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On Nonhomogeneous Elliptic System with Changing Sign Data

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Abstract

In this paper, we give some existence results of multiple positive solutions to the following nonhomogeneous elliptic system

$$\begin{cases} -\Delta u = (\alpha+1)\,|u|^{\alpha-1}\,u\,|v|^{\beta+1} + \lambda f\,(x) & \text{in } \Omega, \\ -\Delta v = (\beta+1)\,|u|^{\alpha+1}\,|v|^{\beta-1}\,v + \mu g\,(x) & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (S_{\lambda,\mu})$$

where Ω is a regular bounded domain in $\mathbb{R}^N$ $(N \geq 3)$; $\alpha, \beta > 0$; λ and μ are positive parameters and f, g are given changing sign functions. We use separately sub – super solutions technique and variational methods.

1 Introduction

This paper is concerned with existence and multiplicity results for positive solutions $(u, v) \in [H_0^1(\Omega)]^2$ of the following system of equations

$$\begin{cases} -\Delta u = (\alpha+1)\,|u|^{\alpha-1}\,u\,|v|^{\beta+1} + \lambda f\,(x) & \text{in } \Omega, \\ -\Delta v = (\beta+1)\,|u|^{\alpha+1}\,|v|^{\beta-1}\,v + \mu g\,(x) & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (S_{\lambda,\mu})$$

where Ω is a bounded regular domain of $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary $\partial\Omega$; $\alpha, \beta > 0$; λ, μ are positive parameters and f, g are changing sign data which will be specified later.

In the nonhomogeneous case of (semilinear or quasilinear) elliptic equations, there is an extensive literature on the existence and multiplicity results (see [7, 8] and [13] and the references cited therein).

Many existence results had been given for a system which is derived from a potential (see for example [1, 4] and also the references cited therein).

$(S_{\lambda,\mu})$ involving pseudo Laplacian operators and critical terms had been discussed in [14]. In this paper the existence results were given by a suitable minimization principle when the nonhomogeneous terms were chosen small in the sense of the dual norm.

Following [5] and [9], we give existence and multiplicity results for $(S_{\lambda,\mu})$ when f and g are changing sign functions. By the sub-supersolution technique, we prove that $(S_{\lambda,\mu})$ admits a minimal solution. Taking account of the fact that $(S_{\lambda,\mu})$ has a variational structure, we obtain a second positive solution by finding the critical points of an energy functional which will be defined later.

The paper is organized as follows. In Section 2 we recall some preliminary results and notation, Section 3 contains the proof of Theorem 1 and in Section 4 we give the proof of Theorem 2.

2 Preliminaries and main results

We consider

$$S := \left\{ h \in C^1(\Omega) \setminus \{0\} \text{ changing sign } \middle| \; -\Delta u = h \text{ has a nonnegative solution in } H_0^1(\Omega) \right\},$$

the set S is not empty (see [6] or [12]).

First we recall some useful lemmas.

A version of the Mountain Pass Lemma (Ambrosetti-Rabinowitz [3]):

Lemma 1 *Let J be a C^1 function on a Banach space E. Assume there are constant $\alpha > 0$ and $\rho > 0$ such that*

$$J(u) \geq \alpha \text{ for all } u \in E \text{ with } \|u\| = \rho$$

and

$$J(0) = 0 \text{ and } J(v_0) \leq 0 \text{ for some } v_0 \in E \text{ with } \|v_0\| > \rho.$$

Set

$$c := \inf_{g \in \Gamma} \left(\max_{t \in [0,1]} J\left[g\left(t\right)\right] \right),$$

where

$$\Gamma := \left\{ g \in \mathcal{C}\left([0,1], E\right); g\left(0\right) = 0, g\left(1\right) = v_0 \right\}.$$

Then there exists a sequence (u_j) in E such that $J\left(u_j\right) \to c$ and $J'\left(u_j\right) \to 0$ in E' (dual of E).

Remark 1 The proof of this lemma has been given also by J. P. Aubin and I. Ekeland [2]. It relies on Ekeland's minimization principle.

Lemma 2 ([1], Theorem 5) *Let Ω be a domain (not necessarily bounded) and $\alpha + \beta \leq 2^* - 2$. Then we have*

$$S_{(\alpha,\beta)}(\Omega) = \left[\left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\beta+1}{\alpha+\beta+2}} + \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{-\alpha-1}{\alpha+\beta+2}}\right] S_{\alpha+\beta+2}(\Omega).$$

Moreover, if ω_0 realizes $S_{\alpha+\beta+2}(\Omega)$, then $(B\omega_0, C\omega_0)$ realizes $S_{(\alpha,\beta)}(\Omega)$ for any constants B and C such that $\frac{B}{C} = \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{1}{2}}$,
where

$$S_{\alpha+\beta+2}(\Omega) := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\left(\int_\Omega |u|^{\alpha+\beta+2} \, dx\right)^{\frac{2}{\alpha+\beta+2}}}$$

and

$$S_{(\alpha,\beta)}(\Omega) := \inf_{(u,v) \in E\setminus\{(0,0)\}} \frac{\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) dx}{\left(\int_\Omega |u|^{\alpha+1} |v|^{\beta+1} \, dx\right)^{\frac{2}{\alpha+\beta+2}}}.$$

Definition 1 J is said to satisfy the Palais–Smale condition $(PS)_c$, where $c \in \mathbb{R}$, if any sequence $(u_n, v_n) \in E$ such that $J(u_n, v_n) \to c$ and $J'(u_n, v_n) \to 0$ in E' as n tends to $+\infty$, contains a convergent subsequence in E.

Notation. We make use of the following notations:
$L^p(\Omega)$, $1 \leq p < \infty$, denote Lebesgue spaces, the norm L^p is denoted by $\|\cdot\|_p$ for $1 \leq p < \infty$ and $|\cdot|$ for $p = \infty$;
$W^{k,p}(\Omega)$ denotes Sobolev spaces; the norm in $W^{k,p}$ is denoted by $\|\cdot\|_{k,p}$;
E denotes $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, endowed with the norm

$$\|(u,v)\|^2 = \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^2.$$

$2^* := \frac{2N}{N-2}$ $(N \geq 3)$ is the Sobolev critical exponent.
A solution is said to be *positive* if each of its components is positive.
We denote by ω^+ the positive part of ω and by $\operatorname{supp}\omega$ the support of ω.

Our results are given by the following theorems.

Theorem 1 *Assume* $0 < \alpha + \beta \leq \frac{4}{N-2}$ *and* $(f, g) \in S^2$. *Then there exists a positive number* Λ_0 *such that* $(S_{\lambda,\mu})$ *has a positive minimal solution* $(u_{\lambda,\mu}, v_{\lambda,\mu})$ *for all* $\lambda, \mu \in (0, \Lambda_0)$ *with* $\max(|u_{\lambda,\mu}|, |v_{\lambda,\mu}|) \to 0$ *when* $\lambda, \mu \to 0$.

Theorem 2 *Under the same hypotheses as in Theorem 1, there exists a second positive solution of* $(S_{\lambda,\mu})$ *for* $N \geq 4$.

3 The existence of a minimal solution

We know from [6] or [12] that the following problem

$$\begin{cases} -\Delta u = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{S_h}$$

admits a nonnegative solution labeled u_h.
Proof of Theorem 1. Set $\underline{u} := \lambda u_f$ and $\underline{v} := \mu v_g$ where u_f (resp. v_g) is the solution of the problem (S_f) (resp. (S_g)).

Thus $(\underline{u}, \underline{v})$ is a subsolution of $(S_{\lambda,\mu})$ for all λ, and μ positive (see [10]) if

$$-\Delta \underline{u} = \lambda(-\Delta u_f) = \lambda f(x) \leq (\alpha + 1)\, \underline{u}^{\alpha} \underline{v}^{\beta+1} + \lambda f(x)$$

and

$$-\Delta \underline{v} = \mu(-\Delta v_g) = \mu g(x) \leq (\beta + 1)\, \underline{u}^{\alpha+1} \underline{v}^{\beta} + \mu g(x).$$

Let e be a solution of the following problem

$$\begin{cases} -\Delta e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega, \end{cases}$$

the existence and uniqueness of e is assured by the Lax–Milgram Lemma. Moreover, by the Maximum Principle, $e(x) > 0$ for all $x \in \Omega$.

We propose to find a supersolution of $(S_{\lambda,\mu})$ in the form $(\overline{u}, \overline{v}) = (Ae, Be)$, where A and B are positive constants.

Thus $(\overline{u}, \overline{v})$ is a supersolution of $(S_{\lambda\mu})$ if A and B satisfy

$$A \geq (\alpha + 1)\, A^{\alpha} B^{\beta+1} |e|^{\alpha+\beta+1} + \lambda |f|$$

and

$$B \geq (\beta + 1)\, A^{\alpha+1} B^{\beta} |e|^{\alpha+\beta+1} + \mu |g|.$$

Let $B = rA$, where r is a positive parameter, then we get

$$A \geq (\alpha+1) A^{\alpha+\beta+1} r^{\beta+1} |e|^{\alpha+\beta+1} + \lambda |f|$$

and

$$rA \geq (\beta+1) A^{\alpha+\beta+1} r^{\beta} |e|^{\alpha+\beta+1} + \mu |g| .$$

Consider the following functions

$$\lambda_r (t) = \frac{1}{|f|} \left(t - (\alpha+1) |e|^{\alpha+\beta+1} r^{\beta+1} t^{\alpha+\beta+1} \right)$$

and

$$\mu_r (t) = \frac{1}{|g|} \left(rt - (\beta+1) |e|^{\alpha+\beta+1} r^{\beta} t^{\alpha+\beta+1} \right).$$

Let $(t_0, \lambda_r (t_0))$ (resp. $(t_1, \mu_r (t_1))$) be the coordinates of the maximum point of the curves $\lambda_r (t)$ (resp. $\mu_r (t)$), where

$$t_0 := \left(\frac{1}{(\alpha+1)(\alpha+\beta+1) |e|^{\alpha+\beta+1} r^{\beta+1}} \right)^{\frac{1}{\alpha+\beta}}$$

and

$$t_1 := \left(\frac{1}{(\beta+1)(\alpha+\beta+1) |e|^{\alpha+\beta+1} r^{\beta-1}} \right)^{\frac{1}{\alpha+\beta}}.$$

$t_0 = t_1$ if we choose

$$r_0 = \left(\frac{\beta+1}{\alpha+1} \right)^{\frac{1}{2}}.$$

Thus we have

$$\lambda_{r_0}(t_0) = \mu_{r_0}(t_0) \left(\frac{\beta+1}{\alpha+1} \right)^{\frac{1}{2}} \frac{|g|}{|f|}.$$

Hence there exists $\Lambda_0 = \min \left(\lambda_{r_0} (t_0), \mu_{r_0} (t_0) \right) > 0$ such that for all $\lambda, \mu \in (0, \Lambda_0)$ there exists $A = A(\lambda, \mu) > 0$ satisfying

$$A \geq (\alpha+1) A^{\alpha+\beta+1} |e|^{\alpha+\beta+1} r_0^{\beta+1} + \lambda |f|$$

and

$$r_0 A \geq (\beta+1) A^{\alpha+\beta+1} |e|^{\alpha+\beta+1} r_0^{\beta} + \mu |g| .$$

Thus $(Ae, r_0 Ae)$ satisfies

$$A = -\Delta (Ae) \geq (\alpha+1)(Ae)^{\alpha} (r_0 Ae)^{\beta+1} + \lambda f(x)$$

and

$$r_0 A = -\Delta (r_0 Ae) \geq (\beta+1)(Ae)^{\alpha+1} (r_0 Ae)^{\beta} + \mu g(x) \quad \text{for all } \ x \in \Omega,$$

and hence $(\overline{u} = Ae, \overline{v} = r_0 Ae)$ is a supersolution of $(S_{\lambda,\mu})$.

Taking λ, μ possibly smaller, we also have

$$\lambda u_f < Ae \quad \text{and} \quad \mu u_g < r_0 Ae.$$

Hence, $(S_{\lambda,\mu})$ admits at least one solution (u, v) such that

$$(\lambda u_f, \mu v_g) \leq (u, v) \leq (Ae, r_0 Ae).$$

By the maximum principle applied to

$$\begin{cases} -\Delta (u - \lambda u_f) = (\alpha+1) u^{\alpha} v^{\beta+1} \geq 0 & \text{in } \ \Omega, \\ u - \lambda u_f = 0 & \text{on } \ \partial\Omega, \end{cases}$$

we obtain

$$u > \lambda u_f \geq 0 \quad \text{in } \ \Omega,$$

i.e., $u_{\lambda\mu} > 0$ in Ω and similarly $v > 0$ in Ω.

The sub and supersolution method can be applied in the process of establishing a minimal solution of $(S_{\lambda,\mu})$ (see [10]).

Let $u_0 := \underline{u} = \lambda u_f$ and $v_0 := \underline{v} = \mu v_g$ and consider the monotone iteration for $\lambda, \mu \in (0, \Lambda_0)$

$$\begin{cases} -\Delta u_{n+1} = (\alpha+1) u_n^{\alpha} v_n^{\beta+1} + \lambda f(x) & \text{in } \Omega, \\ -\Delta v_{n+1} = (\beta+1) u_n^{\alpha+1} v_n^{\beta} + \mu g(x) & \text{in } \Omega, \\ u_{n+1} > 0, v_{n+1} > 0 & \text{in } \Omega, \\ u_{n+1} = v_{n+1} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $n = 0, 1, 2, \ldots$

For $n = 1$, we get

$$\begin{cases} -\Delta (u_1 - u_0) = (\alpha+1) u_0^{\alpha} v_0^{\beta+1} + \lambda f(x) - \lambda f(x) = (\alpha+1) u_0^{\alpha} v_0^{\beta+1} & \text{in } \Omega, \\ u_1 - u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

by the maximum principle, $u_1 > u_0$ in Ω.

By induction on n and the maximum principle, we obtain

$$u_0 < u_1 < u_2 < \cdots < u_n < \cdots < u \quad \text{in } \Omega.$$

Similarly, we have

$$v_0 < v_1 < v_2 < \cdots < v_n < \cdots < v \quad \text{in } \Omega.$$

We may assume that

$$u_n \to u_{\lambda,\mu} \text{ and } v_n \to v_{\lambda,\mu} \text{ in } \Omega.$$

Then $(u_{\lambda,\mu}, v_{\lambda,\mu})$ is a solution of $(S_{\lambda,\mu})$. Furthermore, by the maximum principle, $0 < u_{\lambda,\mu} \leq u$ and $0 < v_{\lambda,\mu} \leq v$ in Ω which imply that $(u_{\lambda,\mu}, v_{\lambda,\mu})$ is the minimal solution of $(S_{\lambda,\mu})$.

Moreover, it follows from the construction of super solution of $(S_{\lambda,\mu})$ that

$$|u_{\lambda,\mu}| \text{ and } |v_{\lambda,\mu}| \to 0 \quad \text{as } \max(\lambda,\mu) \to 0. \qquad \square$$

4 Existence of the second solution

In the sequel λ and μ are fixed. We look for a second solution of $(S_{\lambda,\mu})$ of the form $(u_0 + u, v_0 + v)$, where (u_0, v_0) denotes the minimal solution given by Theorem 1 and (u, v) is a positive solution of the following problem

$$\begin{cases} -\Delta u = & (\alpha+1)\,|u+u_0|^{\alpha-1}(u+u_0)\,|v+v_0|^{\beta+1} \\ & \qquad -(\alpha+1)\,u_0^{\alpha}v_0^{\beta+1} \qquad \text{in } \Omega, \\ -\Delta v = & (\beta+1)\,|u+u_0|^{\alpha+1}\,|v+v_0|^{\beta-1}(v+v_0) \\ & \qquad -(\beta+1)\,u_0^{\alpha+1}v_0^{\beta} \qquad \text{in } \Omega, \\ u = v = 0 & \qquad \text{on } \partial\Omega. \end{cases} \tag{Σ}$$

$(u, v) \in E$ is said to be a *weak solution* of (Σ) if

$$\begin{aligned} \int_\Omega \{\nabla u \nabla \varphi + \nabla v \nabla \psi\}\,dx - \int_\Omega \Big\{(\alpha+1)\,|u+u_0|^{\alpha-1}(u+u_0)\,|v+v_0|^{\beta+1}\varphi\Big\}\,dx \\ - \int_\Omega \Big\{(\beta+1)\,|u+u_0|^{\alpha+1}\,|v+v_0|^{\beta-1}(v+v_0)\,\psi\Big\}\,dx \\ + \int_\Omega \Big\{(\alpha+1)\,u_0^{\alpha}v_0^{\beta+1}\varphi + (\beta+1)\,u_0^{\alpha+1}v_0^{\beta}\psi\Big\}\,dx = 0, \end{aligned}$$

for all $(\varphi, \psi) \in E$.

The corresponding energy functional of (Σ) is

$$J(u,v) = \frac{1}{2}\|(u,v)\|^2 - \int_\Omega G(u,v)\,dx,$$

where

$$G(u,v) = \begin{cases} (u+u_0)^{\alpha+1}(v+v_0)^{\beta+1} - u_0^{\alpha+1}v_0^{\beta+1} & \\ -(\alpha+1)\,u_0^{\alpha}v_0^{\beta+1}u - (\beta+1)\,u_0^{\alpha+1}v_0^{\beta}v & \text{if } u \geq 0 \text{ and } v \geq 0, \\ 0 & \text{if } u \leq 0 \text{ or } v \leq 0. \end{cases}$$

We note that $J \in C^1(E)$.

It is well known that the nontrivial weak solutions of (Σ) are equivalent to the nonzero critical points of J in E. By the regularity theory of elliptic systems each solution of (Σ) is classical.

For proving Theorem 2, we need the following lemmas.

Lemma 3 $(0,0)$ *is a local minimum of J in E.*

Proof. We know that

$$I(u,v) = \frac{1}{2}\|(u,v)\|^2 - \int_\Omega F(u,v)\,dx,$$

where

$$F(u,v) = \begin{cases} u^{\alpha+1}v^{\beta+1} + \lambda f(x)\,u + \mu g(x)\,v & \text{for } u \geq 0 \text{ and } v \geq 0, \\ 0 & \text{for } u \leq 0 \text{ or } v \leq 0, \end{cases}$$

is the associated energy functional to $(S_{\lambda,\mu})$.

Since

$$\begin{aligned} & G(u^+,v^+) - F(u_0+u^+, v_0+v^+) \\ = \; & -u_0^{\alpha+1}v_0^{\beta+1} - \lambda f(x)\left(u_0+u^+\right) - \mu g(x)\left(v_0+v^+\right) \\ & +(\alpha+1)\,u_0^{\alpha}v_0^{\beta+1}u^+ + (\beta+1)\,u_0^{\alpha+1}v_0^{\beta}v^+, \end{aligned}$$

one has

$$\begin{aligned} J(u,v) \; = \; & \frac{1}{2}\|(u,v)\|^2 - \int_\Omega F\left(u_0+u^+, v_0+v^+\right)\,dx \\ + \; & \int_\Omega F(u_0,v_0)\,dx - \int_\Omega \left(\lambda f(x)\,u^+ + \mu g(x)\,v^+\right)\,dx \\ - \; & \int_\Omega \Big((\alpha+1)u_0^{\alpha}v_0^{\beta+1}u^+ + (\beta+1)\,u_0^{\alpha+1}v_0^{\beta}v^+\Big)\,dx. \end{aligned}$$

By a straightforward computation, we obtain

$$\begin{aligned} I\left(u_0+u^+,v_0+v^+\right) &= \frac{1}{2}\left\|(u_0,v_0)\right\|^2+\frac{1}{2}\left\|v^+\right\|^2+\frac{1}{2}\left\|u^+\right\|^2 \\ +\int_\Omega\left(\nabla u_0\nabla u^++\nabla v_0\nabla v^+\right)dx &- \int_\Omega F\left(u_0+u^+,v_0+v^+\right)dx. \end{aligned}$$

Hence

$$J(u,v)=\frac{1}{2}\left\|(u,v)\right\|^2+I\left(u_0+u^+,v_0+v^+\right)-I(u_0,v_0).$$

From the fact that (u_0,v_0) is a local minimum of I in E, it follows that

$$J(u,v)\geq\frac{1}{2}\left\|\left(u^-,v^-\right)\right\|^2 \text{ for all } (u,v) \text{ such that } \left\|(u,v)\right\|\leq\varepsilon$$

with $\varepsilon>0$ small enough. □

Proof of Theorem 2.

Subcritical case. We first deal with the case $\alpha+\beta<\frac{4}{N-2}$ when $(PS)_c$ holds for all c. Since $\alpha+\beta>0$, it follows that for every positive (u,v), $J(tu,tv)\to-\infty$ as $t\to+\infty$ and there exists $(u_1,v_1)\in E$ such that $J(u_1,v_1)<0$.

Let

$$c:=\inf_{\gamma\in\Gamma}\max\left\{J(\gamma(t)):t\in[0,1]\right\}>0,$$

where

$$\Gamma=\left\{\gamma\in C([0,1],E)\mid\gamma(0)=(0,0),\ \gamma(1)=(u_1,v_1)\right\}.$$

Then we apply the Mountain Pass Theorem.

Critical case (*i.e.*, $\alpha+\beta=\frac{4}{N-2}$). Without loss of generality, we suppose that $0\in\Omega$.

Following the method in [5], we use the function

$$\omega_\varepsilon(x)=\frac{\varphi(x)}{\left(\varepsilon^2+|x^2|\right)^{\frac{N-2}{2}}},\quad\varepsilon>0,$$

where φ is a cut-off positive function such that $\varphi\equiv1$ in a neighborhood of 0.

Let B and C be positive constants such that $\frac{B}{C}=\left(\frac{\alpha+1}{\beta+1}\right)^{\frac{1}{2}}$, then $(B\omega_\varepsilon,C\omega_\varepsilon)$ is a solution of

$$\begin{cases}-\Delta u=(\alpha+1)\,u^\alpha v^{\beta+1} & \text{in } \mathbb{R}^N,\\ -\Delta v=(\beta+1)\,u^{\alpha+1}v^\beta & \text{in } \mathbb{R}^N,\\ \quad u(x)=v(x)=0 & \text{as } |x|\to+\infty.\end{cases}$$

A careful expansion as $\varepsilon \to 0$ shows that

$$\sup_{t\geq 0} J\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right) < \frac{2^*}{N}\left(\frac{1}{2^*}S_{\alpha,\beta}(\Omega)\right)^{\frac{N}{2}}.$$

In fact, for u and v nonnegative, we have

$$\begin{aligned}
&(u+u_0)^{\alpha+1}(v+v_0)^{\beta+1}\\
\geq\ & \left(u^{\alpha+1}+u_0^{\alpha+1}+(\alpha+1)\,u_0^{\alpha}u\right)\left(v^{\beta+1}+v_0^{\beta+1}+(\beta+1)\,v_0^{\beta}v\right)\\
\geq\ & u^{\alpha+1}v^{\beta+1}+u_0^{\alpha+1}v_0^{\beta+1}+(\alpha+1)\,u_0^{\alpha}uv_0^{\beta+1}+(\beta+1)\,u_0^{\alpha+1}v_0^{\beta}v+h(u,v),
\end{aligned}$$

where

$$\begin{aligned}
h(u,v) &= u_0^{\alpha+1}v^{\beta+1}+u^{\alpha+1}v_0^{\beta+1}+(\alpha+1)\,u_0^{\alpha}uv^{\beta+1}+(\beta+1)\,u^{\alpha+1}v_0^{\beta}v\\
&+ (\alpha+1)\,(\beta+1)\,u_0^{\alpha}v_0^{\beta}uv.
\end{aligned}$$

We know that

$$J\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right)=\frac{1}{2}\left(B^2+C^2\right)t^2\left|\nabla\omega_\varepsilon\right|_2^2-\int_\Omega G\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right)dx \leq \tilde{J}\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right)$$

$$:=\frac{1}{2}\left(B^2+C^2\right)t^2\left|\nabla\omega_\varepsilon\right|_2^2-t^{2^*}B^{\alpha+1}C^{\beta+1}\left|\nabla\omega_\varepsilon\right|_{2^*}^{2^*}-\int_\Omega h\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right)dx.$$

Note that

$$\tilde{J}\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right)\leq\frac{1}{2}\left(B^2+C^2\right)t^2\left|\nabla\omega_\varepsilon\right|_2^2-t^{2^*}B^{\alpha+1}C^{\beta+1}\left|\omega_\varepsilon\right|_{2^*}^{2^*}$$

and thus $\lim\limits_{t\to+\infty} J\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right)=-\infty$. Therefore, $\sup\limits_{t\geq 0} J\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right)$ is achieved at some $t_\varepsilon>0$ (if $t_\varepsilon=0$, then $\sup\limits_{t\geq 0} J\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right)=0$ and there is nothing to prove).

Since the derivative of the function $t\mapsto \tilde{J}\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right)$ vanishes at $t=t_\varepsilon$, we have

$$\begin{aligned}
&\left(B^2+C^2\right)t_\varepsilon^2\left|\nabla\omega_\varepsilon\right|_2^2-2^*t_\varepsilon^{2^*-1}B^{\alpha+1}C^{\beta+1}\left|\omega_\varepsilon\right|_{2^*}^{2^*}\\
&-\int_\Omega\left[B\omega_\varepsilon\frac{\partial h\left(t_\varepsilon B\omega_\varepsilon, t_\varepsilon C\omega_\varepsilon\right)}{\partial u}+C\omega_\varepsilon\frac{\partial h\left(t_\varepsilon B\omega_\varepsilon, t_\varepsilon C\omega_\varepsilon\right)}{\partial v}\right]dx=0
\end{aligned}$$

and therefore

$$t_\varepsilon\leq\left(\frac{\left(B^2+C^2\right)\left|\nabla\omega_\varepsilon\right|_2^2}{2^*\ B^{\alpha+1}C^{\beta+1}\left|\omega_\varepsilon\right|_{2^*}^{2^*}}\right)^{\frac{1}{2^*-2}}.$$

Thus

$$\begin{aligned}\sup_{t\geq 0} J\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right) &\leq \sup_{t\geq 0}\widetilde{J}\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right) = \widetilde{J}\left(t_\varepsilon B\omega_\varepsilon, t_\varepsilon C\omega_\varepsilon\right)\\ &\leq \frac{2^*}{N}\left(\frac{1}{2^*}S_{(\alpha,\beta)}\left(\Omega\right)\right)^{\frac{N}{2}} + O\left(\varepsilon^{\frac{N-2}{2}}\right) - \int_\Omega h\left(t_\varepsilon B\omega_\varepsilon, t_\varepsilon C\omega_\varepsilon\right)dx.\end{aligned}$$

Using lemma 1 of [5], we have

$$\int_\Omega h\left(t_\varepsilon B\omega_\varepsilon, t_\varepsilon C\omega_\varepsilon\right)dx = \begin{cases} K_1\varepsilon^{\frac{N+2-\beta(N-2)}{4}} + K_2\varepsilon^{\frac{N+2-(\beta+1)(N-2)}{4}} + K_3\varepsilon^{\frac{N+2-(\alpha+1)(N-2)}{4}} \\ \quad +K_4\varepsilon^{\frac{N+2-\alpha(N-2)}{4}} + K_5\varepsilon \quad \text{if} \quad N\geq 5, \\ \\ K_1\varepsilon^{\frac{6-2\beta}{4}} + K_2\varepsilon^{\frac{6-2(\beta+1)}{4}} + K_3\varepsilon^{\frac{6-2(\alpha+1)}{4}} + K_4\varepsilon^{\frac{6-2\alpha}{4}} \\ \quad +K_5\varepsilon\left|\log\varepsilon\right| \quad \text{if} \quad N=4, \end{cases}$$

where K_i, $i=\overline{1,5}$ are positive constants independent of ε.

Then

$$\sup_{t\geq 0} J\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right) \leq \frac{2^*}{N}\left(\frac{1}{2^*}S_{(\alpha,\beta)}\left(\Omega\right)\right)^{\frac{N}{2}} + O\left(\varepsilon^{\frac{N-2}{2}}\right) - O\left(\varepsilon^{\theta}\right)$$

where

$$\theta := \min\left(\begin{array}{c}\frac{N+2-\beta(N-2)}{4}, \frac{N+2-(\beta+1)(N-2)}{4}, \frac{N+2-(\alpha+1)(N-2)}{4}, \\ \frac{N+2-\alpha(N-2)}{4}, 1\end{array}\right).$$

We note that $\theta < \frac{N-2}{2}$ for $N>4$.

Therefore

$$\sup_{t\geq 0} J\left(tB\omega_\varepsilon, tC\omega_\varepsilon\right) < \frac{2^*}{N}\left(\frac{1}{2^*}S_{(\alpha,\beta)}\left(\Omega\right)\right)^{\frac{N}{2}} \quad \text{for} \quad N\geq 4$$

for $\varepsilon>0$ small enough.

Then the problem $(S_{\lambda,\mu})$ has a second solution. □

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