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Relation between Different Types of Stabilities of Linear Nonautonomous Dynamical Systems

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Abstract

This paper is devoted to the study of the linear nonautonomous dynamical systems (LNDS) possessing the property of asymptotic (uniform asymptotic, uniform exponential) stability. We establish the relation between different types of stabilities for infinite-dimensional LNDS. We give applications of the general results for different class of linear nonautonomous differential equations (ordinary differential equations, retarded and neutral functional differential equations and some classes of partial differential equations) and linear difference inclusions.

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1 Introduction

In 1962 W. Hahn [17] posed the problem of whether asymptotic stability implies uniform stability for the linear equation

$$x' = A(t)x \qquad (x \in \mathbb{R}^n) \tag{1}$$

with almost periodic coefficients. C. C. Conley and R. K. Miller [12] gave a negative answer to this by constructing a scalar equation $x' = a(t)x$ with the property that every solution $\varphi(t, x, a) \to 0$ as $t \to +\infty$, but the null solution is not uniformly stable (see also [6]). From the results of R. J. Sacker and G. R. Sell [25] and I. U. Bronstein [2, p.141] it follows that for the linear system (1) with recurrent (in particular, almost periodic) matrix from the asymptotic stability of the null solution of system (1) and all systems

$$x' = B(t)x, \tag{2}$$

where $B \in H(A) := \overline{\{A_\tau : \tau \in \mathbb{R}\}}$, A_τ is the shift of the matrix A by τ, and by bar the closure in the topology of uniform convergence on every compact from $\mathbb{R}$ is denoted, there follows the uniform stability of the null solution of system (1). Finally we note that from the results of D. N. Cheban [3] there follows the validity of the above mentioned result for an arbitrary system (1) with a compact matrix (*i.e.*, when $H(A)$ is compact). Below we study the relationship between the asymptotic stability and uniform stability of the null solution of system (1) in an arbitrary Banach space.

Our main result is that for the linear system (1) with compact coefficients in an arbitrary Banach space the following statement takes place: if the cocycle φ generated by equation (1) is asymptotically compact and the null solution of equation (1) and all the equations (2) are asymptotically stable, then the null solution of equation (1) is uniformly stable (Theorem 2.20).

This paper is organized as follows.

In Section 2 we study the general (abstract) linear nonautonomous dynamical systems having the property of asymptotic stability. The main results in this section are Theorems 2.20 and 2.23 which establish the relation between asymptotic stability, uniform asymptotic stability and uniform exponential stability for asymptotically compact linear non-autonomous dynamical systems (linear cocycles).

Section 3 is devoted to the applications of the general results from Section 2 for linear nonautonomous differential equations (ordinary differential equations, retarded and neutral functional differential equations).

In Section 4 we study the linear difference inclusions. The aim of this section is the study of the problem of the absolute asymptotic stability of the discrete linear inclusion (see, for example, Gurvits [14] and the references therein)

$$x_{t+1} \in F(x_t),$$

where $F(x) = \{A_1x, A_2x, \dots, A_mx\}$ for all $x \in E$ (E is a Banach space) and A_i ($1 \le i \le m$) is a linear bounded operator acting in E. We establish the relation between absolute asymptotic stability (AAS), asymptotic stability (AS), uniform asymptotic stability (UAS) and uniform exponential stability (UES). It is proved that for asymptotically compact discrete linear inclusions these notions of stability are equivalent. We study this problem in the framework of nonautonomous dynamical systems (cocyles)

2 Nonautonomous dynamical systems and their compact global attractors

Assume that X and Y are complete metric spaces, $\mathbb{R}$ ($\mathbb{Z}$) is the group of real numbers (integers), $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{S}_+ = \{s \in \mathbb{S} \mid s \geq 0\}$, $\mathbb{S}_- = \{s \in \mathbb{S} \mid s \leq 0\}$, and $\mathbb{T}$ is a semigroup of the group $\mathbb{S}$ (for example, $\mathbb{T} = \mathbb{S}_+$ or $\mathbb{S}$).

Let (X, ρ) be a complete metric space and $(X, \mathbb{T}, \pi)$ be a dynamical system in X.

Definition 2.1 The system $(X, \mathbb{T}, \pi)$ is called:

– point dissipative if there exist a compact $K \subseteq X$ such that for every $x \in X$

$$\lim_{t \to +\infty} \rho(xt, K) = 0; \tag{3}$$

– compact dissipative if the equality (3) takes place uniformly w.r.t. x on the compacts from X;

– locally dissipative if for any point $p \in X$ there exist $\delta_p > 0$ such that the equality (3) takes place uniformly w.r.t. $x \in B(p, \delta_p)$.

Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of the group $\mathbb{S}$ and $(X, \mathbb{T}_1, \pi)$ $((Y, \mathbb{T}_2, \sigma))$ be a dynamical system on $X(Y)$.

Definition 2.2 A triple $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$, where h is a homomorphism of $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, is called [4] a *nonautonomous dynamical system.*

Definition 2.3 A nonautonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is said [2, 25] to be *linear* if the map $\pi^t : X_y \to X_{yt}$ is linear for every $t \in \mathbb{T}_1$ and $y \in Y$, where $X_y := h^{-1}(y) = \{x \in X \ : \ h(x) = y\}$, $yt := \sigma(t, y)$ and $\pi^t := \pi(t, \cdot)$.

Let (X, h, Y) be a locally trivial Banach fiber bundle [1, 19] and Θ be its trivial section, *i.e.*, $\Theta := \{\theta_y \ : \ y \in Y\}$ (where θ_y is a null element of the linear space X_y).

Definition 2.4 The fiber bundle (X, h, Y) is said to be *normed* if there exists a continuous mapping $|\cdot| : X \to \mathbb{R}_+$ such that $|\cdot|_y := |\cdot|_{X_y}$ is a norm on X_y and $|x|_y = \rho(x, \theta_y)$ for all $x \in X_y$ and $y \in Y$.

Theorem 2.5 [9, 10] *Let Y be compact, then the following assertions are equivalent:*

(i) *the dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is pointwise dissipative;*

(ii) $X^s = X$, *where* $X^s := \{x \in X \ : \ \lim\limits_{t\to+\infty} \|xt\| = 0\}$, *where* $\|x\| := \rho(x, \theta_{h(x)})$.

Theorem 2.6 [9, 10] *Let Y be compact, then the following statements are equivalent:*

(i) *the dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is compactly dissipative;*

(ii) *$X^s = X$ and the trivial section Θ of the fiber bundle (X, h, Y) is uniformly stable, that is, for all $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $\|x\| < \delta$ implies $\|xt\| < \varepsilon$ for all $t \ge 0$;*

(iii) *if the fiber bundle (X, h, Y) is normed, then there is a positive number $\mathcal{N}$ such that*

$$|xt| \le \mathcal{N}|x|$$

for all $x \in X$, $t \ge 0$ and $X^s = X$.

Theorem 2.7 [9, 10] *Let Y be compact, then the following statements are equivalent:*

(i) *the dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is locally dissipative;*

(ii) *$X^s = X$ and the trivial section Θ of the fiber bundle (X, h, Y) is uniformly attracting, i.e., there is $\gamma > 0$ such that*

$$\lim_{t\to+\infty} \sup_{\|x\|\le\gamma} \|\pi(t, x)\| = 0;$$

(iii) *if the fiber bundle (X, h, Y) is normed, then the linear nonautonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is uniformly exponentially stable, i.e., there are positive numbers $\mathcal{N}$ and ν such that*

$$|\pi(t, x)| \le \mathcal{N} e^{-\nu t}|x| \tag{4}$$

for all $x \in X$, $t \ge 0$.

Let W, Y be two complete metric spaces and $(Y, \mathbb{T}, \sigma)$ be a dynamical system on Y.

Definition 2.8 Recall [28] that a triplet $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ (or briefly φ) is called a *cocycle* over $(Y, \mathbb{T}_2, \sigma)$ with the fiber W, if φ is a mapping from $\mathbb{T}_1 \times W \times \Omega$ to W satisfying the following conditions:

1. $\varphi(0, x, y) = x$ for all $(x, y) \in W \times Y$;
2. $\varphi(t + \tau, x, y) = \varphi(t, \varphi(\tau, x, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{T}_1$ and $(x, y) \in W \times Y$;
3. the mapping φ is continuous.

If W is a real or complex Banach space and

4. $\varphi(t, \lambda x_1 + \mu x_2, y) = \lambda\varphi(t, x_1, \omega) + \mu\varphi(t, x_2, \omega)$ for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $t, \tau \in \mathbb{T}_1$, $x_1, x_2 \in W$ and $y \in Y$,

then the cocycle φ is called *linear*.

Let $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ be a cocycle (respectively, linear cocycle) over $(Y, \mathbb{T}_2, \sigma)$ with the fiber W (or shortly φ). If $X := W \times Y, \pi := (\varphi, \sigma)$, *i.e.*, $\pi((u, y), t) := (\varphi(t, x, y), \sigma(t, y))$ for all $(u, y) \in W \times Y$ and $t \in \mathbb{T}_1$, then the dynamical system $(X, \mathbb{T}_2, \pi)$ is called [28] a *skew product* over $(Y, \mathbb{T}_2, \sigma)$ with the fiber W and the triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is a nonautonomous (respectively, linear nonautonomous) dynamical system generated by the cocycle φ, where $h := pr_2 : X \to Y$.

Definition 2.9 The mapping $\lambda : B(X) \to \mathbb{R}_+$, satisfying the following conditions:

(i) $\lambda(A) = 0$ if and only if $A \in B(X)$ is relatively compact;

(ii) $\lambda(A \cup B) = \max(\lambda(A), \lambda(B))$ for every A, $B \in B(X)$.

is called [15, 16, 26] a *measure of non-compactness* on X.

Definition 2.10 The *measure of non-compactness of Kuratowski* $\alpha : B(X) \to \mathbb{R}_+$ is defined by the equality $\alpha(A) := \inf\{\varepsilon > 0 \mid A$ admits a finite ε-covering $\}$.

Definition 2.11 The dynamical system $(X, \mathbb{T}, \pi)$ is said to be *asymptotically compact* [16, 21] if for any bounded subset $M \subseteq X$ with a bounded semi-trajectory $\Sigma_M^+ := \bigcup\{\pi^t M \ : \ t \geq 0\}$ there exists a compact subset $K \subseteq X$ such that $\lim\limits_{t\to+\infty} \beta(\pi^t M, K) = 0$.

Definition 2.12 Recall that a dynamical system $(X, \mathbb{T}_1, \pi)$ is said to be *conditionally β-condensing* [16] if there exists $t_0 > 0$ such that $\beta(\pi^{t_0} B) < \beta(B)$ for all bounded sets B in X with $\beta(B) > 0$. The dynamical system $(X, \mathbb{T}_1, \pi)$ is said to be *β-condensing* if it is conditionally β-condensing and the set $\pi^{t_0} B$ is bounded for all bounded sets $B \subseteq X$.

According to Lemma 2.3.5 in [16, p.15] and Lemma 3.3 in [5] the conditional condensing dynamical system $(X, \mathbb{T}_1, \pi)$ is asymptotically compact.

Let W be a metric space, $X := W \times Y$, $A \subset X$, and $A_y := \{x \in A : pr_2 x = y\}$. Then $A = \cup\{A_y : y \in Y\}$. Let $\tilde{A}_y := pr_1 A_y$ and $\tilde{A} = \cup\{\tilde{A}_y : y \in Y\}$. Note that if the space Y is compact, then a set $A \subset X$ is bounded in X if and only if the set $\tilde{A}$ is bounded in W.

Lemma 2.13 [8, 10] *The equality $\alpha(A) = \alpha(\tilde{A})$ takes place for all bounded sets $A \subset X$, where $\alpha(A)$ (respectively $\alpha(\tilde{A})$) is the Kuratowski measure of non-compactness for the sets $A \subset X$ (respectively $\tilde{A} \subset W$).*

Definition 2.14 A cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ is called *conditionally α-condensing* if there exists $t_0 > 0$ such that for any bounded set $B \subseteq W$ the inequality $\alpha(\varphi(t_0, B, Y)) < \alpha(B)$ holds if $\alpha(B) > 0$. The cocycle φ is called *α-condensing* if it is a conditionally α-condensing cocycle and the set $\varphi(t_0, B, Y) = \cup\{\varphi(t_0, u, Y) | u \in B, y \in Y\}$ is bounded for all bounded set $B \subseteq W$.

Definition 2.15 A cocycle φ is called *conditional α-contraction of order $k \in [0, 1)$* if there exists $t_0 > 0$ such that for any bounded set $B \subseteq W$ for which $\varphi(t_0, B, Y) = \cup\{\varphi(t_0, u, Y) | u \in B, y \in Y\}$ is bounded the inequality $\alpha(\varphi(t_0, B, Y)) \leq k\alpha(B)$ holds. The cocycle φ is called *α-contraction* if it is a conditional α-contraction cocycle and the set $\varphi(t_0, B, Y) = \cup\{\varphi(t_0, u, Y) | u \in B, y \in Y\}$ is bounded for all bounded sets $B \subseteq W$.

Lemma 2.16 [8, 10] *Let Y be compact and the cocycle φ be α-condensing. Then the skew-product dynamical system $(X, \mathbb{T}_1, \pi)$, generated by the cocycle φ, is α-condensing too.*

Theorem 2.17 [8, 10] *Let E be a Banach space, φ be a cocycle on $(Y, \mathbb{T}_2, \sigma)$ with fiber E and the following conditions be fulfilled:*

(i) *$\varphi(t, u, y) = \psi(t, u, y) + \gamma(t, u, y)$ for all $t \in \mathbb{T}_1, u \in E$ and $y \in Y$.*

(ii) *There exists a function $m : \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfying the condition $m(t, r) \to 0$ as $t \to +\infty$ (for every $r > 0$) such that $|\psi(t, u_1, y) - \psi(t, u_2, y)| \leq m(t, r)|u_1 - u_2|$ for all $t \in \mathbb{T}_1, u_1, u_2 \in B[0, r]$ and $y \in Y$.*

(iii) *$\gamma(t, A, Y)$ is compact for all bounded $A \subset X$ and $t > 0$.*

Then the cocycle φ is an α-contraction.

Definition 2.18 The cocycle $\langle E, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ is said to be *asymptotically compact* if the bounded sequence $\{\varphi(t_n, u_n, y_n)\}$ is relatively compact for any $\{t_n\} \to +\infty$ $(t_n \in \mathbb{T})$, bounded sequence $\{u_n\} \subset E$ and $\{y_n\} \subset Y$.

Remark 2.19 If the space Y is compact, then the cocycle $\langle E, \varphi, (Y, \mathbb{T}_2, \sigma)\rangle$ is asymptotically compact if and only if the skew-product dynamical system $(X, \mathbb{T}_1, \pi)$ $(X := E \times Y,\ \pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y)$, generated by the cocycle φ is such.

Theorem 2.20 *Let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a linear cocycle. Suppose that the following conditions are fulfilled:*

(i) *Y is a compact metric space;*

(ii) *the cocycle φ is asymptotically stable, i.e., $\lim\limits_{t\to+\infty} |\varphi(t, u, y)| = 0$ for all $(u, y) \in E \times Y$.*

Then the cocycle φ is uniformly stable, i.e., there exists a positive constant M such that $|\varphi(t, u, y)| \le M|u|$ for $t \in \mathbb{T}_1^+ := \{t \in \mathbb{T}_1\ : t \ge 0\}$ and $(u, y) \in E \times Y$.

Proof. Consider the family of linear bounded operators $\{U(t, y)\ :\ t \in \mathbb{T}_1^+, y \in Y\}$ acting on the space E, where $U(t, y) := \varphi(t, \cdot, y)$. By the principle of uniform boundedness it is sufficient to show that for each $u \in E$ there exists a constant $M_u > 0$ such that

$$|U(t, y)u| \le M_u \tag{5}$$

for all $t \in \mathbb{T}_+$ and $y \in Y$. If we suppose that it is not true, then there are $u_0 \in E$ $(|u_0| = 1)$, $t_n \in \mathbb{T}_+$ and $y_n \in Y$ such that

$$|U(t_n, y_n)u_0| \ge n \quad \text{and} \quad |U(t, y_n)u_0| < n \quad (0 \le t \le t_n). \tag{6}$$

Since the space Y is compact, we may suppose that the sequence $\{t_n\} \to +\infty$. Denote by

$$\alpha_n := \sup\{t :\ |U(t, y_n)u_0| \le 1\}.$$

It is clear that $\alpha_n \le t_n$ and

$$|U(t, y_n)u_0| \ge 1$$

for all $t \in (\alpha_n, t_n]$.

Logically there are two possibilities:

1. The sequence $\{\alpha_n\}$ is bounded and, consequently, we may suppose that it is convergent. Let $\alpha := \lim\limits_{n\to+\infty} \alpha_n$, then we have

$$|U(t + \alpha_n, y_n)u_0| = |U(t, y_n\alpha_n)U(\alpha_n, y_n)u_0| \ge 1 \tag{7}$$

for all $t \in (0, t_n - \alpha_n]$. Since the space Y is compact, then we may suppose that $\{y_n\}$ and $\{y_n\alpha_n\}$ are convergent. Let $y_0 := \lim\limits_{n\to+\infty} y_n$ and $\bar{y}_0 := \lim\limits_{n\to+\infty} y_n\alpha_n$. Passing to the limit in (7) as $n \to +\infty$, we obtain

$$|U(t, \bar{y}_0)\bar{u}_0| \ge 1 \tag{8}$$

for all $t \in (0, +\infty)$, where $\bar{u}_0 := \lim_{n \to +\infty} U(\alpha_n, y_n)u_0 = U(\alpha, y_0)u_0$. But the condition (8) contradicts the asymptotic stability of the cocycle φ.

2. The sequence $\{\alpha_n\}$ is unbounded and, consequently, we may suppose that $\{\alpha_n\} \to +\infty$. We will show that in this case the sequence $\{t_n - \alpha_n\}$ is unbounded too and we may suppose that $\{t_n - \alpha_n\} \to +\infty$. In fact, if it is not true, then we may consider that this sequence is convergent. Let $0 \le \beta := \lim_{n \to +\infty} t_n - \alpha_n$. We observe that

$$|U(t_n, y_n)u_0| = |U(t_n - \alpha_n, y_n\alpha_n)U(\alpha_n, y_n)u_0|. \tag{9}$$

Since the sequence $\{U(\alpha_n, y_n)u_0\}$ is bounded and the cocycle φ is asymptotically compact, then we may suppose that the sequence $\{U(\alpha_n, y_n)u_0\}$ is convergent. By the compactness of Y we may consider that the sequence $\{y_n\alpha_n\}$ is convergent too. Let $\bar{u}_0 := \lim_{n \to +\infty} U(\alpha_n, y_n)u_0$ and $\bar{y}_0 := \lim_{n \to +\infty} y_n\alpha_n$. From the equality (9) it follows that the sequence $\{U(t_n, y_n)u_0\}$ is convergent and $\lim_{n \to +\infty} |U(t_n, y_n)u_0| = |U(\beta, \bar{y}_0)\bar{u}_0|$. In particular, the sequence $\{|U(t_n, y_n)u_0|\}$ is bounded, *i.e.*,

$$\sup\{|U(t_n, y_n)u_0| \ : n \in \mathbb{N}\} < +\infty. \tag{10}$$

But relations (6) and (10) contradict each other.

The contradiction obtained proves the relation (5). $\square$

Remark 2.21 1. If $\mathbb{T} = \mathbb{S}$ and Y is a compact and minimal set, then Theorem 2.20 holds without the assumption of asymptotical compactness of the cocycle φ (see [8]).

2. If the Banach space E is reflexive, then using the results [30] it is possible to show that Theorem 2.20 is still true without the asymptotical compactness of the cocycle φ.

Theorem 2.22 [8, 9, 10] *Let $(X, \mathbb{T}, \pi)$ be an asymptotically compact dynamical system. Then the following statements are equivalent:*

(i) *the dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative;*

(ii) *the dynamical system $(X, \mathbb{T}, \pi)$ is locally dissipative.*

Theorem 2.23 *Let $\langle E, \varphi, (Y, \mathbb{T}_2, \sigma), h\rangle$ be a linear asymptotically compact cocycle and Y be compact. Then the following assertions are equivalent:*

(i) *the cocycle φ is asymptotically stable (convergent), i.e.,*

$$\lim_{t \to +\infty} |\varphi(t, u, y)| = 0$$

for all $(u, y) \in E \times Y$;

(ii) *the cocycle* φ *is uniformly exponentially stable, i.e., there are two positive constants* $\mathcal{N}$ *and* ν *such that*

$$|\varphi(t,u,y)| \le \mathcal{N} e^{-\nu t}|u|$$

for all $t \ge 0$ *and* $(u,y) \in E \times Y$.

Proof. Let $X := E \times Y$, $(X, \mathbb{T}_1, \pi)$ (respectively $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$) be the skew-product (respectively, linear nonautonomous) dynamical system, generated by the linear cocycle φ. Under the conditions of the Theorem the skew-product dynamical system $(X,\mathbb{T}_1,\pi)$ is asymptotically compact and pointwise dissipative. By Theorem 2.20 the linear nonautonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is uniformly stable and, consequently, according to Theorem 2.6 it is compactly dissipative. According to Theorem 2.22 the dynamical system $(X,\mathbb{T},\pi)$ is locally dissipative. Now to finish the proof of Theorem it is sufficient to refer to Theorem 2.7. $\square$

3 Linear nonautonomous differential equations

3.1 Linear ordinary differential equations.

Let $\Lambda = [E]$ be a set of all linear bounded operators acting in the Banach space E and consider the linear differential equation

$$u' = A(t)u, \tag{11}$$

where $A \in C(\mathbb{R},\Lambda)$. Along with equation (11), we shall also consider its H-class, that is, the family of equations

$$v' = B(t)v, \tag{12}$$

where $B \in H(A) := \overline{\{A_\tau : \tau \in \mathbb{R}\}}$, $A_\tau(t) = A(t+\tau)$ $(t \in \mathbb{R})$, and the bar denotes closure in $C(\mathbb{R},\Lambda)$. Let $\varphi(t,u,B)$ be the solution of equation (12) that satisfies the condition $\varphi(0,u,B) = u$. Let $(C(\mathbb{R},\Lambda),\mathbb{R},\sigma)$ be a shifts dynamical system on $C(\mathbb{R},\Lambda)$ (see, for example, [28, 29]). We put $Y := H(A)$ and denote the dynamical system of shifts on $H(A)$ by $(Y,\mathbb{R},\sigma)$. Then the triple $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ is a linear nonautonomous dynamical system, where $X := E \times Y$, $\pi := (\varphi,\sigma)$; *i.e.*, $\pi((v,B),\tau) := (\varphi(\tau,v,B),B_\tau)$ and $h := pr_2 : X \to Y$).

Theorem 3.1 *Let* $A \in C(\mathbb{R},\Lambda)$ *be compact (i.e.,* $H(A)$ *is a compact subset of* $(C(\mathbb{R},\Lambda),\mathbb{R},\sigma)$ *), the nonautonomous system* $\langle (X,\mathbb{R}_+,\pi)$, $(Y,\mathbb{R},\sigma),h\rangle$ *generated by equation* (11) *is asymptotically compact. Then the following conditions are equivalent:*

1. *The trivial solution of equation* (11) *is uniformly exponentially stable, i.e., there exist positive numbers N and ν such that $\|U(t,A)U(\tau,A)^{-1}\| \le Ne^{-(t-\tau)}$ for all $t \ge \tau$.*

2. $\lim\limits_{t\to+\infty} |\varphi(t,u,B)| = 0$ *for every $u \in E$ and $B \in H(A)$.*

Proof. Applying Theorem 2.23 to the nonautonomous system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by equation (11), we obtain the equivalence of conditions 1. and 2. The theorem is proved. □

We now formulate some sufficient conditions for the α-condensingness (in particular, asymptotic compactness) of the linear non-autonomous dynamical system generated by equation (11).

Theorem 3.2 [8, 10] *Let $A \in C(\mathbb{R},[E]), A(t) = A_1(t) + A_2(t)$ for all $t \in \mathbb{R}$, and assume that $H(A_i)$ $(i = 1,2)$ are compact and the following conditions hold:*

(i) *The null solution of the equation*

$$u' = A_1(t)u \tag{13}$$

is uniformly exponentially stable, that is, there are positive numbers $\mathcal{N}$ and ν such that

$$\|U(t,A_1)U^{-1}(\tau,A_1)\| \le \mathcal{N}e^{-\nu(t-\tau)} \tag{14}$$

for all $t \ge \tau$ $(t,\tau \in \mathbb{R})$, where $U(t,A_1)$ is the Cauchy operator of equation (13)*.*

(ii) *The family of operators $\{A_2(t) : t > 0\}$ is uniformly completely continuous, that is, for any bounded set $A \subset E$ the set $\{A_2(t)A : t > 0\}$ is relatively compact.*

Then the linear nonautonomous dynamical system generated by equation (11) *is an α-contraction.*

Theorem 3.3 [8, 10] *Let $H(A)$ be compact and assume that there is a finite-dimensional projection $P \in [E]$ such that*

(i) *the family of projections $\{P(t) : t \in \mathbb{R}\}$, where $P(t) := U(t,A)PU^{-1}(t,A)$, is relatively compact in $[E]$, and*

(ii) *there are positive numbers $\mathcal{N}$ and ν such that*

$$\|U(t,A)QU^{-1}(\tau,A)\| \le \mathcal{N}e^{-\nu(t-\tau)}$$

for all $t \ge \tau$, where $Q := I - P$.

Then the linear nonautonomous dynamical system generated by equation (11) *is an α-contraction.*

3.2 Linear functional differential equations

Let $r > 0$, $C([a,b],\mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi : [a,b] \to \mathbb{R}^n$ with the sup norm. For $[a,b] := [-r,0]$ we put $\mathcal{C} := C([-r,0],\mathbb{R}^n)$. Let $c \in \mathbb{R}$, $a \geq 0$, and $u \in C([c-r,c+a],\mathbb{R}^n)$. We define $u_t \in \mathcal{C}$ for any $t \in [c,c+a]$ by the relation $u_t(\theta) := u(t+\theta), -r \geq \theta \geq 0$. Let $\mathfrak{A} = \mathfrak{A}(\mathcal{C},\mathbb{R}^n)$ be the Banach space of all linear operators that act from $\mathcal{C} \to \mathbb{R}^n$ equipped with the operator norm, let $C(\mathbb{R},\mathfrak{A})$ be the space of all operator-valued functions $A : \mathbb{R} \to \mathfrak{A}$ with the compact-open topology, and let $(C(\mathbb{R},\mathfrak{A}),\mathbb{R},\sigma)$ be the dynamical system of shifts on $C(\mathbb{R},\mathfrak{A})$. Let $H(\mathfrak{A}) := \overline{\{\mathfrak{A}_\tau \mid \tau \in \mathbb{R}\}}$, where $\mathfrak{A}_\tau$is the shift of the operator-valued function $\mathfrak{A}$ by τ and the bar denotes closure in $C(\mathbb{R},\mathfrak{A})$.

Consider the equation

$$u' = A(t)u_t, \tag{15}$$

where $A \in C(\mathbb{R},\mathfrak{A})$. We put $H(A) := \overline{\{A_\tau : \tau \in \mathbb{R}\}}$, $A_\tau(t) = A(t+\tau)$, where the bar denotes closure in the topology of uniform convergence on every compact of $\mathbb{R}$.

Along with equation (15) we also consider the family of equations

$$u' = B(t)u_t, \tag{16}$$

where $B \in H(A)$. Let $\varphi_t(v,B)$ be a solution of equation (16) with the condition $\varphi_0(v,B) = v$ defined on $\mathbb{R}_+$. We put $Y := H(A)$ and denote by $(Y,\mathbb{R},\sigma)$ the dynamical system of shifts on $H(A)$. Let $X := \mathcal{C} \times Y$ and $\pi := (\varphi,\sigma)$ the dynamical system on X defined by the equality $\pi(\tau,(v,B)) := (\varphi_\tau(v,B),B_\tau)$. The nonautonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ ($h := pr_2 : X \to Y$) is linear. The following assertion is valid.

Lemma 3.4 [8, 10] *Let $H(A)$ be compact in $C(\mathbb{R},\mathfrak{A})$. Then the linear nonautonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by equation* (15) *is completely continuous (in particularly, asymptotically compact), that is, for any bounded set $\mathcal{A} \subseteq X$ there is an $l = l(\mathcal{A}) > 0$ such that $\pi^l A$ is relatively compact.*

Theorem 3.5 *Let $H(A)$ be compact. Then the following assertions are equivalent:*

(i) *For any $B \in H(A)$ the zero solution of equation* (16) *is asymptotically stable, i.e., $\lim\limits_{t\to+\infty} |\varphi_t(v,B)| = 0$ for all $v \in \mathcal{C}$ and $B \in H(A)$.*

(ii) *The null solution of equation* (15) *is uniformly exponentially stable, i.e., there are positive numbers $\mathcal{N}$ and ν such that $|\varphi_t(v,B)| \leq \mathcal{N}e^{-\nu t}|v|$ for all $t \geq 0, v \in \mathcal{C}$ and $B \in H(A)$.*

Proof. Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ be the linear nonautonomous dynamical system generated by equation (15). According to Lemma 3.4 this system is completely continuous and to finish the proof it is sufficient to refer to Theorem 2.23. $\square$

Consider the neutral functional differential equation

$$\frac{d}{dt} D x_t = A(t) x_t, \tag{17}$$

where $A \in C(\mathbb{R}, \mathfrak{A})$ and $D \in \mathfrak{A}$ is an operator non-atomic at the zero [15, p.67]. As well as in the case of equation (15), the equation (17) generates a linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$, where $X = \mathcal{C} \times Y$, $Y = H(A)$ and $\pi = (\varphi, \sigma)$. The following statement takes place.

Lemma 3.6 [8, 10] *Let $H(A)$ be compact and the operator D is stable; i.e., the null solution of the homogeneous difference equation $Dy_t = 0$ is uniformly asymptotically stable. Then the linear nonautonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$, generated by equation* (17)*, is conditionally α-condensing.*

Theorem 3.7 *Let $A \in C(\mathbb{R}, \mathfrak{A})$ be compact (i.e., $H(A)$ is a compact set of the shifts dynamical system $(C(\mathbb{R}, \mathfrak{A}), \mathbb{R}, \sigma)$) and D is stable, then the following assertions are equivalent:*

(i) *The null solution of equation* (15) *and all equations*

$$\frac{d}{dt} D x_t = B(t) x_t, \tag{18}$$

where $B \in H(A)$, is asymptotically stable, i.e., $\lim_{t \to +\infty} |\varphi_t(v, B)| = 0$ for all $v \in \mathcal{C}$ and $B \in H(A)$ ($\varphi_t(v, B)$ is the solution of equation (18) *with condition $\varphi_0(v, B) = v$);*

(ii) *The null solution of equation* (17) *is uniformly exponentially stable, i.e., there are positive numbers N and ν such that $|\varphi_t(v, B)| \le N e^{-\nu t} |v|$ for all $t \ge 0, v \in \mathcal{C}$ and $B \in H(A)$.*

Proof. Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ be the linear nonautonomous dynamical system generated by equation (17). According to Lemma 3.6 this system is conditionally α condensing. To finish the proof of Theorem 3.7 it is sufficient to refer to Theorem 2.23. $\square$

3.3 Linear partial differential equations

Let Λ be some complete metric space of linear closed operators acting in a Banach space E (for example $\Lambda = \{A_0 + C | C \in [E]\}$, where A_0 is a closed operator that acts in E). We assume that the following conditions are fulfilled for equation (11) and its H-class (12):

a. for any $v \in E$ and $B \in H(A)$ equation (12) has exactly one mild solution $\varphi(t,v,B)$ (*i.e.*, $\varphi(\cdot,v,B)$ is continuous, defined on $\mathbb{R}_+$ and satisfies the equation
$$\varphi(t,v,B) = U(t,B)v + \int_0^t U(t-\tau,B)\varphi(\tau,v,B)\,d\tau \tag{19}$$
and the condition $\varphi(0,v,B) = v$);

b. the mapping $\varphi : (t,v,B) \to \varphi(t,v,B)$ is continuous in the topology of $\mathbb{R}_+ \times E \times C(\mathbb{R};\Lambda)$.

Under the above assumptions the equation (11) generates a linear nonautonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$, where $X := E \times Y, \pi = (\varphi,\sigma)$ and $h := pr_2 : X \to Y$. Applying the results from Section 2 to this system, we will obtain the analogous assertions for different classes of partial differential equations.

We will consider examples of partial differential equations which satisfy the above conditions a.-b.

Example 3.8 Consider the differential equation
$$u' = (A_1 + A_2(t))u, \tag{20}$$
where A_1 is a sectorial operator that does not depend on $t \in \mathbb{R}$, and $A_2 \in C(\mathbb{R},[E])$. The results of [18, 22] imply that equation (20) satisfies conditions a.-b.

Example 3.9 Let H be a Hilbert space with a scalar product $\langle\cdot,\cdot\rangle = |\cdot|^2$, $\mathcal{D}(\mathbb{R}_+,H)$ be the set of all infinitely differentiable, bounded functions on $\mathbb{R}_+$ with values in H.

Denote by $(C(\mathbb{R},[H]),\mathbb{R},\sigma)$ a dynamical system of shifts on $C(\mathbb{R},[H])$. Consider the equation
$$\int\limits_{\mathbb{R}_+} [\langle u(t),\varphi'(t)\rangle + \langle A(t)u(t),\varphi(t)\rangle]\,dt = 0, \tag{21}$$
along with equation (21) we consider also the family of equations
$$\int\limits_{\mathbb{R}_+} [\langle u(t),\varphi'(t)\rangle + \langle B(t)u(t),\varphi(t)\rangle]\,dt = 0, \tag{22}$$

where $B \in H(A) := \overline{\{A_\tau | \tau \in \mathbb{R}\}}$, $A_\tau(t) := (t+\tau)$ and the bar denotes closure in $C(\mathbb{R}, [H])$.

The function $u \in C(\mathbb{R}_+, H)$ is called a *solution* of equation (21) if (21) takes place for all $\varphi \in \mathcal{D}(\mathbb{R}_+, H)$.

Let $(H(A), \mathbb{R}, \sigma)$ be a dynamical system of shifts on $H(A)$, $\varphi(t, v, B)$ be a solution of (22) with condition $\varphi(0, v, B) = v$, $\tilde{X} := H \times H(A)$, X be a set of all the points $\langle u, B \rangle \in \tilde{X}$ such that through the point $u \in H$ there passes a solution $\varphi(t, u, A)$ of equation (21) defined on $\mathbb{R}_+$. According to Lemma 2.21 in [7] the set X is closed in $\tilde{X}$. By virtue of Lemma 2.22 in [7] the triple $(X, \mathbb{R}_+, \pi)$ is a dynamical system on X (where $\pi := (\varphi, \sigma)$) and $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a linear non-autonomous dynamical system, where $h := pr_2 : X \to Y := H(A)$.

Theorem 3.10 *Let $A \in C(\mathbb{R}, \Lambda)$ be compact (i.e., the set $H(A)$ is compact in $C(\mathbb{R}, \Lambda)$) and the cocycle φ generated by equation* (11) *be asymptotically compact, then the following assertion are equivalent:*

(i) *The trivial solution of equation* (11) *is uniformly exponentially stable, i.e., there exist positive numbers N and ν such that $\|U(t, B)\| \leq Ne^{-\nu t}$ for all $t \geq 0$ and $B \in H(A)$.*

(ii) $\lim\limits_{t \to +\infty} |\varphi(t, u, B)| = 0$ *for all $(u, B) \in E \times H(A)$.*

Proof. This statement follows directly from Theorem 2.23. □

4 Discrete linear inclusions

4.1 Discrete linear inclusions and cocycles

Consider a compact set of operators $\mathcal{M} \subseteq [E]$.

Definition 4.1 A *discrete linear inclusion* $DLI(\mathcal{M})$ is called (see, for example, [14]) the set of all sequences $\{\{x_j\} \mid j \geq 0\}$ of vectors in E such that

$$x_j = A_{i_j} x_{j-1} \tag{23}$$

for some $A_{i_j} \in \mathcal{M}$ (trajectory of $DLI(\mathcal{M})$), *i.e.*,

$$x_j = A_{i_j} A_{i_{j-1}} \dots A_{i_1} x_0, \text{ all } A_{i_k} \in \mathcal{M}. \tag{24}$$

Definition 4.2 A bilateral sequence $\{\{x_j\} \mid j \in \mathbb{Z}\}$ of vectors in E is called a *full trajectory* of $DLI(\mathcal{M})$ (*entire trajectory* or *trajectory on* $\mathbb{Z}$), if $x_{n+j} = A_{i_j} x_{n+j-1}$ for all $n \in \mathbb{Z}$.

We may consider that it is a discrete control problem, where at each moment of time j we may apply a control from the set $\mathcal{M}$, and $DLI(\mathcal{M})$ is the set of possible trajectories of the system. The basic issue for any control system concerns its stability. One of the most important types of stability is the so called absolute asymptotic stability (AAS).

Definition 4.3 $DLI(\mathcal{M})$ is called *absolutely asymptotically stable* (AAS) (or *convergent*), if for any trajectory $\{x_j\}$ of its we have

$$\lim_{j\to\infty} x_j = 0.$$

Equivalently, all operator products

$$\lim_{j\to\infty} A_{i_j}A_{i_{j-1}}\dots A_{i_1}x = 0 \quad (\text{all } A_{i_j} \in \mathcal{M})$$

for every $x \in E$.

Definition 4.4 A set $\mathcal{M} \subseteq [E]$ of operators is called *product bounded* if there exists $M > 0$ such that $\|A_{i_n}A_{i_{n-1}}\dots A_{i_1}\| \le M$ for all finite sequences $\{i_1, i_2, \dots, i_n\}$ $(n \in \mathbb{N})$.

Definition 4.5 $DLI(\mathcal{M})$ is said to be *asymptotically stable* (AS) if it is product bounded (or uniformly stable) and is convergent.

Definition 4.6 $DLI(\mathcal{M})$ is said to be *uniformly asymptotically stable* (or *uniformly convergent*) (UAS) if

$$\lim_{n\to+\infty} \|A_{i_n}A_{i_{n-1}}\dots A_{i_1}\| = 0$$

uniformly with respect to the sequences $\{A_{i_n}\}$ $(A_{i_n} \in \mathcal{M}$ for all $n \in \mathbb{N})$.

Definition 4.7 $DLI(\mathcal{M})$ is said to be *uniformly exponentially stable* (UES) if there are two positive constants $\mathcal{N}$ and $\alpha \in (0,1)$ such that

$$\|A_{i_n}A_{i_{n-1}}\dots A_{i_1}x\| \le \mathcal{N}\alpha^n|x|$$

for all $x \in E,\ n \in \mathbb{N}$ and any sequence $\{A_{i_n}\}$ $(A_{i_n} \in \mathcal{M}$ for all $n \in \mathbb{N})$.

Let (X,ρ) be a complete metric space with the metric ρ. Denote by $K(X)$ the family of all compact subsets of X. Consider the set-valued function $F : E \to K(E)$

defined by the equality $F(x) := \{Ax \,|A \in \mathcal{M}\}$. Then the discrete linear inclusion $DLI(\mathcal{M})$ is equivalent to the difference inclusion

$$x_j \in F(x_{j-1}). \tag{25}$$

Below we will give a new approach concerning the study of discrete linear inclusions $DLI(\mathcal{M})$ (or difference inclusion (25)). Denote by $C(\mathbb{Z}_+, X)$ the space of all continuous mappings $f : \mathbb{Z}_+ \to X$ equipped with the compact-open topology. This topology can be metrized. For instance, by the equality

$$d(f^1, f^2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f^1, f^2)}{1 + d_n(f^1, f^2)}$$

$$(d_n(f^1, f^2) := \max\{|f^1(k) - f^2(k)| \mid 0 \le k \le n\})$$

a complete metric on $C(\mathbb{Z}_+, X)$ is defined, which generates a compact-open topology. Denote by $(C(\mathbb{Z}_+, X), \mathbb{Z}_+, \sigma)$ a dynamical system of translations (shifts dynamical system or dynamical system of Bebutov [28, 29]) on $C(\mathbb{Z}_+, X)$, *i.e.*, $\sigma(k, f) := f_k$ and f_k is a $k \in \mathbb{Z}_+$ shift of f (*i.e.*, $f_k(n) := f(n + k)$ for all $n \in \mathbb{Z}_+$).

We may now rewrite equation (23) in the following way:

$$x_{j+1} = \omega(j)x_j (\omega \in \Omega), \tag{26}$$

where $\omega \in \Omega$ is the operator-function defined by the equality $\omega(j) := A_{i_{j+1}}$ for all $j \in \mathbb{Z}_+$. Denote by $\varphi(n, x_0, \omega)$ a solution of equation (26) issuing from the point $x_0 \in E$ at the initial moment $n = 0$. Note that $DLI(\mathcal{M})$ (or inclusion (25)) is equivalent to the family of linear nonautonomous equations (26) ($\omega \in \Omega$).

From the general properties of linear difference equations it follows that the mapping $\varphi : \mathbb{Z}_+ \times E \times \Omega \to E$ satisfies the following conditions:

(i) $\varphi(0, x_0, \omega) = x_0$ for all $(x_0, \omega) \in E \times \Omega$;

(ii) $\varphi(n+\tau, x_0, \omega) = \varphi(n, \varphi(\tau, x_0, \omega), \sigma(\tau, \omega))$ for all $n, \tau \in \mathbb{Z}_+$ and $(x_0, \omega) \in E\times\Omega$;

(iii) the mapping φ is continuous;

(iv) $\varphi(n, \lambda x_1 + \mu x_2, \omega) = \lambda\varphi(n, x_1, \omega) + \mu\varphi(n, x_2, \omega)$ for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $x_1, x_2 \in E$ and $\omega \in \Omega$.

From what was presented above it follows that every $DLI(\mathcal{M})$ (respectively, inclusion (25)) in a natural way generates a linear cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$, where $\Omega = C(\mathbb{Z}_+, \mathcal{M})$, $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system of shifts on Ω and $\varphi(n, x, \omega)$ is a solution of equation (26) issuing from the point $x \in E$ at the initial moment $n = 0$. Notice that by the theorem of Tychonoff [20] the space Ω is compact in $C(\mathbb{Z}_+, [E])$. Thus, we can study the inclusion (25) (respectively, $DLI(\mathcal{M})$) in the framework of the theory of linear cocycles with discrete time.

4.2 Absolute asymptotic stability of discrete linear inclusions in Banach spaces

In this section we will study $DLI(\mathcal{M})$ in an arbitrary Banach space. Let E be a real or complex Banach space with the norm $|\cdot|$ and $[E]$ be a Banach space of all linear bounded operators acting on the space E and equipped with the operator norm. Below we suppose that $\mathcal{M} := \{A_1, A_2, \dots, A_m\}$ and $A_i \in [E]$.

Note that for infinite-dimensional discrete linear inclusions $DLI(\mathcal{M})$ ($\dim(E) < +\infty$) the notion of absolute asymptotic stability (AAS) and the equality

$$\lim_{n\to+\infty} \|A_{i_n}A_{i_{n-1}}\dots A_{i_1}\| = 0 \tag{27}$$

are equivalent. It is well known (see, for example, [11]) that for infinite-dimensional $DLI(\mathcal{M})$ ($\dim(E) = +\infty$) it is not true. This fact is confirmed by the following example.

Example 4.8 Let $E := c_0$, $A \in [c_0]$ be the operator defined by the equality

$$A\xi := \{\xi_{k+1}\}$$

for all $\xi := \{\xi_k\} \in c_0$. It is easy to check that the operator A has the following properties:

(i)
$$A^n\xi \to 0 \tag{28}$$
as $n \to \infty$ for each $\xi \in c_0$, where $A^n := A \circ A^{n-1}$ $(n \geq 1)$ and $A^0 := Id_E$;

(ii)
$$A^n e_{n+1} = e_1, \tag{29}$$
where $e_1 = (1,0,0,\dots),\ e_2 = (0,1,0,\dots),\dots\ (n = 1,2,\dots)$.

Let $\mathcal{M} := \{A\}$, *i.e.*, $m = 1$. In this case $DLI(\mathcal{M})$ is equivalent to the linear autonomous difference equation

$$x_{n+1} = Ax_n.$$

From (28) it follows that $DLI(\mathcal{M})$ (with $\mathcal{M} = \{A\}$) is absolutely asymptotically stable. On the other hand, from equality (29) we have that $\|A^n\| \geq 1$ and, consequently, equality (27) does not hold.

Let $\mathcal{M} \subseteq [E]$ be a nonempty bounded set of operators and denote by $\mathcal{S} = \mathcal{S}(\mathcal{M})$ the semi-group generated by $\mathcal{M}$ and augmented with the identity operator $I := Id_E$, so that $\mathcal{S} = \bigcup_{n=0}^{\infty} \mathcal{M}^n$, where $\mathcal{M}^n := \{\prod_{i=1}^n A_i \mid A_i \in \mathcal{M}, 1 \leq i \leq n\}$.

Definition 4.9 The number

$$\rho(\mathcal{M}) := \limsup_{n\to\infty} \|\mathcal{M}^n\|^{1/n}, \quad \text{where} \quad \|\mathcal{M}\| := \sup\{\|A\| \ : \ A \in \mathcal{M}\}$$

is called [13, 14, 24] a *joint spectral radius* of a bounded subset of linear operators $\mathcal{M}$.

Definition 4.10 The subset $\mathcal{M} \subseteq [E]$ of linear bounded operators is said to be *generally contracting* if there are positive numbers $\mathcal{N}$ and $\alpha < 1$ such that

$$\|A_{i_n} A_{i_{n-1}} \dots A_{i_1}\| \le \mathcal{N}\alpha^n \tag{30}$$

for all $A_{i_1}, A_{i_2}, \dots, A_{i_n} \in \mathcal{M}$ and $n \in \mathbb{N}$.

Example 4.11 Let $E := C[0,1]$ and $A \in [E]$ be defined by the equality

$$(A\varphi)(t) := \frac{3}{2}\int_0^t \varphi(s)\,ds$$

($t \in [0,1]$ and $\varphi \in C[0,1]$). It is easy to see that $\|A^n\| = (\frac{3}{2})^n \frac{1}{n!}$. In particular, $\|A\| = \frac{3}{2}$, $\|A^2\| = \frac{9}{8}$ and $\|A^3\| = \frac{27}{32} < 1$. In addition, $\|A^n\| \le 2(\frac{3}{4})^n$ for all $n \in \mathbb{N}$. Thus, the set $\mathcal{M} = \{A\}$ is generally contracting.

Theorem 4.12 *Let $\mathcal{M} \subset [E]$ be a compact subset (in particular, the set $\mathcal{M}$ may consist of a finite number of elements, i.e., $\mathcal{M} = \{A_1, A_2, \dots, A_m\}$ with $A_i \in [E]$ $(1 \le i \le m)$). Then the following statements are equivalent:*

a) the discrete linear inclusion $DLI(\mathcal{M})$ is uniformly asymptotically stable;

b) the set $\mathcal{M} \subseteq [E]$ is generally contracting;

c) $\rho(\mathcal{M}) < 1$.

Proof. Let $\Omega := C(\mathbb{Z}_+, \mathcal{M})$. Then Ω is a compact subset of $C(\mathbb{Z}_+, [E])$ and on Ω there is defined a dynamical system of shifts $(\Omega, \mathbb{Z}_+, \sigma)$ induced by the Bebutov's dynamical system $(C(\mathbb{Z}_+, [E]), \mathbb{Z}_+, \sigma)$. Consider the cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ generated by $DLI(\mathcal{M})$, *i.e.*, $\varphi(n, \omega, x) := A_{i_n} A_{i_{n-1}} \dots A_{i_1} x$, where $A_{i_n} \in \mathcal{M}$ and $\omega \in \Omega$ with $\omega(n) := A_{i_n}$ (for all $n \in \mathbb{N}$). Applying to the cocycle φ Theorem 2.6, we obtain the equivalence of the first two conditions.

Now we note that if $\mathcal{M}$ is a generally contracting subset of $[E]$, then from inequality (30) it follows that $\rho(\mathcal{M}) \le \alpha < 1$. Thus the condition b) implies c). Let c) be fulfilled, then for all $\varepsilon \in (0, 1 - \rho(\mathcal{M}))$ there exists a number $n_\varepsilon \in \mathbb{N}$ such that

$$\|A_{i_n} A_{i_{n-1}} \dots A_{i_1}\| \le (\alpha + \varepsilon)^n \tag{31}$$

for all $n \ge n_\varepsilon$. Since $\beta := \alpha + \varepsilon < 1$, then from inequality (31) the condition a) follows. The theorem is proved. □

Remark 4.13 The statements close to Theorem 4.12 were established before in [14, 30] for infinite-dimensional DLIs and in [23] for finite-dimensional DLIs.

4.3 Asymptotically compact discrete linear inclusions

Lemma 4.14 *Let $A', A'' \in [E]$, $A := A' + A''$ and the following conditions hold:*

(i) *the operator A' is contracting, i.e., $\|A'\| < 1$;*

(ii) *the operator A'' is compact.*

Then the operator A is an α-contraction and $\alpha(A(B)) \le k\alpha(B)$ for all bounded subsets $B \subseteq E$, where $k := \|A'\|$.

Proof. Since $A(B) \subseteq A'(B) + A''(B)$, then according to Lemma 22.2 [27] $\alpha(A(B)) \le \alpha(A'(B)) + \alpha(A''(B)) \le \|A'\|\alpha(B) + \alpha(A''(B))$. To finish the proof of our lemma it is sufficient to note that under the conditions of the lemma $\alpha(A''(B)) = 0$. □

Lemma 4.15 *Let $\mathcal{M}$ be a compact subset of $[E]$. Suppose that each operator A of $\mathcal{M}$ may be presented as a sum $A' + A''$, where A' is a contraction and A'' is a compact operator. Then for any bounded set $B \subset E$ and $n \in \mathbb{N}$ we have $\alpha(U(n,\Omega)B) \le k\alpha(B)$ for any bounded subset $B \subseteq E$, where $U(n,\omega) := \varphi(n,\cdot,\omega) = A_{i_n}A_{i_{n-1}}\ldots A_{i_1}$ ($\omega(j) := A_{i_j} \in \mathcal{M}$ for all $j \in \mathbb{N}$) and $k := \prod_{j=1}^{n}\|A'_{i_j}\| < 1$.*

Proof. Since the set Ω is compact and $U(n,\omega) = \prod_{k=1}^{n}\omega(k)$ ($\omega \in \Omega$), then for each n the mapping $U(n,\cdot) : \Omega \to [E]$ is continuous. Note that $A_{i_j} = A'_{i_j} + A''_{i_j}$ and, consequently, we have

$$U(n,\omega) := \prod_{j=1}^{n} A_{i_j} = \prod_{j=1}^{n}(A'_{i_j} + A''_{i_j}) = \prod_{j=1}^{n} A'_{i_j} + C,$$

where $C \in [E]$ is some compact operator. By Lemma 4.14 we have

$$\alpha(U(n,\omega)B) \le k_0\alpha(B)$$

for all bounded subsets $B \subseteq E$, where $k_0 = \|\prod_{j=1}^{n} A'_{i_j}\| \le \prod_{j=1}^{n}\|A'_{i_j}\| = k < 1$. The lemma is proved. □

Theorem 4.16 *Let $\mathcal{M}$ be a compact subset of $[E]$. Suppose that each operator A of $\mathcal{M}$ may be represented as a sum $A' + A''$, where A' is a contraction and A'' is a compact operator. Then the following assertions are equivalent:*

(i) *the discrete linear inclusion $DLI(\mathcal{M})$ is absolutely asymptotically stable;*

(ii) *the discrete linear inclusion $DLI(\mathcal{M})$ is uniformly exponentially stable.*

Proof. Consider the cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ generated by $DLI(\mathcal{M})$. By Lemma 4.15, under the conditions of the Theorem, this cocycle is an α-contraction. Now to finish the proof it is sufficient to apply Theorem 2.23, because every α-contracting cocycle φ is α-condensing and, in particular, it is asymptotically compact. $\square$

Theorem 4.17 *Let $\mathcal{M}$ be a compact subset of $[E]$. Suppose that the following conditions hold:*

(i) *each operator A of $\mathcal{M}$ may be represented as a sum $A' + A''$, where A' is a contraction and A'' is a compact operator;*

(ii) *the discrete linear inclusion $DLI(\mathcal{M})$ does not admit nontrivial bounded trajectories on $\mathbb{Z}$;*

(iii) *the set $\mathcal{M} \subseteq [E]$ of operators is product bounded.*

Then the discrete linear inclusion $DLI(\mathcal{M})$ is uniformly exponentially stable.

Proof. Consider the cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ generated by $DLI(\mathcal{M})$ and the corresponding skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$, where $X := E \times Y$ and $\pi := (\varphi, \sigma)$. By Lemma 4.15, under the conditions of the Theorem, this cocycle is an α-contraction and, consequently, the dynamical system $(X, \mathbb{Z}_+, \pi)$ also is. It is easy to check that under the conditions of the theorem this nonautonomous dynamical system is uniformly stable, *i.e.*, $|\pi(n,x)| \le M|x|$ for all $x := (u,y) \in X$ and $n \in \mathbb{Z}_+$, because $|\pi(n,x)| = |U(n,\omega)u| \le M|u| = M|x|$, where $U(n,\omega) := \varphi(n,\cdot,\omega) = A_{i_n}A_{i_{n-1}}\ldots A_{i_1}$ ($\omega(j) := A_{i_j} \in \mathcal{M}$ for all $j \in \mathbb{N}$) and $\mathcal{M} \subset [E]$ is product bounded.

Now we will show that the nonautonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is convergent. In our case this means that $\lim\limits_{n\to+\infty} |\pi(n,x)| = 0$ for all $x \in X$. Really, the system $(X, \mathbb{Z}_+, \pi)$ is an α-contraction and, consequently, it is asymptotically compact. The trajectory $\{\pi(n,x) \mid n \in \mathbb{Z}_+\}$ is bounded and consequently it is relatively compact. Denote by ω_x an ω-limit set of the point x. This set is nonempty, compact and invariant. In particular, ω_x consists of full trajectories of $DLI(\mathcal{M})$ bounded on $\mathbb{Z}$. Under the conditions of our theorem, $\omega_x \subseteq \Theta := \{(0,y) \mid y \in Y\}$ and, hence, $\lim\limits_{n\to+\infty} |\pi(n,x)| = 0$. Now to finish the proof it is sufficient to apply Theorem 2.23. $\square$

Theorem 4.18 *Let $\mathcal{M}$ be a compact subset of $[E]$ and each operator A of $\mathcal{M}$ may be represented as a sum $A' + A''$, where A' is a contraction and A'' is a compact operator.*

Then the following assertions are equivalent:

(i) *the discrete linear inclusion $DLI(\mathcal{M})$ is product bounded and is absolutely asymptotically stable;*

(ii) *the set $\mathcal{M}$ is generally contracting.*

Proof. Consider the cocycle φ generated by $DLI(\mathcal{M})$. By Lemma 4.15, under the conditions of the Theorem, this cocycle is α-condensing. Now to finish the proof it is enough to refer to Theorem 2.23. $\square$

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