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Nonlinear Second Order Periodic Boundary Value Problems

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Abstract

We are concerned with the solvability of nonlinear second order periodic boundary value problems. We shall provide sufficient conditions on the nonlinearity in order to obtain an *a priori* bound on the solutions of a one-parameter family of problems related to the original one. We then use the topological transversality theorem to prove the existence of at least one solution.

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1 Introduction

Since its introduction by Granas, in 1959, the topological transversality theorem has been very effective in proving the existence of solutions of periodic boundary value problems for ordinary differential equations of the form

$$\begin{cases} y''(t) = f(t, y(t), y'(t)), & 0 < t < 1, \\ y(0) = y(1), \\ y'(0) = y'(1) \end{cases} \tag{1}$$

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(see for instance [2, 3, 5, 6, 7] and the references therein). Also, the method of upper and lower solutions and the monotone method have been successfully used to establish the existence of at least one solution of problem (1) (see [10, 11] and the references therein). Our aim in this paper is to establish the existence of solutions for the following periodic boundary value problems (1) under fairly simple and quite general conditions on the nonlinearity f, which is assumed to be an L^1-Carathéodory function; *i.e.*, f satisfies

(i) $f(\cdot, y, z)$ is measurable for all $(y, z) \in \mathbb{R}^2$.

ii) $f(t, \cdot, \cdot)$ is continuous for almost all $t \in [0, 1]$.

(iii) for each $\rho > 0$ there exists $h_\rho \in L^1(0, 1)$ such that $|y| + |z| \leq \rho$ implies that $|f(t, y, z)| \leq h_\rho(t)$ for almost all $t \in [0, 1]$.

2 Preliminaries

Let I denote the real interval $[0, 1]$. $X = AC^1(I)$ denotes the Banach space of absolutely continuous real-valued functions together with their first derivatives on I, equipped with the norm

$$\|y\| = \max\left\{|y(t)| + |y'(t)|; \quad t \in I\right\} \qquad \forall y \in X.$$

Car $(I \times \mathbb{R}^2)$ is the set of all real-valued functions satisfying the Carathéodory conditions (i), (ii), (iii).

By a *solution* of (1) we mean a function

$$y \in X_0 = \{u \in X; \quad u(0) - u(1) = u'(0) - u'(1) = 0\}$$

satisfying the differential equation in (1) almost everywhere on I.

Since the homogeneous problem $y'' = 0,\ y(0) - y(1) = y'(0) - y'(1) = 0$ has nontrivial solutions, we shall deal with the following problem, for $m > 1$

$$\begin{cases} y''(t) = \frac{1}{m}\, y(t) + f(t, y(t),\, y'(t)), \qquad 0 < t < 1, \\ y(0) = y(1), \\ y'(0) = y'(1). \end{cases} \tag{1_m}$$

Problem (1) is considered as a limiting case of (1_m) as $m \to +\infty$.

Lemma 1 *The problem* $y'' = \frac{1}{m}y,\ y(0) - y(1) = y'(0) - y'(1) = 0$ *has only the trivial solution. Green's function* $G_m(t,s)$ *exists and there exists a constant* $\gamma_m > 0$ *such that*

$$|G_m(t,s)| + \left|\frac{\partial G_m}{\partial t}(t,s)\right| \leq \gamma_m \qquad \forall (t,s) \in I^2.$$

Proof. Suppose, on the contrary, that the homogeneous problem has a nontrivial solution y_0. Then

$$\int_0^1 y_0''(t)y_0(t)\,dt = \frac{1}{m}\int_0^1 y_0(t)^2\,dt.$$

A simple integration by parts of the left-hand side and the boundary conditions lead to

$$-\int_0^1 y_0'(t)^2\,dt = \frac{1}{m}\int_0^1 y_0(t)^2\,dt,$$

which is impossible.

As a consequence, Green's function, $G_m(t,s)$, exists and has the following representation (see [1]),

$$G_m(t,s) = \begin{cases} -\dfrac{\sqrt{m}}{2}\,\dfrac{\cosh \frac{1}{\sqrt{m}}\left(t - s + \frac{1}{2}\right)}{\sinh \frac{1}{2\sqrt{m}}}, & 0 \leq t \leq s, \\ -\dfrac{\sqrt{m}}{2}\,\dfrac{\cosh \frac{1}{\sqrt{m}}\left(t - s - \frac{1}{2}\right)}{\sinh \frac{1}{2\sqrt{m}}}, & s < t \leq 1. \end{cases}$$

The properties of the hyperbolic cosine function and the fact that $\sinh\theta > \theta$ for all $\theta > 0$ imply that

$$|G_m(t,s)| \leq m \cosh \frac{1}{2\sqrt{m}} \quad \text{and} \quad \left|\frac{\partial G_m}{\partial t}(t,s)\right| \leq \frac{1}{2}$$

for all $(t,s) \in I^2$.

Letting $\gamma_m = \dfrac{1}{2} + m\cosh \dfrac{1}{2\sqrt{m}}$, we get the desired inequality, and the lemma is proved.

Lemma 2 *Assume there exists* $h \in L^1(I)$ *such that*

$$|f(t,y,z)| \leq h(t) \quad \forall t \in I.$$

Then problem (1_m) *has at least one solution.*

Proof. It follows from Lemma 1 that problem (1) is equivalent to the nonlinear integral equation

$$y(t) = \int_0^1 G_m(t,s) f(s, y(s), y'(s))\, ds \quad \forall\, t \in I,$$

so that

$$y'(t) = \int_0^1 \frac{\partial G_m(t,s)}{\partial t} f(s, y(s),\, y'(s))\, ds.$$

Therefore

$$\|y\| \le \gamma_m \|h\|_{L^1}\,.$$

Define a nonlinear operator $T : X \longrightarrow X_0$ by

$$(Ty)(t) = \int_0^1 G_m(t,s) f(s, y(s),\, y'(s))\, ds.$$

Let $D := \{y \in X_0;\, \|y\| \le \gamma_m \|h\|_{L^1}\}\,.$

One can easily show that T is completely continuous and maps the closed convex set D into itself. By the Schauder fixed point theorem T has a fixed point in D, which is a solution of problem (1_m).

3 Main results

In this section we shall state sufficient conditions in order to obtain an *a priori* bound on solutions of a one-parameter family of problems related to (1_m). We then prove that for each $m > 1$, problem (1_m) has at least one solution y_m such that $\|y_m\|$ is uniformly bounded and, moreover, the *a priori* bound does not depend on m. Going to subsequences, if necessary, we see that $y = \lim\limits_{m\to+\infty} y_m$ is a solution of problem (1).

Since our arguments are based on the topological transversality theorem (see [5, 6] for definitions and properties) we shall consider the following one-parameter family of problems

$$\begin{cases} y''(t) = \frac{1}{m}\, y(t) + \lambda f(t, y(t),\, y'(t)), \quad 0 < t < 1, \\ y(0) = y(1), \\ y'(0) = y'(1), \end{cases} \tag{$1_m \cdot \lambda$}$$

where $0 \le \lambda \le 1$.

Notice that $(1_m \cdot 1)$ is exactly (1_m) and $(1_m \cdot 0)$ has only the trivial solution. Moreover, $(1_m \cdot \lambda)$ is equivalent to the abstract equation

$$y = H_m(\lambda, y), \tag{$2_m \cdot \lambda$}$$

where $H_m(\lambda, y)(t) = \lambda \int_0^1 G_m(t,s) f(s, y(s), y'(s)\, ds$ and for each λ, $H_m(\lambda, \cdot)$ is a compact operator. Moreover, the fixed points of $H_m(1, \cdot)$ are solutions of (1_m).

We have the following existence principle.

Theorem 1 *Assume that there exists an open bounded subset Ω of X_0 such that*

(i) $0 \in \Omega$,

(ii) $H_m(\lambda, y) \neq y$ for all $y \in \partial\Omega$, $0 \leq \lambda \leq 1$.

Then Problem (1_m) has at least one solution in Ω.

Proof. Consider $H_m : [0,1] \times \overline{\Omega} \to X_0$ defined as above. Then, it can be easily shown that $H_m(\lambda, \cdot)$ is a compact homotopy between the zero map $H_m(0, \cdot)$ and $H_m(1, \cdot)$. Moreover, this homotopy has no fixed points on $\partial\Omega$. This implies that this homotopy is admissible. Since $0 \in \Omega$, $H_m(0, \cdot)$ is essential. By the topological transversality theorem $H_m(1, \cdot)$ is essential. Therefore it has a fixed point.

Notice the important role played by the open bounded set Ω. This set will be called admissible for (1_m).

We shall state sufficient conditions on the nonlinearity that will allow us to construct the open bounded set Ω having the required properties.

Theorem 2 *Assume $f \in \mathrm{Car}(I \times \mathbb{R}^2)$ satisfies*

$(H1)$ there exists $r > 0$ such that $f(t, y, 0)\,\mathrm{sgn}\, y > 0$ whenever $|y| > r$;

$(H2)$ there exists $\gamma > 0$, $\ell \in L^1(I)$ and $\psi : [0, +\infty) \to (0, +\infty)$ with $\dfrac{1}{\psi}$ integrable over bounded intervals and $\displaystyle\int_0^{+\infty} \frac{d\sigma}{\psi(\sigma)} = +\infty$ such that $\sup\limits_{|y| \leq r} |y + f(t, y, z)| \leq (\ell(t) + \gamma|z|)\psi(|z|)$ for all $(t, z) \in I \times \mathbb{R}$.

Then there exists $M_0 > 0$, independent of m and λ, such that $\Omega_0 := \{y \in X_0;\ \|y\| < 1 + M_0\}$ is admissible for (1_m).

Remark 1 Condition (H2) is known as the Nagumo-Wintner condition, which is more general than the Nagumo or Nagumo-Bernstein condition.

Proof. Let y be a solution of $(1_m \cdot \lambda)$ for $0 < \lambda \leq 1$. Condition (H1) implies that $|y(t)| \leq r$ for all $t \in I$. Suppose, on the contrary, that there exists $\tau_1 \in I$ such that $|y(\tau_1)| > r$. It follows from the continuity of y on I that there exists $\tau_2 \in I$ such that $|y(\tau_2)| = \max\{|y(t)|;\ t \in I\} > r$. Assume for definiteness that

$|y(\tau_2)| = y(\tau_2)$ (the other case can be handled similarly). Hence $y(\tau_2) > r$. Then $f(t, y(\tau_2), 0) \operatorname{sgn} y(\tau_2) > 0$, $y'(\tau_2) = 0$ and $y''(\tau_2) \leq 0$. The differential equation in $(1_m \cdot \lambda)$ yields

$$0 \geq y''(\tau_2)y(\tau_2) = \frac{1}{m}y(\tau_2)^2 + f\left(t, y\left(\tau_2\right), 0\right) y(\tau_2) > 0.$$

This is a contradiction. Hence, $|y(t)| \leq r$ for all $t \in I$.
Let y be a solution of $(1_m \cdot \lambda)$ such that

$$|y(t)| \leq r \quad \forall\, t \in I.$$

Choose

$$C_1 > 0 \quad \text{ so that } \quad \int_0^{C_1} \frac{d\sigma}{\psi(\sigma)} > \|\ell\|_{L^1} + 2r\,\gamma.$$

We want to show that $|y'(t)| \leq C_1$ for all $t \in I$. Suppose, on the contrary, that there exists $\overline{t} \in I$ such that $|y'(\overline{t})| > C_1$. Since $y(1) = y(0)$, there exists $\underline{t} \in I$ such that $y'(\underline{t}) = 0$.

Thus, we have $|y'(\underline{t})| = 0$ and $|y'(\overline{t})| > C_1$. Since $y \in AC^1(I)$, there exists an interval $[\sigma_1, \sigma_2] \subset I$ such that one of the following situations hold:

(i) $y'(\sigma_1) = 0, \quad y'(\sigma_2) = C_1$ and $0 < y'(t) < C_1$ for all $t \in (\sigma_1, \sigma_2)$.

(ii) $y'(\sigma_1) = C_1, \quad y'(\sigma_2) = 0$ and $0 < y'(t) < C_1$ for all $t \in (\sigma_1, \sigma_2)$.

(iii) $y'(\sigma_1) = 0, \quad y'(\sigma_2) = -C_1$ and $-C_1 < y'(t) < 0$ for all $t \in (\sigma_1, \sigma_2)$.

(iv) $y'(\sigma_1) = C_1, \quad y'(\sigma_2) = 0$ and $-C_1 < y'(t) < 0$ for all $t \in (\sigma_1, \sigma_2)$.

We consider only the first case, since the other cases can be handled similarly.
We have

$$y''(t) = \frac{y(t)}{m} + \lambda f(t, y(t),\, y'(t)).$$

Hence, since $m > 1$ and $0 < \lambda \leq 1$,

$$\begin{aligned} y''(t) &\leq |y''(t)| \leq |y(t)| + |f(t, y(t),\, y'(t))| \\ &\leq (\ell(t) + \gamma |y'(t)| \psi(|y'(t)|) \quad \forall\, t \in I, \end{aligned}$$

so that, for $t \in [\sigma_1, \sigma_2]$ we have

$$y''(t) \leq [\ell(t) + \gamma\, y'(t)]\psi(y'(t)),$$

which gives

$$\frac{y''(t)}{\psi(y'(t))} \leq \ell(t) + \gamma\, y'(t).$$

Thus

$$\begin{aligned}\int_{\sigma_1}^{\sigma_2} \frac{y''(t)\,dt}{\psi(y'(t))} &\leq \int_{\sigma_1}^{\sigma_2} \ell(t)\,dt + \gamma \int_{\sigma_1}^{\sigma_2} y'(t)\,dt \\ &\leq \int_0^1 \ell(t)\,dt + \gamma(y(\sigma_2) - y(\sigma_1)) \\ &\leq \|\ell\|_{L^1} + 2r\gamma.\end{aligned}$$

A change of variables in the left-hand side gives

$$\int_0^{C_1} \frac{d\sigma}{\psi(\sigma)} \leq \|\ell\|_{L^1} + 2r\gamma.$$

This is clearly in contradiction with the definition of C_1. Therefore,

$$|y'(t)| \leq C_1 \quad \forall t \in I.$$

Set $M_0 := r + C_1$ and let $\Omega_0 := \{y \in X_0;\ \|y\| < 1 + M_0\}$. Then $0 \in \Omega_0$ and $H_m :\ [0,1] \times \overline{\Omega}_0 \longrightarrow X_0$ is a compact homotopy without fixed points on $\partial\Omega_0$, the boundary of Ω_0. Then $\Omega_0 := \{y \in X_0;\ \|y\| < M_0 + 1\}$ is admissible for problem (1_m).

This completes the proof of the theorem.

Theorem 3 *Assume that $f \in \mathrm{Car}\,(I \times \mathbb{R}^2)$ satisfies*

$(H3)$ *there exists $k_0 > 0$ such that $\int_0^1 f(t, k_0, 0)\,dt > 0$ and $\int_0^1 f(t, -k_0, 0)\,dt < 0$;*

$(H4)$ *there exists $p \in L^1(I;\mathbb{R}_+)$, $\Psi :\ [0, +\infty) \to (0, +\infty)$ nondecreasing, with $1/\Psi$ integrable over bounded intervals and $\int_{k_0}^{+\infty} \dfrac{d\sigma}{\Psi\,(\sigma)} > \|p\|_{L^1}$ such that $|f(t, y, z)| \leq p(t)\Psi\,(|z|)\ \forall\,(t, y) \in I \times [-k_0, k_0]$ and $z \in \mathbb{R}$.*

Then there exists $M_1 > 0$, independent of m and λ, such that the set $\Omega_1 := \{y \in X_0; \|y\| < M_1 + 1\}$ is admissible for (1_m).

Proof. Consider the modified problem

$$\begin{cases} y''(t) = \frac{1}{m}\, y(t) + \lambda f_1(t, y(t), y'(t)), \quad 0 < t < 1, \\ y(0) = y(1), \\ y'(0) = y'(1), \end{cases} \tag{$3_m \cdot \lambda$}$$

where $f_1 : I \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given by

$$f_1(t,y,z) = \begin{cases} \max\left\{f(t,y,z),\ -\frac{k_0}{m} + \int_0^1 f(t,k_0,0)\,dt\right\}, & y > k_0, \\ f(t,y,z), & -k_0 \le y \le k_0, \\ \min\left\{f(t,y,z),\ \frac{k_0}{m} + \int_0^1 f(t,-k_0,0)\,dt\right\}, & y < -k_0. \end{cases}$$

Notice that any solution y of $(3_m \cdot \lambda)$ satisfying

$$|y(t)| \le k_0 \quad \text{for all} \quad t \in I \tag{$*\cdot 1$}$$

is a solution of $(1_m \cdot \lambda)$ and, conversely, any solution y of $(1_m \cdot \lambda)$ satisfying $(* \cdot 1)$ is a solution of $(3_m \cdot \lambda)$ since $f(t,y,z) \equiv f_1(t,y,z)$ when $|y| \le k_0$.

We show that any solution y of $(3_m \cdot \lambda)$ satisfies $(* \cdot 1)$.

For $\lambda = 0$, problem $(3_m \cdot \lambda)$ has only the trivial solution which, clearly, satisfies $(* \cdot 1)$.

Let y be a possible solution of $(3_m \cdot \lambda)$ for $0 < \lambda \le 1$.

Let $t_0 \in I$ be such that y takes on its positive maximum at t_0. Then $y'(t_0) = 0$.

Suppose $y(t_0) > k_0$ and $t_0 \in (0,1)$. Then there exists $a > 0$ such that $y(t) > k_0$ for all $t \in [t_0, t_0 + a]$. It follows from the differential equation in $(3_m \cdot \lambda)$ and the definition of f_1 that

$$y''(t) \ge \frac{y(t)}{m} - \frac{k_0}{m} + \int_0^1 f(t,k_0,0)\,dt, \qquad t_0 \le t \le t_0 + a.$$

Hence $y''(t) > 0$ for all $t \in [t_0, t_0 + a]$.

Since $y(t) - y(t_0) = \int_{t_0}^t (t-s) y''(s)\,ds$ for all $t \ge t_0$, we see that $y(t) > y(t_0)$ and this contradicts the fact that $y(t_0)$ is the maximum value of y on I.

If it happens that $t_0 = 0$, then assuming $y(0) > k_0$ we also arrive at a contradiction. By periodicity, we have $y(1) = y(0)$ and so, $y(1) > k_0$ will also lead to a contradiction.

Therefore,

$$y(t) \le k_0 \quad \text{for all} \ \ t \in I.$$

Next, if y takes on a negative minimum at $t = \tau_0$ such that $y(\tau_0) < -k_0$, then we can find $b > 0$ such that $y(t) < -k_0$ for all $t \in [\tau_0, \tau_0 + b]$.

Hence

$$y''(t) \le \frac{y(t)}{m} + \frac{k_0}{m} + \int_0^1 f(s,-k_0,0)\,ds < 0, \qquad \tau_0 \le t \le \tau_0 + b.$$

This implies that

$$y(t) - y(\tau_0) = \int_{\tau_0}^{t} (t-s) y''(s)\, ds < 0, \qquad t \geq \tau_0,$$

which contradicts the fact that $y(\tau_0)$ is the minimum value of y on I.

Hence

$$y(t) \geq -k_0 \ \text{ for all } \ t \in I.$$

Therefore, we have proved that the estimate $(*\cdot 1)$ holds for any solution y of $(3_m \cdot \lambda)$.

Remark 2 The authors in [7] and [8] assume the existence of $M > 0$ such that $|y| \geq M$ implies that $yf(t, y, 0) > 0$ for almost all $t \in I$. It is clear that our assumption (H3) is much more general than this sign condition.

Next, we obtain an *a priori* bound on y' for any solution y of $(3_m \cdot \lambda)$ satisfying $(* \cdot 1)$, *i.e.*, there exists $C_0 > 0$, independent of m and λ, such that for any solution y of $(3_m \cdot \lambda)$ satisfying the estimate $(* \cdot 1)$ it holds

$$|y'(t)| \leq C_0 \quad \text{ for all } \quad t \in I. \tag{$*\cdot 2$}$$

Proof. Let y be a solution of $(3_m \cdot \lambda)$ such that

$$|y(t)| \leq k_0 \quad \text{ for all } \quad t \in I.$$

We have

$$\begin{aligned} y''(t) &= \frac{y(t)}{m} + \lambda f_1(t, y(t), y'(t)) \\ &= \frac{y(t)}{m} + \lambda f(t, y(t), y'(t)) \end{aligned}$$

(Since f and f_1 coincide when $|y| \leq k_0$).

Hence, since $m > 1$ and $0 < \lambda \leq 1$,

$$\begin{aligned} y''(t) &\leq |y''(t)| \leq |y(t)| + |f(t, y(t), y'(t))| \\ &\leq k_0 + p(t)\Psi\left(|y'(t)|\right) \quad \forall\, t \in I, \end{aligned}$$

so that

$$|y'(t)| \leq k_0 t + \int_0^t p(s)\Psi(|y'(s)|)\, ds \qquad \forall\ t \in I,$$

which gives (since $0 \leq t \leq 1$)

$$|y'(t)| \leq k_0 + \int_0^t p(s)\Psi|y'(s)|)\, ds \qquad \forall\ t \in I.$$

Let
$$u(t) = k_0 + \int_0^t p(s)\Psi(|y'(s)|)\,ds \qquad \forall\, t \in I.$$
Then
$$|y'(t)| \leq u(t) \qquad \forall\, t \in I$$
and
$$u'(t) = \ p(t)\Psi(|y'(t)|) \quad \forall\, t \in I.$$
Since Ψ is nondecreasing,
$$u'(t) \leq \ p(t)\Psi(u(t)) \quad \forall\, t \in I.$$
This last inequality gives
$$\frac{u'(t)}{\Psi(u(t))} \leq p(t) \quad \forall\, t \in I.$$
Thus
$$\int_0^t \frac{u'(s)}{\Psi(u(s))}\,ds \leq \int_0^t p(s)\,ds \leq \int_0^1 p(s)\,ds := \|p\|_{L^1}\,.$$
Using a change of variables in the left-hand side, we obtain
$$\int_{k_0}^{u(t)} \frac{d\sigma}{\Psi(\sigma)} \leq \|p\|_{L^1}\,.$$
It follows from the condition on Ψ that there exists a constant N_1, independent of m and λ, such that
$$u(t) \leq N_1 \quad \forall\, t \in I.$$
Consequently,
$$|y'(t)| \leq N_1 \qquad \forall\, t \in I.$$

Now, let $M_1 = k_0 + N_1$ and $\Omega_1 := \{y \in X_0;\ \|y\| < M_1 + 1\}$. It follows from the above discussion that Ω_1 is admissible for (1_m). This completes the proof of the theorem.

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