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Contemporary Theory of Functional Differential Equations and Some Classical Problems

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A survey of the recent trends and ideas of the Perm Seminar

Let $\mathbf{D}$ be a linear manifold of elements x, $\mathbf{B}$ be a Banach space of elements z, $\mathbb{R}^n$ be the space of vectors $\alpha = \mathrm{col}\{\alpha^1, \dots \alpha^n\}$ with real elements. Further assume that there exists a pair of linear operators $\delta : \mathbf{D} \to \mathbf{B}$ and $r : \mathbf{D} \to \mathbb{R}^n$ such that the systems of equations

$$\delta x = z, \quad rx = \alpha, \tag{1}$$

has a unique solution $x \in \mathbf{D}$ for any $z \in \mathbf{B}$, $\alpha \in \mathbb{R}^n$. We will write down the solution of (1) in the form

$$x = Wz + U\alpha. \tag{2}$$

The "$W-$substitution" (2) puts an element $x \in \mathbf{D}$ in correspondence to any pair $\{z, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$ and to any $x \in \mathbf{D}$ it puts in correspondence a pair $z \in \mathbf{B}$ and $\alpha \in \mathbb{R}^n$ by (1).

Thus the manifold $\mathbf{D}$ is isomorphic to the direct product $\mathbf{B} \times \mathbb{R}^n$ ($\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$). The linear isomorphism $\mathcal{J} = \{W, U\} : \mathbf{B} \times \mathbb{R}^n \to \mathbf{D}$ is defined by the operators $W : \mathbf{B} \to \mathbf{D}$ and $U : \mathbb{R}^n \to \mathbf{D}$. The space $\mathbf{D}$ is a Banach one under the norm $\|x\|_{\mathbf{D}} = \|\delta x\|_{\mathbf{B}} + \|rx\|_{\mathbb{R}^n}$. An example of such a $\mathbf{D}$ is the space of absolutely continuous functions (the functions $x : [a, b] \to \mathbb{R}^n$ such that $x(t) = \int\limits_a^b \dot{x}(s)\,ds + x(a)$). Here $(Wz)(t) = \int\limits_a^t z(s)\,ds$, $U\alpha = \alpha$, $\mathbf{B}$ is the space $\mathbf{L}$ of summable $z : [a, b] \to \mathbb{R}^n$.

The equation

$$\dot{x} = Fx \tag{3}$$

in the space $\mathbf{D}$ of absolutely continuous functions was thoroughly studied by the Perm Seminar in [1, 2]. This equation is called "the functional differential" one. It is a natural generalization of the classical ordinary differential equation and contains in itself the equations with deviating argument, integro-differential and many other with respect to a differentiable unknown x.

The theory of $\dot{x} = Fx$ is founded on the isomorphism $\mathcal{J} : \mathbf{L} \times \mathbb{R}^n \to \mathbf{D}$ and the specific character of the Lebesgue space $\mathbf{L}$ shows itself only in some questions. On the whole, the fact that $\mathbf{L}$ is a Banach space was used.

Therefore in replacement of the space $\mathbf{L}$ by an arbitrary Banach space $\mathbf{B}$ most of the fundamental statements of the general theory of equation (3) still hold. Thus a further generalization of the differential equation arises. Namely, the theory of the equation $\mathcal{L}x = Fx$ in the space $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$ with a linear operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ and a compact operator $F : \mathbf{D} \to \mathbf{B}$. To the theory of such a generalization the monographs [3, 4] are devoted.

In the mentioned monographs the role of the choice of the adequate space to the problem considered is emphasized . Moreover, there are proposed new effective approaches to some classical problems on the base of the general theory. Let us make myself more explicit using some examples.

1 Stability and asymptotic behavior of solutions

Let $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$ be a space of functions with given asymptotics. We say that the linear equation $\mathcal{L}x = f$ with the linear operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ is $\mathbf{D}-$stable if the Cauchy problem

$$\mathcal{L}x = f, \quad x(0) = \alpha \tag{4}$$

has a unique solution $x \in \mathbf{D}$ for each pair $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$.

Thus, the equation $\mathcal{L}x = f$ (for instance $\mathcal{L}x \stackrel{\text{def}}{=} \dot{x} + Ax$) is exponentially stable, if problem (4) is correctly solvable in the space $\mathbf{D} \simeq \mathbf{L}^\infty \times \mathbb{R}^n$, with elements defined by

$$x(t) = \int_0^t e^{s-t} z(s)\, ds + \alpha e^{-t}, \quad z \in \mathbf{L}^\infty$$

(by the $W-$substitution (2) $x = Wz + U\alpha$ with $(Wz)(t) = \int_0^t e^{s-t} z(s)\, ds$, $(U\alpha)(t) = \alpha e^{-t}$).

The general theory offers some theorems on solvability of problem (4). For instance,

Theorem. *Let $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ be bounded. Problem* (4) *is correctly solvable if and only if the operator $\mathcal{L}W : \mathbf{B} \to \mathbf{B}$ has the bounded inverse $[\mathcal{L}W]^{-1}$* ([4]–[6]).

Let us emphasize that the operator $\mathcal{L}W$ has an explicit form and by means of this theorem we obtain both known and some new effective tests of stability.

2 A new effective approach to some problems of the calculus of variations

Let $\mathbf{L}_2$ be the Banach space of functions $z: [a,b] \to \mathbb{R}^1$ under the norm

$$\|z\|_{\mathbf{L}_2} = \left(\int_a^b z^2(s)\, ds \right)^{1/2}$$

and $\mathbf{D}$ be isomorphic to $\mathbf{L}_2 \times \mathbb{R}^n$. Consider the problem of minimization in the space $\mathbf{D}$ of the functional with linear restrictions

$$\begin{gathered} \mathcal{I}(x) = \int_a^b f(s, (T_1x)(s), \dots, (T_m x)(s))\, ds \;\to\; \min, \\ \ell x \overset{\text{def}}{=} \operatorname{col}\{\ell^1 x, \dots, \ell^n x\} = \alpha \overset{\text{def}}{=} \operatorname{col}\{\alpha^1, \dots, \alpha^n\}. \end{gathered} \tag{5}$$

Here $T_i: \mathbf{D} \to \mathbf{L}_2$ and $\ell^i: \mathbf{D} \to \mathbb{R}^1$ are linear bounded operators.

Let us choose the linear operator $\delta: \mathbf{D} \to \mathbf{L}_2$ such that the problem

$$\delta x = z, \quad \ell^i x = \alpha^i, \quad i = 1, \dots, n, \tag{1 bis}$$

has a unique solution

$$x = Wz + U\alpha. \tag{2 bis}$$

Let us further construct an auxiliary functional $\mathcal{I}_1(z) \overset{\text{def}}{=} \mathcal{I}(Wz + U\alpha)$ in the space $\mathbf{L}_2$. If $x_0 \in \mathbf{D}$ is a point of minimum to problem (5), then $z_0 = \delta x_0$ is a point of minimum to the functional $\mathcal{I}_1(z)$. And if $z_0 \in \mathbf{L}_2$ is a point of minimum of the functional $\mathcal{I}_1(z)$, then $x_0 = Wz_0 + U\alpha$ ($\alpha = \operatorname{col}\{\alpha^1, \dots, \alpha^n\}$) is a point of minimum of problem (5).

Thus the $W-$substitution (2 bis) reduces problem (5) in the space $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbb{R}^n$ to the much simpler one in the "convenient" space $\mathbf{L}_2$. It is relevant to say that the space $\mathbf{L}_2$ is very convenient for problems of the calculus of variations. In particular, the theorems of Fermat and Weierstrass on the minimum are still valid for functionals in the space $\mathbf{L}_2$. On the base of these theorems there effective tests for convexity of functionals are established.

3 Regularization of singular equations

One and the some equation may be singular in one space and regular in another. For instance, the equation

$$(\mathcal{L}x)(t) \overset{\text{def}}{=} t(1-t)\ddot{x}(t) + \frac{q(t)}{t}\dot{x}(t) + \frac{p(t)}{1-t}x(t) = f(x), \quad t \in [0,1],$$

with summable $p(\cdot)$ and $q(\cdot)$ is singular in the traditional space W^2 of $x : [0,1] \to \mathbb{R}^1$ with absolutely continuous $\dot{x}$ and summable $\ddot{x}$. Nevertheless, it becomes regular (complies with a theorem of the general theory) in the space of $\mathbf{D} \simeq \mathbf{L}^\gamma \times \mathbb{R}^2$, where $\mathbf{L}^\gamma$ is the weighted space $\mathbf{L}^\gamma$ ($z \in \mathbf{L}^\gamma$ if the product $t(1-t)z(t)$ is summable).

The ideas and applications of the general theory had explicit influence on many other actual problems. In the works of G. G. Islamov [7, 8] an original view on the construction of the spaces $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$ and on the problem of equations of the first order was proposed. The general theory had a very essential impact on our assumption about stochastic equations [1, 4].

Constructive methods, control theory and economic dynamics obtain many new approaches under the influence of the ideas and methods of the contemporary theory of functional differential equations [1, 4].

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