$, two roots $(-0.727 \pm 0.934i)$ have absolute values strictly greater than 1.

The procedure for obtaining such solutions works equally well for more general equations. For instance, in [30], a similar procedure is applied to the equation

$$x^{(n)}(z) = az^j \left(x^{[m]}(z)\right)^k, \tag{3}$$

where k, m, n are positive integers, j is a nonnegative integer, $m \geq 2$, and $a \neq 0$ is a complex number.

Let $\mu_1, \dots, \mu_m$ be the roots of

$$k\mu^m + j = \mu - n, \tag{4}$$

and $\lambda_1, \dots, \lambda_m$ be defined by

$$\lambda_i = \left[\frac{\mu_i(\mu_i - 1) \cdots (\mu_i - n + 1)}{a}\right]^{(1-\mu_i)/(k+n+j-1)}, \qquad i = 1, \dots, m. \tag{5}$$

Theorem 1 [30] *Let Ω be a domain of the complex plane $\mathbb{C}$ which does not include the negative real axis (nor the origin). Then there exist m distinct (single valued and analytic) power functions of the form*

$$x_i(z) = \lambda_i z^{\mu_i}, \quad i = 1, 2, \dots, m, \tag{6}$$

which are solutions of (3) *defined on Ω.*

Three other equations can be handled in similar manners.

The first is of the form [31]

$$x^{(n)}(z) = a \prod_{i=1}^{l} \left(x^{[m_i]}(q_i z)\right)^{k_i},$$

where $n, l, k_1, \ldots, k_l$ are positive integers, $m_1, m_2, \ldots, m_l$ are nonnegative integers such that $m_l \geq 2$ and

$$0 \leq m_1 < m_2 < \cdots < m_l,$$

and a as well as $q_1, \ldots, q_l$ are nonzero numbers.

The second is of the form [75]

$$\left(x^{(n_1)}(z)\right)^{N_1} \cdots \left(x^{(n_a)}(z)\right)^{N_a} = Az^j \left(x^{[m_1]}(z)\right)^{M_1} \cdots \left(x^{[m_b]}(z)\right)^{M_b} \qquad (7)$$

where $N_1, \ldots, N_a, M_1, \ldots, M_b,\ a, b,\ n_1, \ldots, n_a,\ m_1, \ldots, m_b$ are positive integers, j is a nonnegative integer, $A \neq 0$ and $n_1 > n_2 > \cdots > n_a$, $m_1 > m_2 > \cdots > m_b$.

The third is of the form [70]

$$x'(z) = \frac{1}{x^{[m]}(z)}, \quad m \geq 2. \qquad (8)$$

Again, we seek solutions of the form $x(z) = \lambda z^\mu$. Setting it into (8), we obtain

$$\lambda \mu z^{\mu-1} = \lambda^{-(1+\mu+\cdots+\mu^{m-1})} z^{-\mu^m}.$$

This prompts us to consider the equations

$$\mu \lambda^{2+\mu+\cdots+\mu^{m-1}} = 1 \qquad (9)$$

and

$$\mu^m + \mu - 1 = 0. \qquad (10)$$

We can find m distinct roots $\mu_1, \ldots, \mu_m$ of the polynomial equation (10), so that from (9) we can then solve

$$\lambda_i = \mu_i^{(\mu_i-1)/(2-\mu_i-\mu_i^m)} = \mu_i^{\mu_i-1}, \qquad i = 1, 2, \ldots, m, \qquad (11)$$

and find m distinct solutions of (8) of the form

$$x_i(z) = \lambda_i z^{\mu_i}, \qquad i = 1, 2, \ldots, m. \qquad (12)$$

Theorem 2 [70] *Let $\mu_1, \ldots, \mu_m$ be the m distinct roots of* (10)*, and let $\alpha_i = 1/\mu_i$ for $i = 1, \ldots, m$. Then in the neighborhood of each point α_i defined by $|z - \alpha_i| < \alpha_i$, equation* (8) *has an analytic solution of the form*

$$\begin{aligned} x_i(z) &= \alpha_i^{1-\mu_i} z^{\mu_i} = \alpha_i \left(1 + \frac{z-\alpha_i}{\alpha_i}\right)^{\mu_i} \\ &= \alpha_i \left[1 + \frac{\mu_i}{1!}\left(\frac{z-\alpha_i}{\alpha_i}\right) + \frac{\mu_i(\mu_i-1)}{2!}\left(\frac{z-\alpha_i}{\alpha_i}\right)^2 + \cdots\right], \end{aligned}$$

which satisfies $x_i(\alpha_i) = \alpha_i$.

For example, consider equation (8) where $m = 2$. In this case, $\mu^2 + \mu - 1 = 0$ has roots $\mu_\pm = (-1 \pm \sqrt{5})/2$. Thus we find two analytic solutions

$$x_+(z) = \mu_+^{\mu_+ - 1} z^{\mu_+}, \text{ and } x_-(z) = \mu_-^{\mu_- - 1} z^{\mu_-},$$

which are already known to Mckiernan [37].

We note that explicit solutions of equations of the form

$$\left(x^{(n_1)}\left(p_1 z\right)\right)^{N_1} \cdots \left(x^{(n_a)}\left(p_a z\right)\right)^{N_a} = \frac{A\left(x^{[s_1]}\left(r_1 z\right)\right)^{T_1} \cdots \left(x^{[s_c]}\left(r_c z\right)\right)^{T_c}}{\left(x^{[m_1]}\left(q_1 z\right)\right)^{M_1} \cdots \left(x^{[m_b]}\left(q_b z\right)\right)^{M_b}}$$

can also be found and the corresponding results are under preparation.

3 Analytic solutions

Before finding analytic solutions of iterative functional differential equations, let us state some preparatory results. First we set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

For motivation, let us consider finding power series solutions of the form

$$x(z) = d_0 + d_1(z - \mu) + d_2(z - \mu)^2 + \cdots$$

for the prototype equation

$$x'(z) = x(x(z)).$$

Note that if $x(\mu) = \mu$, then we may calculate $d_0 = 0$,

$$d_1 = x'(\mu) = x(x(\mu)) = x(\mu) = \mu,$$

$$d_2 = \frac{x''(\mu)}{2} = \frac{x'(x(\mu))x'(\mu)}{2} = \frac{\mu^2}{2},$$

etc. so that

$$x(z) = \mu z + \frac{\mu^2}{2} z^2 + \cdots .$$

The problem now is to determine whether the formal power series function converges in a neighborhood of the number μ.

This is not an easy question since the coefficient d_n is not known explicitly. Fortunately, we may employ a transformation technique in the form

$$x(z) = y(\mu y^{-1}(z))$$

that transforms composition of functions into products of functions and then seek majorizing power series functions to show convergence.

Majorizing functions can be obtained in many ways. Two basic types can be obtained from the following two theorems.

Analytic Implicit Function Theorem *Suppose $F = F(x, y)$ is analytic at the point (x_0, y_0), $F(x_0, y_0) = 0$ and $F_y(x_0, y_0) \neq 0$. Then there exists a unique function*

$$f(x) = y_0 - \frac{F_x(x_0, y_0)}{F_y(x_0, y_0)}(x - x_0) + \sum_{k=2}^{\infty} a_k (x - x_0)^k$$

which is analytic on a neighborhood of x_0 and satisfies $F(x, f(x)) = 0$ for x near x_0.

The above analytic implicit function theorem is well known and can be used to obtain analytic solutions for a polynomial functional equation. Given a polynomial $P(G)$ in the function $G = G(z)$ and a function $h = h(z)$, the equation

$$P(G)(z) = h(z)$$

is called a *polynomial functional equation.* For instance,

$$G^m(z) + a_{m-1}G^{m-1}(z) + \cdots + a_1 G(z) + a_0 z = h(z)$$

is such an equation. For example, we may show that the equation

$$G^2(z) - MG(z) - M\,|\eta|\,z = 0, \quad M > 0,$$

has a solution

$$G(z) = \sum_{n=0}^{\infty} g_n z^n$$

which is analytic on a neighborhood of the origin and the sequence $g = \{g_n\}_{n \in \mathbb{N}}$ is given by $g_0 = 0$, $g_1 = |\eta|$ and

$$g_{n+1} = M^{-1} g_{n+1} g_{n-k}, \quad n \geq 1.$$

A complex number α is called a *Siegel number* if $|\alpha| = 1$, $\alpha^n \neq 1$ for $n \in \mathbb{Z}^+$ and

$$\log |\alpha^n - 1|^{-1} \leq T \log n, \quad n = 2, 3, \ldots, \tag{13}$$

for some positive constant T.

The following property of the Siegel number is also important in obtaining majorants of power series functions. Its proof is a variation of that of Siegel in [73].

Generalized Siegel Theorem *Let α be a Siegel number. For each $n = 2, 3, \ldots$, let $P_n = P_n(x_1, \ldots, x_{n-1})$ be a function of n variables such that (i) if $0 \leq x_i \leq y_i$*

for $i = 1, \ldots, n-1$, *then* $0 \leq P_n(x_1, \ldots, x_{n-1}) \leq P_n(y_1, \ldots, y_{n-1})$, *and (ii) if* $d_1 = 1$ *and* d_n *is defined by*

$$d_n = \frac{1}{|\alpha^{n-1} - 1|} \max_{n_1 + \cdots + n_t = n;\ 0 < n_1 \leq \cdots \leq n_t;\ 2 \leq t \leq n} \{d_{n_1} \cdots d_{n_t}\},\ n \geq 2, \tag{14}$$

then

$$P_n(d_1 x_1, \ldots, d_{n-1} x_{n-1}) \leq d_n P_n(x_1, \ldots, x_{n-1}).$$

Let $u = \{u_n\}_{n \in \mathbb{N}}$ *be a complex sequence which satisfies* $u_1 = \mu > 0$ *and*

$$u_n = \frac{1}{|\alpha^{n-1} - 1|} P_n(u_1, \ldots, u_{n-1}),\ n \geq 2.$$

Let $v = \{v_n\}_{n \in \mathbb{N}}$ *be a complex sequence which satisfies* $v_1 = \eta \geq \mu$ *and*

$$v_n = P_n(v_1, \ldots, v_{n-1}),\ n \geq 2.$$

If there is $r > 0$ *such that* $v_n \leq r^n$ *for* $n \in \mathbb{Z}^+$, *then there is* $\delta > 0$ *such that*

$$u_n \leq r^n \left(2^{5\delta+1}\right)^{n-1} n^{-2\delta},\ n \geq 2.$$

3.1 Equations involving first derivatives

Next, we are interested in finding analytic solutions.

Theorem 3 [55] *Suppose the complex number* μ *satisfies either* (A) $0 < |\mu| < 1$; *or* (B) μ *is a Siegel number. Then equation* (1) *has an analytic solution of the form*

$$x(z) = \mu + \mu(z - \mu) + \frac{\mu^m}{2!}(z - \mu)^2 + \frac{\mu^{2m-1}(\mu^m - 1)}{3!(\mu - 1)}(z - \mu)^3 + \cdots$$

in a neighborhood of μ.

In deriving the above theorem, we need to find (formal) power series solutions of the functional differential equation

$$y'(\mu z) = \frac{1}{\mu} y'(z)\, y(\mu^m z), \tag{15}$$

$$y(0) = \mu. \tag{16}$$

Then we show that such a power series solution is majorized by a convergent power series. Then we show that

$$x(z) = y\left(\mu y^{-1}(z)\right) \tag{17}$$

is an analytic solution of (1) in a neighborhood of μ. These arguments are presented in [54] and repeated here since similar ideas can be used to obtain analytic solutions of other iterative functional differential equations.

We first show that when $0 < |\mu| < 1$, then for each complex number $\eta \neq 0$, equation (15) has a solution of the form

$$y(z) = \sum_{n=0}^{\infty} b_n z^n \tag{18}$$

which is analytic on a neighborhood of the origin and satisfies $b_0 = \mu$ and $b_1 = \eta$.

Indeed, assume (15) has a solution of the form (18) which is analytic at 0 and satisfies $b_0 = \mu$ and $b_1 = \eta$. Substituting (18) into (15) and then comparing coefficients, we see that the sequence $\{b_n\}_{n=2}^{\infty}$ is successively determined by the condition

$$\left(\mu^{n+1} - \nu\right)(n+1)\, b_{n+1} = \sum_{k=0}^{n-1} (k+1)\, \mu^{m(n-k)} b_{k+1} b_{n-k}, \quad n \in \mathbb{Z}^+, \tag{19}$$

in a unique manner. Furthermore, there is some $M > 0$ such that

$$\left| \frac{(k+1)\mu^{m(n-k)}}{(n+1)\left(\mu^{n+1} - \mu\right)} \right| \leq \frac{1}{|\mu^n - 1|} \leq M^{-1}, \quad n \geq 2, \quad 0 \leq k \leq n-1.$$

Thus if we define a sequence $\{B_n\}_{n=0}^{\infty}$ by $B_0 = \mu$, $B_1 = |\eta|$ and $B_{n+1} = M^{-1} B_{n+1} B_{n-k}$ for $n \in \mathbb{Z}^+$, then in view of (19),

$$|b_n| \leq B_n, \quad n \in \mathbb{Z}^+,$$

that is, $\{b_n\}$ is majorized by the sequence $\{B_n\}_{n \in \mathbb{N}}$. Therefore our proof will be complete if we can show that the radius of convergence of $\{B_n\}_{n \in \mathbb{N}}$ is positive.

To this end, we may check by means of the analytic implicit function theorem that the equation

$$G^2(z) - MG(z) - M\,|\eta|\, z = 0$$

has a solution

$$G(z) = \sum_{n=0}^{\infty} g_n z^n$$

which is analytic on a neighborhood of the origin and the sequence $g = \{g_n\}_{n \in \mathbb{N}}$ is given by $g_0 = 0$, $g_1 = |\eta|$ and

$$g_{n+1} = M^{-1} g_{n+1} g_{n-k}, \quad n \geq 1.$$

Since $B_0 = g_0$ and $B_1 = g_1$, it is clear that $\{B_n\}_{n\in\mathbb{N}} = g$ so that $\{B_n\}_{n\in\mathbb{N}}$ has a positive radius of convergence. The proof is complete.

Next we suppose μ is a Siegel number. If $\eta = 1$, then equation (15) has a solution of the form (18) which is analytic on a neighborhood of the origin and satisfies $b_0 = \mu$ and $b_1 = 1$.

Indeed, as in the previous proof, assume the existence of an analytic solution of the form (18) with $b_0 = \mu$ and $b_1 = 1$. Then (19) holds again, so that

$$|b_{n+1}| \le \frac{1}{|\mu^n - 1|} \sum_{k=0}^{n-1} |b_{k+1}|\,|b_{n-k}|\,, \quad n \in \mathbb{Z}^+. \tag{20}$$

By the analytic implicit function theorem, the equation

$$G^2(z) - G(z) + z = 0$$

has a solution $G(z)$ which is analytic on a neighborhood of the origin and

$$G(z) = \sum_{n=0}^{\infty} C_n z^n,$$

where the sequence $C = \{C_n\}_{n\in\mathbb{N}}$ is defined by $C_0 = 0$, $C_1 = 1$ and

$$C_{n+1} = \sum_{k=0}^{n-1} C_{k+1} C_{n-k} = \sum_{n_1+n_2=n+1;\ 1\le n_1,n_2\le n} C_{n_1} C_{n_2}, \quad n \in \mathbb{Z}^+.$$

Thus by the Generalized Siegel Theorem, we may easily see that the sequence $\{b_n\}$ has a positive radius of convergence. Indeed, it suffices to take

$$P_n(x_1, \ldots, x_{n-1}) = \sum_{k=0}^{n-2} x_{k+1} x_{n-1-k}$$

and check directly that the conditions imposed in the Generalized Siegel Theorem are satisfied. The proof is complete.

We may now prove our Theorem 3 in two steps. First, we show that the power series function $y(z)$ generated by the sequence $\{b_n\}$ defined by $b_0 = \alpha$, $b_1 = \eta \ne 0$, and (19) satisfies (1). Indeed, since $y'(0) = \eta \ne 0$, the function $y^{-1}(z)$ is analytic in a neighborhood of the point $y(0) = \mu$. If we now define $x(z)$ by means of (17), then

$$\begin{aligned} x'(z) &= \mu y'\left(\mu y^{-1}(z)\right)\left(y^{-1}\right)'(z) = \mu y'\left(\mu y^{-1}(z)\right)\frac{1}{y'\left(y^{-1}(z)\right)} \\ &= y\left(\mu^m y^{-1}(z)\right) = x^{[m]}(z)\,, \end{aligned}$$

as required. Second, note that

$$y\left(\mu y^{-1}(\mu)\right) = y(\mu \cdot 0) = \mu,$$

or, μ is a fixed point of the solution $x(z)$, we may assume (1) has an analytic solution of the form

$$x(z) = \mu + d_1(z-\mu) + d_2(z-\mu)^2 + d_3(z-\mu)^3 + \cdots$$

and then $d_1 = x'(\mu) = x^{[m]}(\mu) = \mu$,

$$d_2 = \frac{1}{2!}x''(\mu) = \frac{1}{2!}\left\{x'(x^{[m-1]}(\mu))(x^{[m-1]})'(\mu)\right\} = \frac{1}{2!}\{\mu\cdots\mu\} = \frac{\mu^m}{2!},$$

etc. will lead to the conclusion of the proof of Theorem 3.

In the rest of this section, we will give several results along similar lines.

For the equation

$$x'(z) = c_1 x(z) + c_2 x^{[2]}(z) + \cdots + c_m x^{[m]}(z), \quad C \equiv c_1 + \cdots + c_m \neq 0, \tag{21}$$

we have the following theorem.

Theorem 4 [52] *Suppose the complex number μ satisfies either (A) $0 < |\mu| < 1$; or (B) μ is a Siegel number. Then equation (21) has an analytic solution of the form*

$$\begin{aligned} x(z) &= \frac{\mu}{C} + \mu\left(z - \frac{\mu}{C}\right) + \frac{1}{2!}\left(\sum_{i=1}^{m} c_i \mu^i\right)\left(z - \frac{\mu}{C}\right)^2 \\ &\quad + \frac{1}{3!}\left(\sum_{i=1}^{m} c_i \mu^{i-1}\right)\left(\sum_{i=1}^{m} c_i \mu^i(\mu^{i-1} + \mu^{i-2} + \cdots + 1)\right)\left(z - \frac{\mu}{C}\right)^3 \\ &\quad + \sum_{n=4}^{\infty} \frac{1}{n!}\lambda_n \left(z - \frac{\mu}{C}\right)^n, \end{aligned}$$

in a neighborhood of μ/C, where $\lambda_4, \lambda_5, \ldots$ are constants.

In deriving the above theorem, we need to find analytic solutions of the initial value problem

$$y'(\mu z) = \frac{1}{\mu} y'(z) \sum_{i=1}^{m} c_i y(\mu^i z), \quad y(0) = \frac{\mu}{C}.$$

Theorem 5 [80] *Suppose the complex number μ satisfies either (A) $0 < |\mu| < 1$ or (B) μ is a Siegel number together with $c_0 = \cdots = c_{r-1} = 0$ where $r \leq m$, or (C) $|\mu| > 1$ and $c_0 = \cdots = c_{r-1} = 0$ where $r \geq m$. Then equation (21) has a solution of the form*

$$x(z) = \frac{\mu}{C} + \mu\left(z - \frac{\mu}{C}\right) + \frac{1}{2!}\left(\sum_{j=0}^{m} c_j \mu^{j-r}\right)\left(z - \frac{\mu}{C}\right)^2$$

$$+\frac{1}{3!\mu^{3r+1}(\mu-1)}\left(\sum_{i=0}^{m}\sum_{j=0}^{m} c_i c_j \mu^{i+j}\left(\mu^j + 2\mu^r\right) - 3\mu^r\left(\sum_{j=0}^{m} c_j \mu^j\right)^2\right)\left(z - \frac{\mu}{C}\right)^3 + \cdots$$

which is analytic in a neighborhood of μ/C.

For the equation

$$x'(z) = x(az + bx(z)), \tag{22}$$

we have the following results.

When $b = 0$ and $|a| \leq 1$, equation (22) has the following entire solution

$$x(z) = \eta \sum_{n=0}^{\infty} \frac{a^{n(n-1)/2}}{n!} z^n.$$

When $a \neq 1$ and $b \neq 0$, we have the following

Theorem 6 [51] *Suppose the number μ satisfies either (A) or (B) in Theorem 3. Then equation (22) has the following analytic solution*

$$\begin{aligned} x(z) &= \frac{\mu - a}{b} + \frac{1}{b}(\mu - a)\left(z - \frac{\mu - a}{1-a}\right) + \frac{\mu(\mu - a)}{2!b}\left(z - \frac{\mu - a}{1-a}\right)^2 \\ &\quad + \frac{\beta(\beta - a)(\mu^2 + \mu - a)}{3!b}\left(z - \frac{\mu - a}{1-a}\right)^3 + \sum_{i=4}^{\infty} \frac{\lambda_{i,0}}{i!}\left(z - \frac{\mu - a}{1-a}\right)^i \end{aligned}$$

in a neighborhood of $(\mu - a)/(1 - a)$, where $\lambda_{4,0}, \lambda_{5,0}, \ldots$ are constants.

In deriving the above theorem, we need to find analytic solutions of the initial value problem

$$y'(\mu z) = \frac{1}{\mu} y'(z)\left\{y(\mu^2 z) - a y(\mu z) + a\right\}, \qquad y(0) = \frac{\mu - a}{1 - a}.$$

For the equation

$$x'(z) = x(x'(z)), \tag{23}$$

we can find the solution
$$x(z) = cz - c^2 + c.$$

For the equation
$$\alpha z + \beta x'(z) = x(az), \quad a \neq 0, \beta \neq 0, \tag{24}$$
we can find an analytic solution of the form
$$x(z) = \sum_{n=0}^{\infty} c_n z^n.$$
Indeed, by the method of undetermined coefficients, we find
$$x(z) = c_0 + \frac{c_0}{\beta} z + \frac{ac_0 - \alpha\beta}{2\beta^2} z^2 + (ac_0 - \alpha\beta) \sum_{n=3}^{\infty} \frac{1}{n!\beta^n} a^{(n-2)(n+1)/2} z^n,$$
which is entire when $0 < |a| \leq 1$.

For the equation
$$\alpha z + \beta x'(z) = x(az + bx'(z)), \quad \alpha, \beta, a, b \in \mathbb{C}, \tag{25}$$
we have the following theorem.

Theorem 7 [64] *If the following conditions hold:*

(1) *If $a = 1$, then $\beta \neq 0$ and $\beta\mu = a\beta - b\alpha$;*
(2) *If $a = 1$, then s is arbitrary;*
(3) *If $a \neq 1$, then $s = (\beta\mu + b\alpha - a\beta)/((1-a)\mu)$;*
(4) *$b \neq 0$ and $b\alpha - a\beta \neq 0$,*

where μ satisfies the conditions (A) or (B) in Theorem 3, then equation (25) *has an analytic solution of the form*
$$\begin{aligned} x(z) \;=\; & \frac{(b\alpha + (1-a)\beta)s}{b} + \frac{(1-a)s}{b}(z-s) + \frac{\mu - a}{2!b}(z-s)^2 \\ & + \frac{\mu^3(\mu - a)}{3!b(a\beta - b\alpha)}(z-s)^3 + \frac{\mu^5(\mu - a)(\mu^2 + 3\mu - 3a)}{4!b(a\beta - b\alpha)^2}(z-s)^4 + \cdots \end{aligned}$$
in a neighborhood of s.

In deriving the above theorem, we need to find analytic solutions of the initial value problem
$$\mu y'(\mu z) \left\{ y(\mu^2 z) - ay(\mu z) - \beta a \right\} = y'(z)(b\alpha - a\beta), \quad y(0) = s.$$

Next, we consider the following equation

$$x'(z) = f\left(\sum_{s=0}^{m} c_s x^{[s]}(z)\right), \tag{26}$$

where $m \geq 1$ is an integer, c_s, $s = 0, 1, \ldots, m$, are constants but not all equal to zero and f is a given function analytic for $|z| < \sigma$.

We reduce the equation (26) to the auxiliary one

$$\alpha\phi'(\alpha z) = \phi'(z) f\left(\sum_{s=0}^{m} c_s \phi(\alpha^s z)\right), \qquad z \in \mathbb{C},\ \alpha = f(0), \tag{27}$$

which is a functional differential equation without the iterates of the unknown function.

Theorem 8 [84] *Assume* $(H1)$ $0 < |\alpha| < 1$, *or* $(H2)$ $|\alpha| = 1$ *and there exist some constants* $\gamma > 0$ *and* $k > 0$ *such that* $|\alpha^n - 1| \geq \gamma^{-1} n^{-k}$ *for* $n \geq 1$. *Then equation* (26) *has an analytic solution of the form* $x(z) = \phi\left(\alpha\phi^{-1}(z)\right)$ *in a neighborhood of the origin, where* $\phi(z)$ *is an analytic solution of the auxiliary equation* (27) *satisfying* $\phi(0) = 0$ *and* $\phi'(0) = \tau$.

Theorem 9 [84] *Assume that* $(H3)$ $\alpha^p = 1$ *for some* $p \in \mathbb{N}$, $p \geq 2$ *and* $\alpha^k \neq 1$ *for all* $1 \leq k \leq p-1$. *Assume further that for some* $\tau \in \mathbb{C} \setminus \{0\}$ *the system*

$$b_1 = \tau,$$

and

$$\begin{aligned}&\left(\alpha^{n+1} - \alpha\right)(n+1) b_{n+1} \\ = &\sum_{k=0}^{n-1} (k+1) b_{k+1} \sum_{1 \leq t \leq n-k,\ l_1,\ldots,l_t > 0,\ l_1+\cdots+l_t = n-k} a_t \prod_{j=1}^{t} \left(\sum_{s=0}^{m} c_s \alpha^{s l_j} b_{l_j}\right)\end{aligned} \tag{28}$$

has a solution $\{b_i\}_{i=1}^{\infty}$ *satisfying* $b_{lp+1} = 0$ *for* $l = 1, 2, \ldots$. *Then equation* (26) *has an analytic solution of the form* $x(z) = \phi\left(\alpha\phi^{-1}(z)\right)$ *in a neighborhood of the origin, where* $\phi(z)$ *is an analytic solution of the auxiliary equation* (27) *satisfying* $\phi(0) = 0$, $\phi'(0) = \tau$ *and* $\phi^{(i)}(0) = i! b_i$ *for* $i = 2, 3, \ldots$.

Example Consider the equation

$$x'(z) = f\left(x^{[2]}(z)\right), \tag{29}$$

where

$$f(z) = \frac{1}{2(1-z)} = \frac{1}{2}\sum_{n=0}^{\infty} z^n,\ |z| < 1. \tag{30}$$

Near the origin, equation (29) with f in (30) has an analytic solution

$$x(z) = \frac{1}{2}z + \frac{1}{16}z^2 + \frac{7}{384}z^3 + \cdots \tag{31}$$

3.2 Equations involving second order derivatives

Consider iterative functional differential equations of the form

$$x''\left(x^{[r]}(z)\right) = c_0 z + c_1 x(z) + \cdots + c_m x^{[m]}(z), \tag{32}$$

where r and m are nonnegative integers, $c_0, c_1, ..., c_m$ are complex constants such that $\sum_{i=0}^{m} |c_i| \neq 0$.

Theorem 10 [59] *Suppose (i) α is a Siegel number; (ii) $|\alpha| > 1$ and $r \geq m$ or (iii) $0 < |\alpha| < 1$ and either $0 < r \leq m$ and $c_0 = 0, \ldots, c_{r-1} = 0$, or $r = 0$. Then, for any μ, (32) has an analytic solution $x(z)$ in a neighborhood of μ satisfying the initial conditions $x(\mu) = \mu$ and $x'(\mu) = \alpha$. This solution has the form $x(z) = y\left(\alpha y^{-1}(z)\right)$, where $y(z)$ is an analytic solution of*

$$\alpha^2 y''\left(\alpha^{r+1}z\right) y'\left(\alpha^r z\right) = \alpha y'\left(\alpha^{r+1}z\right) y'\left(\alpha^r z\right) + \left[y'\left(\alpha^r z\right)\right]^3 \left[\sum_{i=0}^{m} c_i y\left(\alpha^i z\right)\right]$$

under the condition

$$y(0) = \mu,\ y'(0) = \eta \neq 0.$$

We note that $x(z)$ is of the form

$$x(z) = u + \alpha(z-u) + \frac{u\sum_{i=0}^{m} c_i}{2!}(z-u)^2 + \frac{\sum_{i=0}^{m} c_i \alpha^{i-t}}{3!}(z-u)^3 + \cdots.$$

Next consider an iterative functional differential equation of the form

$$x''(z) = \left(x^{[m]}(z)\right)^2. \tag{33}$$

Theorem 11 [62] *Suppose $0 < |\mu| < 1$. Then equation (33) has an analytic solution $x(z)$ in a neighborhood of s. This solution has the form $x(z) = y\left(\mu y^{-1}(z)\right)$, where $y(z)$ is an analytic solution of the equation*

$$\mu^2 y''(\mu z) y'(z) = \mu y'(\mu z) y''(z) + \left[y'(z)\right]^3 \left[y\left(\mu^m z\right)\right]^2 \tag{34}$$

satisfying the condition

$$y(0) = s,\ y'(0) = \eta \neq 0. \tag{35}$$

Theorem 12 [62] *Suppose μ is a Siegel number. Suppose further that $0 < |\eta| \leq 1$. Then equation* (33) *has an analytic solution $x(z)$ in a neighborhood of s. This solution has the form $x(z) = y\left(\mu y^{-1}(z)\right)$, where $y(z)$ is an analytic solution of the equation* (34) *satisfying* (35)*.*

We note that $x(z)$ is of the form

$$x(z) = s + \mu(z-s) + \frac{s^2}{2!}(z-s)^2 + \frac{2s\mu^m}{3!}(z-s)^3 + \sum_{n=1}^{\infty} \frac{\lambda_n}{(n+3)!}(z-s)^{n+3}.$$

Existence of analytic solutions of second-order functional differential equations of the form

$$x''(z) = x(az + bx(z))$$

and

$$x''(z) = x\left(az + bx'(z)\right),$$

where a, b are complex numbers, have also been established in [60] and [58] respectively.

4 Differentiable solutions

We have discussed analytic solutions of iterative functional differential equations. Next, we will give some examples of $C^{(n)}$-solutions.

Again, let us consider the prototype equation

$$x'(t) = cx(x(t))$$

under the additional condition

$$x(\mu) = \mu.$$

We note that the above condition is motivated by the fixed point condition (2).

A standard approach is to transform the above problem into a fixed point problem

$$(Tx)(t) = \mu + \int_{\mu}^{t} cx^{[2]}(s)\, ds$$

in the Banach space $C^{(n)}[\mu - \delta, \mu + \delta]$ under the usual maximum norm. Under appropriate conditions on c, μ and δ as well as $x^{(1)}(\mu), \ldots, x^{(n)}(\mu)$, we may then show that T is a contraction mapping and find a corresponding solution.

The same principle can be applied to the equation

$$x'(t) = c_1 x(t) + c_2 x^{[2]}(t) + \cdots + c_m x^{[m]}(t) + F(t) \tag{36}$$

and we can establish the following local existence theorem.

Theorem 13 [52] *Let $I = [\xi - \delta, \xi + \delta]$ where ξ and δ satisfy*

$$|\xi| < \frac{1}{1 + |c_1| + \cdots + |c_m|},$$

and

$$0 < \delta < \frac{1 - |\xi|\,(1 + |c_1| + \cdots + |c_m|)}{1 + |c_1| + \cdots + |c_m|}.$$

Then when F is a $C^{(n-1)}(I, I)$ function satisfying

$$\left|F^{(i)}(t)\right| \leq M_i, \qquad M_i > 0, \quad i = 1, \ldots, n-1, \quad t \in I,$$

and

$$\left|F^{(n-1)}(s) - F^{(n-1)}(t)\right| \leq M_n\,|s - t|\,, \quad M_n > 0, \quad s, t \in I,$$

$$F^{(i)}(\xi) = \eta_i, \quad i = 0, 1, \ldots, n-1,$$

equation (36) *has a solution $x \in C^{(n)}(I, I)$ which satisfies $x(\xi) = \xi$, provided that the constants M_i, η_i, c_i satisfy some compatibility conditions.*

The idea of the proof is based on finding fixed points using Schauder's theorem:

$$(Tx)\,(t) = \xi + \sum_{j=1}^{m} c_j \int_{\xi}^{t} x^{[j]}(s)\,ds + \int_{\xi}^{t} F(s)\,ds.$$

We can also show that under more restrictive conditions on the constants M_i, η_i, c_i, the solution is unique and depends continuously on the nonhomogeneous function F.

By means of a more general fixed point problem of the form [56]

$$(Tx)\,(t) = \xi + \sum_{j=1}^{m} \int_{\xi}^{t} a_j(s) x^{[j]}(s)\,ds + \int_{\xi}^{t} F(s)\,ds$$

and Schauder's theorem, we may show the following: Let $I = [\xi - \delta, \xi + \delta]$ where ξ and δ satisfy

$$|\xi| < \frac{\sqrt{1 + 4m} - 1}{2m}$$

and

$$0 < \delta < \frac{\sqrt{1 + 4m} - 1 - 2m\,|\xi|}{2m}.$$

Then when $a_1(t), a_2(t), \ldots, a_m(t)$ and $F(t)$ are $C^{(n-1)}(I, I)$ functions satisfying

$$\left|F^{(i)}(t)\right| \le N_i, \quad N_i > 0, \quad i = 1, 2, \ldots, n-1, \quad t \in I$$

and

$$\left|F^{(n-1)}(s) - F^{(n-1)}(t)\right| \le N_n \left|s - t\right|, \quad N_n > 0, \quad s, t \in I,$$

$$F^{(i)}(\xi) = \eta_i, \quad i = 0, 1, \ldots, n-1,$$

$$\left|a_j^{(i)}(t)\right| \le L_{ji}, \quad L_{ji} > 0, \quad j = 1, 2, \ldots, m, \quad i = 1, \ldots, n-1, \quad t \in I,$$

$$\left|a_j^{(n-1)}(s) - a_j^{(n-1)}(t)\right| \le L_{jn} \left|s - t\right|, \quad L_{jn} > 0, \quad s, t \in I,$$

and

$$a_j^{(i)}(\xi) = \xi_i, \quad i = 0, 1, \ldots, n-1.$$

Then the following equation

$$x'(t) = \sum_{j=1}^{m} a_j(t) x^{[j]}(t) + F(t)$$

has a solution $x \in C^{(n)}(I, I)$ and $x(\xi) = \xi$, where N_i, ξ_i and L_{ji} satisfy some "compatibility conditions".

We will be concerned with $C^{(n)}$-solutions of the initial value problem

$$x'(t) = f\left(x^{[m]}(t)\right), \tag{37}$$

$$x(a_0) = a_0, \tag{38}$$

where we will also assume that $m, n \ge 1$ to avoid degenerate and trivial cases.

We write $g \in C^{(n)}(I, \mathbb{R})$ if $g = g^{(0)}, g' = g^{(1)}, \ldots, g^{(n)}$ are continuous on the interval I, and we write $g \in C^{(n)}(I, I)$ if $g \in C^{(n)}(I, \mathbb{R})$ and maps the closed interval I into I. It is well known that when endowed with the usual operations and the norm

$$\|g\|_n = \sum_{k=0}^{n} \sup_{t \in I} \left|g^{(k)}(t)\right|,$$

$C^{(n)}(I, \mathbb{R})$ is a Banach space. Let $\mathbf{a} = (a_0, a_1, \ldots, a_n)$ and let $\mathbf{M} = (M_1, \ldots, M_{n+1})$ be a vector of $n+1$ positive numbers $M_1, \ldots, M_{n+1}$, We will denote by $\Omega_n(\mathbf{M}, I)$ the subset of all $x \in C^{(n)}(I, I)$ each of which satisfies

$$\left|x^{(i)}(t)\right| \le M_i, \quad i = 1, 2, \ldots, n, \quad t \in I,$$

and

$$\left|x^{(n)}(t_1) - x^{(n)}(t_2)\right| \le M_{n+1} \left|t_1 - t_2\right|, \quad t_1, t_2 \in I;$$

and by $\Phi_n(\mathbf{a}, \mathbf{M}, I)$ the subset of all $x \in \Omega_n(\mathbf{M}, I)$ each of which satisfies

$$x^{(i)}(a_0) = a_i, \quad i = 0, 1, \ldots, n.$$

We will need a corresponding space for the function f in (37). Let $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$, and let $\mathbf{N} = (N_1, \ldots, N_n)$ be a vector of positive numbers. The subset $\Omega_{n-1}(\mathbf{N}, I)$ has been defined. The subset of all $f \in \Omega_{n-1}(\mathbf{N}, I)$ each of which satisfies

$$f^{(i)}(a_0) = b_i, \; i = 0, 1, \ldots, n-1,$$

and

$$|f(t)| \le \lambda |t|, \quad t \in I,$$

for some $\lambda > 0$, will be denoted by $\Psi_{n-1}(\mathbf{b}, \mathbf{N}, I, \lambda)$.

Introduce the notations

$$x_{ij}(t) = x^{(i)}\left(x^{[j]}(t)\right), \qquad x^*_{jk}(t) = \left(x^{[j]}(t)\right)^{(k)},$$

then we can show that for each $x^*_{jk}(t)$, there corresponds a unique and nontrivial multivariate polynomial $P_{jk}(\mathbf{z})$, where

$$\mathbf{z} = (z_{10}, \ldots, z_{1,j-1}; z_{20}, \ldots, z_{2,j-1}; \ldots; z_{k0}, \ldots, z_{k,j-1}),$$

such that P_{jk} can be expressed as a nonnegative linear combinations of the products of powers of the components of $\mathbf{z}$, and such that

$$x^*_{jk}(t) = P_{jk}(x_{10}(t), \ldots, x_{1,j-1}(t); x_{20}(t), \ldots, x_{2,j-1}(t); \ldots; x_{k0}(t), \ldots, x_{k,j-1}(t)).$$

Theorem 14 [5] *Let $\lambda > 0$ and $I = [a_0 - \delta, a_0 + \delta]$, where $0 < \delta < 1/\lambda$ and $|a_0| < 1/\lambda - \delta$. Let $\mathbf{a} = (a_0, a_1, \ldots, a_n)$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$, and let $\mathbf{M} = (M_1, M_2, \ldots, M_{n+1})$ as well as $\mathbf{N} = (N_1, \ldots, N_n)$ be two vectors of positive numbers. Let $\Lambda_{jk} = [0, M_1]^j \times [0, M_2]^j \times \cdots \times [0, M_k]^j$,*

$$H_k = P_{mk}(M_1, \ldots, M_1; M_2, \ldots, M_2; \ldots; M_k, \ldots, M_k), \quad k = 1, 2, \ldots, n,$$

and

$$W_{lk} = \max_{(w_1, \ldots, w_k) \in \Xi_k} \left|Q_{lk}(w_1, \ldots, w_k)\right|, \quad l = 1, 2, \ldots, k; \; k = 1, 2, \ldots, n,$$

where $\Xi_k = [0, H_1] \times [0, H_2] \times \cdots \times [0, H_k]$. *Let further*

$$N_{uv}(\Lambda_{jk}) = \max_{\mathbf{z} \in \Lambda_{jk}} \left| \frac{\partial P_{jk}(\mathbf{z})}{\partial z_{uv}} \right|, \quad 1 \le u \le k,\ 0 \le v \le j-1, \tag{39}$$

and

$$K_{st}(\Xi_k) = \max_{\mathbf{w} \in \Xi_k} \left| \frac{\partial Q_{sk}(\mathbf{w})}{\partial w_t} \right|. \tag{40}$$

Suppose f *is a function in* $\Psi_{n-1}(\mathbf{b}, \mathbf{N}, I, \lambda)$. *Suppose further the following conditions hold:*

(i) $M_1 = 1, N_1 \le M_2$;

(ii) $a_1 = b_0, a_2 = b_1 c_1^m$, *and*

$$a_r = \sum_{v=0}^{k-2} \sum_{l=1}^{k-2-v} \binom{k-2}{v} \rho_{l,k-2-v} b_{l+1} c_{v+1}, \quad r = 3, 4, \ldots, n,$$

where $c_k = P_{mk}(a_1, \ldots, a_1; \ldots; a_k, \ldots, a_k)$ *for* $k = 1, 2, \ldots, n$, *and* $\rho_{lk} = Q_{lk}(c_1, c_2, \ldots, c_k)$;

(iii)

$$\sum_{v=0}^{k-2} \sum_{l=1}^{k-2-v} \binom{k-2}{v} W_{l,k-2-v} N_{l+1} H_{v+1} \le M_k, k = 3, 4, ..., n;$$

(iv)

$$\sum_{v=0}^{n-2} \sum_{l=1}^{n-2-v} \binom{n-2}{v} \Bigg(N_{l+1} N_{v+1} \sum_{s=1}^{n-2-v} \sum_{\zeta=1}^{s} \sum_{\tau=0}^{m-1} K_{ls}(\Xi_{n-2-v}) N_{\zeta\tau}(\Lambda_{ms}) M_{\zeta+1}$$
$$+ W_{l,n-2-v} H_{v+1} N_{l+2} + W_{l,n-2-v} H_{l+1} \sum_{i=1}^{v+1} \sum_{j=1}^{m-1} N_{ij}(\Lambda_{m,v+1}) M_{i+1} \Bigg) \le M_{n+1}.$$

Then the initial value problem (37), (38) *has a solution in* $\Phi_n(\mathbf{a}, \mathbf{M}, I)$.

The proof is accomplished by considering an operator T from Φ_n into $C^{(n)}(I, I)$ defined by

$$(Tx)(t) = a_0 + \int_{a_0}^{t} f(x^{[m]}(s))\, ds, \quad x \in \Phi_n(\mathbf{a}, \mathbf{M}, I),$$

and apply Schauder's fixed point theorem.

There are other similar approaches such as the monotone method, continuation theorems, etc. Some of them can be found in the following references.

References

[1] Aiello W. G., Freedman H. I. and Wu J., *Analysis of a model representing stage-structured population growth with state-dependent time delay*, SIAM J. Appl. Math. **52** (1992), No. 3, 855–869.

[2] Bélair J., *Population models with state-dependent delays*, in: Mathematical Population Dynamics (New Brunswick, NJ, 1989), pp. 165–176, Lecture Notes in Pure and Appl. Math., **131**, Dekker, New York, 1991.

[3] Bélair J., *Stability analysis of an age-structured model with a state-dependent delay*, Geoffrey J. Butler Memorial Conference in Differential Equations and Mathematical Biology (Edmonton, AB, 1996). Canad. Appl. Math. Quart., **6** (1998), No. 4, 305–319.

[4] Bessis D., Marmi S. and Turchetti G., *On the singularities of divergent majorant series arising from normal form theory*, Rendiconti di Matematica, Serie VII, **9** (1989), 645–659.

[5] Cheng S. S., Si J. G. and Wang X. P., *Recent studies on smooth solutions of iterative functional differential equations*, J. Binzhou Teachers College, **14** (1998), No. 4, 17–21. (in Chinese)

[6] Cheng S. S., Si J. G. and Wang X. P., *An existence theorem for iterative functional differential equations*, Acta Math. Hungarica, **94** (2002), No. 1–2, 1–17.

[7] Cooke K. L., *Functional differential systems: some models and perturbation problems*, Inter. Symp. Diff. Eqs. Dynamical Systems, Puerto Rico, 1965.

[8] Driver R. D., *Existence theory for a delay-differential system*, Contributions to Differential Equations, **1** (1963), 317–336.

[9] Driver R. D., *A two-body problem of classical electrodynamics: the one-dimensional case*, Ann. Physics, **21** (1963), 122–142.

[10] Driver R. D., *A functional-differential system of neutral type arising in a two-body problem of classical electrodynamics*, in: International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics, pp. 474–484, Academic Press, New York, 1963.

[11] Driver R. D. and Norris M. J., *Note on uniqueness for a one-dimensional two-body problem of classical electrodynamics*, Ann. Physics, **42** (1967), 347–351.

[12] Driver R. D., *The "backwards" problem for a delay-differential system arising in classical electrodynamics*, in: Proceedings of the Fifth International Conference on Nonlinear Oscillations (Kiev, 1969), Vol. 2: Qualitative methods in the theory of nonlinear oscillations, pp. 137–143. Izdanie Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1970. (in Russian)

[13] Driver R. D., *Can the future influence the present?*, Phys. Rev. D, **19** (1979), No. 4, 1098–1107.

[14] Dunkel G. M., *On nested functional differential equations*, SIAM J. Appl. Math., **18** (1970), No. 2, 514–525.

[15] Eder E., *Existence, uniqueness and iterative construction of motions of charged particles with retarded interactions*, Ann. Inst. H. Poincaré, **39** (1983), No. 1, 1–27.

[16] Eder E., *The functional differential equation $x'(t) = x(x(t))$*, J. Diff. Eq., **54** (1984), 390–400.

[17] Fiagbedzi Y. A. and El-Gebeily M. A., *Existence and uniqueness of the solution of a class of nonlinear functional differential equations*, Ann. of Diff. Eq., **16** (2000), 381–390.

[18] Fine N. J., *Solution to problem 5407*, Amer. Math. Monthly, **74** (1967), 740–743.

[19] Ge W. G., *A transform theorem for differential-iterative equations and its application*, Acta Math. Sinica, **40** (1997), No. 6, 881–888. (in Chinese)

[20] Ge W. E. and Mo Y. R., *Lyapunov stability of an iterative functional differential equation*, Proc. Fifth National Conference on Stability Theory of Differential Equations and its Applications, Dailien, 1996.

[21] Ge W. E. and Mo Y. R., *Existence and bifurcation of solutions to differential-iterative equations*, Acta Math. Appl. Sinica, **21** (1998), No. 1, 113–122. (in Chinese)

[22] Golomb S. W., *Problem 5407*, Amer. Math. Monthly, **73** (1966), 674.

[23] Grimm L. J., *Existence and continuous dependence for a class of nonlinear equations*, Proc. Amer. Math. Soc., **29** (1971), 467–473.

[24] Hale J. K. and Cruz M. A., *Existence, uniqueness, and continuous dependence for hereditary systems*, Ann. Mat. Pura Appl., **85** (1970), 63–81.

[25] He Z. M. and Ge W. G., *The monotone iterative technique and periodic boundary value problem for first order impulsive functional differential equations*, Acta Math. Sin. (Engl. Ser.), **18** (2002), No. 2, 253–262.

[26] Hsing D. P. K., *Existence and uniqueness theorem for the one dimensional backwards two-body problem of electrodynamics*, Phy. Rev. D., **16** (1977), No. 4, 974–982.

[27] Jackiewicz Z., *Existence and uniqueness of solutions of neutral delay-differential equations with state dependent delays*, Funk. Ekv., **30** (1987), 9–17.

[28] Jia M., Ge W. G. and Liu X. P., *Existence and continuation of solutions for a class of nonautonomous differential-iterative equations*, J. Beijing Inst. Technol. (Chin. Ed.), **18** (1998), No. 1, 5–10. (in Chinese)

[29] Kuczma M., *Functional Equation in a Single Variable*, Polish Scientific Publishers, Warsaw, 1968.

[30] Li W. R., Cheng S. S. and Lu T. T., *Closed form solutions of iterative functional differential equations*, Appl. Math. E-Notes, **1** (2001), 1–4.

[31] Li W. R., Cheng S. S. and Cheng L. J., *Analytic solution of an iterative differential equation*, Math. Practice Theory, **32** (2002), No. 6, 994–998. (in Chinese)

[32] Li W. R. and Cheng S. S., *Analytic solutions of an iterative functional equation*, Aequationes Math., to appear.

[33] Li C. Z. and Ge W. G., *Existence of periodic solutions to a type of differential-iterative equations*, J. Beijing Inst. Technol. (Chin. Ed.), **20** (2000), No. 5, 534–538. (in Chinese)

[34] Li W. R. and Si J. G., *The theory, applications, and development of functional equations in one variable*, Qufu Shifan Daxue Xuebao Ziran Kexue Ban, **21** (1995), No. 1, 34–44. (in Chinese)

[35] Liu X. P., Jia M. and Ge W. G., *A class of nonautonomous functional-differential iterative equations*, Gaoxiao Yingyong Shuxue Xuebao Ser. A, **13** (1998), No. 3, 249–256. (in Chinese)

[36] Marcus D., *Solution to problem 5407*, Amer. Math. Monthly, **74** (1967), 740.

[37] McKiernan M. A., *The functional differential equation $Df = 1/ff$*, Proc. Amer. Math. Soc., **8** (1957), 230–233.

[38] Oberg R. J., *On the local existence of solutions of certain functional differential equations*, Proc. Amer. Math. Soc., **20** (1969), 295–302.

[39] Petermann Y. F. S., *On Golomb's self describing sequence II*, Arch. Math. **67** (1996), 473–477.

[40] Sarkovskii A. N., *Functional and functional-differential equations in which the deviation of the argument depends on the unknown function*, in: Functional and Differential-Difference Equations, pp. 148–155, Izdanie Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1974. (in Russian)

[41] Shi B. and Li Z. X., *Existence of solutions and bifurcation of a class of first order functional differential equations*, Acta Math. Appl. Sinica, **18** (1995), No. 1, 83–89.

[42] Shi B., Wang Z. C. and Li Z. X., *Asymptotic behavior of the solutions of a class of new type of functional differential equations*, J. Hunan Univ., **23** (1996), No. 2, 1–5.

[43] Si J. G., *On analytic solutions of the equation of invariant curves*, C. R. Math. Rep. Acad. Sci. Canada, **17** (1995), No. 1, 49–52.

[44] Si J. G., *Continuous solutions of the iterative equation* $G(f(x), f^{n_1}(x), \ldots, f^{n_k}(x)) = F(x)$, J. Math. Res. Exposition, **15** (1995), No. 1, 149–150. (in Chinese)

[45] Si J. G., *On the monotonic continuous solutions of some iterated equation*, Demonstratio Math., **29** (1996), No. 3, 543–547.

[46] Si J. G., $C^{(1)}$*-solutions of a class of iterative equations*, Acta Math. Sinica, **39** (1996), No. 2, 247–256. (in Chinese)

[47] Si J. G., *The existence of analytic solutions for two classes of linear functional differential equations of neutral type*, J. Qingdao Univ. Nat. Sci. Ed., **9** (1996), No. 1, 20–33.

[48] Si J. G., *Discussion on the periodic solutions for higher-order linear equation of neutral type with constant coefficients*, Appl. Math. Mech. (English Ed.), **17** (1996), No. 1, 29–37.

[49] Si J. G., *Analytic solutions of a nonlinear functional differential equation with proportional delays*, Demonstratio Math., **33** (2000), No. 4, 747–752.

[50] Si J. G., *Analytic solutions of linear neutral functional differential-difference equations*, Appl. Math. E-Notes, **1** (2001), 11–17.

[51] Si J. G. and Cheng S. S., *Analytic solutions of a functional-differential equation with state dependent argument*, Taiwanese J. Math., **1** (1997), No. 4, 471–480.

[52] Si J. G. and Cheng S. S., *Note on an iterative functional-differential equation*, Demonstratio Math., **31** (1998), No. 3, 609–614.

[53] Si J. G. and Cheng S. S., *Smooth solutions of a nonhomogeneous iterative functional-differential equation*, Proc. Roy. Soc. Edinburgh Sect. A, **128** (1998), No. 4, 821–831.

[54] Si J. G. and Cheng S. S., *Analytic solutions of a functional differential equation with proportional delays*, Bull. Korean Math. Soc., **39** (2002), No. 2, 225–236.

[55] Si J. G., Li W. R. and Cheng S. S., *Analytic solutions of an iterative functional-differential equation*, Comput. Math. Appl., **33** (1997), No. 6, 47–51.

[56] Si J. G. and Wang X. P., *Smooth solutions of a nonhomogeneous iterative functional-differential equation with variable coefficients*, J. Math. Anal. Appl., **226** (1998), No. 2, 377–392.

[57] Si J. G. and Wang X. P., *Analytic solutions of a polynomial-like iterative functional equation*, Demonstratio Math., **32** (1999), No. 1, 95–103.

[58] Si J. G. and Wang X. P., *Analytic solutions of a second-order functional-differential equation with a state derivative dependent delay*, Colloq. Math., **79** (1999), No. 2, 273–281.

[59] Si J. G. and Wang X. P., *Analytic solutions of a second-order iterative functional differential equation*, J. Comput. Appl. Math., **126** (2000), No. 1–2, 277–285.

[60] Si J. G. and Wang X. P., *Analytic solutions of a second-order functional differential equation with a state dependent delay*, Results Math., **39** (2001), No. 3–4, 345–352.

[61] Si J. G. and Wang X. P., *Differentiable solutions of a polynomial-like iterative equation with variable coefficients*, Publ. Math. Debrecen, **58** (2001), No. 1–2, 57–72.

[62] Si J. G. and Wang X. P., *Analytic solutions of a second-order iterative functional differential equation*, Comput. Math. Appl., **43** (2002), No. 1–2, 81–90.

[63] Si J. G., Wang X. P and Cheng S. S., *Nondecreasing and convex $C^{(2)}$-solutions of an iterative functional-differential equation*, Aequationes Math., **60** (2000), No. 1–2, 38–56.

[64] Si J. G., Wang X. P and Cheng S. S., *Analytic solutions of a functional-differential equation with a state derivative dependent delay*, Aequationes Math., **57** (1999), No. 1, 75–86.

[65] Si J. G., Wang X. P and Zhang W. N., *Analytic invariant curves for a planar map*, Appl. Math. Lett., **15** (2002), No. 5, 567–573.

[66] Si J. G. and Zhang W. N., *On $C^{(2)}$ solutions of a functional equation, Acta Math. Sinica (N. S.)*, **14** (1998), suppl., 705–710.

[67] Si J. G. and Zhang W. N., *$C^{(2)}$ solutions of a functional equations*, Acta Math. Sinica, **41** (1998), No. 5, 1061–1064. (in Chinese)

[68] Si J. G. and Zhang W. N., *Analytic solutions of a functional equation for invariant curves*, J. Math. Anal. Appl., **259** (2001), No. 1, 83–93.

[69] Si J. G. and Zhang W. N., *Analytic solutions of a nonlinear iterative equation near neutral fixed points and poles*, J. Math. Anal. Appl., **284** (2003), No. 1, 373–388.

[70] Si J. G. and Zhang W. N., *Analytic solutions of a class of iterative functional differential equations*, J. Comput. Appl. Math., **162** (2004), No. 2, 467–481.

[71] Si J. G., Zhang W. N. and Cheng S. S., *Smooth solutions of an iterative functional equation*, in: Functional Equations and Inequalities, pp. 221–232, Math. Appl., **518**, Kluwer Acad. Publ., Dordrecht, 2000.

[72] Si J. G., Zhang W. N. and Cheng S. S., *Continuous solutions of an iterative functional inequality on compact interval*, Nonlinear Stud., **7** (2000), No. 1, 105–108.

[73] Siegel C. L., *Iteration of analytic functions*, Ann. Math., **43** (1942), No. 2, 607–612.

[74] Stephan B. H., *On the existence of periodic solutions of $z'(t) = -az(t - r + \mu k(t, z(t))) + F(t)$*, J. Diff. Eq., **6** (1969), 408–419.

[75] Talwong S., Laohakosol V. and Cheng S. S., *Power function solutions of iterative functional differential equations*, Applied Math. E-Notes, **4** (2004), 160–163.

[76] Tavernini L., *The approximate solution of Volterra differential systems with state dependent time lags*, SIAM J. Numer. Anal., **15** (1978), 1039–1052.

[77] Wang K., *On the equation $x'(t) = f(x(x(t)))$*, Funk. Ekv., **33** (1990), 405–425.

[78] Wang X. P. and Si J. G., *Continuous solutions of an iterative functional equation*, Acta Math. Sinica, **42** (1999), No. 5, 945–950. (in Chinese)

[79] Wang X. P. and Si J. G., *Differentiable solutions of an iterative functional equation*, Aequationes Math., **61** (2001), No. 1–2, 79–96.

[80] Wang X. P. and Si J. G., *Analytic solutions of an iterative functional differential equation*, J. Math. Anal. Appl., **262** (2001), No. 2, 490–498.

[81] Wu H. Z., *On existences and asymptotic behaviors of strong solutions of equation $x'(t) = f(x(x(t)))$*, Ann. Diff. Eqs., **9** (1993), No. 3, 336–351.

[82] Wu H. Z., *A class of functional differential equations with deviating arguments depending on the state*, Acta Math. Sinica, **38** (1995), No. 6, 803–809.

[83] Xiang X. F. and Ge W. G., *On the periodic solution of the differential-iterative equation $x'(t) = \omega(t)(ax(t) - bx(x(t)))$ $(a > b > 0)$*, J. Systems Sci. Math. Sci., **19** (1999), No. 4, 457–464. (in Chinese)

[84] Xu B., Zhang W. N. and Si J. G., *Analytic solutions of an iterative functional differential equation which may violate the Diophantine condition*, J. Difference Eq. Appl., **10** (2004), No. 2, 201–211.

[85] Yang W. X. and Ge, W. G., *Periodic solutions for the differential-iterative equation $\ddot{x} + g(x(x(t))) = p(t)$*, J. Beijing Inst. Technol. (Chin. Ed.), **22** (2002), No. 5, 537–539. (in Chinese)

[86] Xu J. H. and Zheng Z. X., *On the continuous solutions of a class of functional equations*, Mathematica Applicata, **11** (1998), No. 4, 1–3. (in Chinese)