

On Difference Schemes for Hyperbolic-Parabolic Equations

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Abstract

The nonlocal boundary value problem for a hyperbolic-parabolic equation in a Hilbert space H is considered. The difference schemes approximately solving this boundary value problem are presented. The stability estimates for the solution of these difference schemes are established. In applications, the stability estimates for the solutions of the difference schemes of the mixed type boundary value problems for hyperbolic-parabolic equations are obtained. The theoretical statements for the solution of these difference schemes for hyperbolic-parabolic equation are supported by the results of numerical experiments.

1 The differential problem

Methods for numerical solutions of the nonlocal boundary value problems for hyperbolic-parabolic equations

$$\begin{cases} \frac{d^2 u(t)}{dt^2} + Au(t) = f(t) & (0 \leq t \leq 1), \\ \frac{du(t)}{dt} + Au(t) = g(t) & (-1 \leq t \leq 0), \\ u(-1) = \alpha u(\mu) + \varphi, & |\alpha| \leq 1, 0 < \mu \leq 1, \end{cases}$$

for differential equations in a Hilbert space H , with the self-adjoint positive definite operator A have been studied extensively (see [7]–[17] and the references therein).

It is known (see [1]–[4]) that various boundary value problems for the hyperbolic-parabolic equations can be reduced to the nonlocal boundary-value problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t) & (0 \leq t \leq 1), \\ \frac{du(t)}{dt} + Au(t) = g(t) & (-1 \leq t \leq 0), \\ u(-1) = \alpha \frac{du(\mu)}{dt} + \varphi, & |\alpha| \leq 1, 0 < \mu \leq 1, \end{cases} \quad (1)$$

for differential equations in a Hilbert space H , with the self-adjoint positive definite operator A .

In the present paper stability estimates for the solution of the nonlocal boundary value problem (1) are established. The difference schemes approximately solving this boundary value problem are presented. The stability estimates for the solution of these difference schemes are established. In applications, the stability estimates for the solutions of the difference schemes of the mixed type boundary value problems for hyperbolic-parabolic equations are obtained. The theoretical statements for the solution of these difference schemes for hyperbolic-parabolic equation are supported by the results of numerical experiments.

Note that methods for the solutions of the nonlocal boundary value problems for partial differential and difference equations have been studied extensively by many researches (see [5, 6], [18]–[29] and the references therein).

A function $u(t)$ is called a *solution* of the problem (1) if the following conditions are satisfied:

- i. $u(t)$ is twice continuously differentiable on the interval $(0,1]$ and continuously differentiable on the segment $[-1,1]$. The derivative at the endpoints of the segment are understood as the appropriate unilateral derivatives.
- ii. The element $u(t)$ belongs to $D(A)$ for all $t \in [-1,1]$, and the function $Au(t)$ is continuous on the segment $[-1,1]$.
- iii. $u(t)$ satisfies the equations and nonlocal boundary condition (1).

Theorem 1 *Suppose that $\varphi \in D(A)$, $g(0) \in D(A^{1/2})$, $g'(0) \in H$, $f(0) \in D(A^{1/2})$ and $f'(0) \in H$. Let $f(t)$ be twice continuously differentiable on $[0,1]$ and $g(t)$ be twice continuously differentiable on $[-1,0]$ functions. Then there is a unique solution of the problem (1) and the stability inequalities hold:*

$$\begin{aligned} & \max_{-1 \leq t \leq 1} \|u(t)\|_H \leq M \left[\|\varphi\|_H + \max_{-1 \leq t \leq 0} \|A^{-1/2}g'(t)\|_H \right. \\ & \left. + \|A^{-1/2}g(0)\|_H + \|A^{-1/2}f(0)\|_H + \max_{0 \leq t \leq 1} \|A^{-1/2}f'(t)\|_H \right], \\ & \max_{-1 \leq t \leq 1} \left\| \frac{du}{dt} \right\|_H + \max_{-1 \leq t \leq 1} \|A^{1/2}u(t)\|_H \leq M \left[\|A^{1/2}\varphi\|_H \right. \end{aligned}$$

$$\begin{aligned}
& + \|g(0)\|_H + \max_{-1 \leq t \leq 0} \|g'(t)\|_H + \|f(0)\|_H + \max_{0 \leq t \leq 1} \|f'(t)\|_H \Big], \\
& \max_{-1 \leq t \leq 0} \left\| \frac{du}{dt} \right\|_H + \max_{0 \leq t \leq 1} \left\| \frac{d^2u}{dt^2} \right\|_H + \max_{-1 \leq t \leq 1} \|Au(t)\|_H \\
& \leq M \left[\|A\varphi\|_H + \|A^{1/2}g(0)\|_H + \|g'(0)\|_H + \max_{-1 \leq t \leq 0} \|g''(t)\|_H \right. \\
& \quad \left. + \|A^{1/2}f(0)\|_H + \|f'(0)\|_H + \max_{0 \leq t \leq 1} \|f''(t)\|_H \right],
\end{aligned}$$

where M does not depend on $f(t)$, $t \in [0, 1]$, $g(t)$, $t \in [-1, 0]$ and φ .

The proof of Theorem 1 is based on the formulas

$$\begin{aligned}
u(t) &= e^{-(t+1)A}u(-1) + \int_{-1}^t e^{-(t-\lambda)A}g(\lambda) d\lambda, \quad -1 \leq t \leq 0, \\
u(t) &= [c(t) - As(t)] \left\{ e^{-A}u(-1) + \int_{-1}^0 e^{\lambda A}g(\lambda) d\lambda \right\} \\
& \quad + s(t)g(0) + \int_0^t s(t-\lambda)f(\lambda) d\lambda, \quad 0 \leq t \leq 1, \\
u(-1) &= T \left\{ -\alpha [Ac(\mu) + As(\mu)] \int_{-1}^0 e^{\lambda A}g(\lambda) d\lambda + \alpha [c(\mu)g(0) + \int_0^\mu c(\mu-\lambda)f(\lambda) d\lambda] + \varphi \right\}, \\
T &= (I + \alpha [Ac(\mu) + As(\mu)] e^{-A})^{-1}, \\
c(t) &= \frac{e^{iA^{1/2}t} + e^{-iA^{1/2}t}}{2}, \quad s(t) = A^{-1/2} \frac{e^{iA^{1/2}t} - e^{-iA^{1/2}t}}{2i}
\end{aligned}$$

and on the estimates

$$\|c(t)\|_{H \rightarrow H} \leq 1, \tau \|A^{1/2}s(t)\|_{H \rightarrow H} \leq 1, \quad t \geq 0, \quad (2)$$

$$\|A^\beta e^{-tA}\|_{H \rightarrow H} \leq Mt^{-\beta} e^{-\delta t}, \quad t > 0, \quad 0 \leq \beta \leq 1, \delta > 0, M > 0, \quad (3)$$

and on the following lemma.

Lemma 1 *The operator $I + \alpha [Ac(\mu) + As(\mu)] e^{-A}$ has an inverse $T = (I + \alpha [Ac(\mu) + As(\mu)] e^{-A})^{-1}$ and the estimate holds:*

$$\|T\|_{H \rightarrow H} \leq M. \quad (4)$$

Proof. The proof of this lemma is based on the estimate

$$\|[Ac(\mu) + As(\mu)]e^{-A}\|_{H \rightarrow H} < 1.$$

Using the definitions of $c(\mu)$ and $s(\mu)$ and the positivity and selfadjointness property of A , we obtain

$$\|[Ac(\mu) + As(\mu)]e^{-A}\|_{H \rightarrow H} < \sup_{\delta \leq \lambda < \infty} |[\lambda \cos(\sqrt{\lambda}\mu) + \sqrt{\lambda} \sin(\sqrt{\lambda}\mu)]e^{-\lambda}|.$$

Since

$$\lambda \cos(\sqrt{\lambda}\mu) + \sqrt{\lambda} \sin(\sqrt{\lambda}\mu) = \sqrt{\lambda}\sqrt{\lambda+1} \cos(\sqrt{\lambda}\mu - \varphi_0),$$

we have that

$$|[\lambda \cos(\sqrt{\lambda}\mu) + \sqrt{\lambda} \sin(\sqrt{\lambda}\mu)]e^{-\lambda}| \leq \sqrt{\lambda}\sqrt{\lambda+1}e^{-\lambda}.$$

It is easy to show that $\sqrt{\lambda}\sqrt{\lambda+1}e^{-\lambda} < 1$. Lemma 1 is proved.

2 The difference schemes. Stability

Let us associate with the boundary-value problem (1) the corresponding first order accuracy difference scheme

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k, \\ f_k = f(t_{k+1}), t_{k+1} = (k+1)\tau, 1 \leq k \leq N-1, \\ \tau^{-1}(u_1 - u_0) = -Au_0 + g_0, \\ \tau^{-1}(u_k - u_{k-1}) + Au_k = g_k, \quad g_k = g(t_k), \\ t_k = k\tau, \quad -(N-1) \leq k \leq 0, \\ u_{-N} = \alpha^{\frac{u_{[\mu/\tau]} - u_{[\mu/\tau]-1}}{\tau}} + \varphi. \end{cases} \quad (5)$$

A study of discretization, over time only, of the nonlocal boundary value problem also permits one to include general difference schemes in applications, if the differential operator in space variables, A , is replaced by the difference operators A_h that act in the Hilbert spaces H_h and are uniformly self-adjoint positive definite in h for $0 < h \leq h_0$.

Theorem 2 *Let $\varphi \in D(A)$, $g_0 \in D(A^{1/2})$ and $f_1 \in D(A^{1/2})$. Then for the solution of the difference scheme (5) the stability inequalities hold:*

$$\max_{-N \leq k \leq N} \|u_k\|_H$$

$$\begin{aligned}
&\leq M \left[\|\varphi\|_H + \|A^{-1/2}f_1\|_H + \max_{2 \leq k \leq N-1} \|A^{-1/2}(f_k - f_{k-1})\tau^{-1}\|_H \right. \\
&\quad \left. + \|A^{-1/2}g_0\|_H + \max_{-(N-1) \leq k \leq 0} \|A^{-1/2}(g_k - g_{k-1})\tau^{-1}\|_H \right], \\
&\quad \max_{-N \leq k \leq N} \|A^{1/2}u_k\|_H \\
&\leq M \left[\|A^{1/2}\varphi\|_H + \|f_1\|_H + \max_{2 \leq k \leq N-1} \|(f_k - f_{k-1})\tau^{-1}\|_H \right. \\
&\quad \left. + \|g_0\|_H + \max_{-(N-1) \leq k \leq 0} \|(g_k - g_{k-1})\tau^{-1}\|_H \right], \\
&\quad \max_{1 \leq k \leq N-1} \|\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\|_H \\
&\quad + \max_{-(N-1) \leq k \leq 0} \|\tau^{-1}(u_k - u_{k-1})\|_H + \max_{-N \leq k \leq N} \|Au_k\|_H \\
&\leq M \left[\|A\varphi\|_H + \|A^{1/2}f_1\|_H + \|(f_2 - f_1)\tau^{-1}\|_H \right. \\
&\quad \left. + \max_{2 \leq k \leq N-2} \|(f_{k+1} - 2f_k - f_{k-1})\tau^{-2}\|_H + \|A^{1/2}g_0\|_H \right. \\
&\quad \left. + \|(g_0 - g_{-1})\tau^{-1}\|_H + \max_{-(N-1) \leq k \leq -1} \|(g_{k+1} - 2g_k - g_{k-1})\tau^{-2}\|_H \right],
\end{aligned}$$

where M does not depend on τ , f_k , $1 \leq k \leq N-1$, g_k , $-N+1 \leq k \leq 0$ and φ .

The proof of Theorem 2 is based on the formulas

$$\begin{aligned}
u_1 &= u_0 + \tau(-Au_0 + g_0) = (I - \tau A)u_0 + \tau g_0, \\
u_k &= \left\{ \frac{1}{2}(I - \tau A) \left(R^{k-1}(\tau A^{1/2}) + R^{k-1}(-\tau A^{1/2}) \right) \right. \\
&\quad \left. + \frac{1}{2i}A^{1/2} \left(R^{k-1}(\tau A^{1/2}) + R^{k-1}(-\tau A^{1/2}) \right) \right\} \\
&\quad \times [R^N u_{-N} + \tau \sum_{s=-N+1}^0 R^{-s+1} g_s] \\
&\quad + 1/(2i)A^{-1/2}(I + \tau^2 A) \left[R^k(-\tau A^{1/2}) - R^k(\tau A^{1/2}) \right] g_0 \\
&\quad + \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2} \left[R^{k-s}(-\tau A^{1/2}) - R^{k-s}(\tau A^{1/2}) \right] f_s, \quad 2 \leq k \leq N,
\end{aligned}$$

where $R(\pm\tau A^{1/2}) = (I \pm i\tau A^{1/2})^{-1}$;

$$u_k = R^{N+k}u_{-N} + \tau \sum_{s=-N+1}^k R^{k-s+1}g_s, \quad -(N-1) \leq k \leq 0,$$

where $R = R(\tau A) = (I + \tau A)^{-1}$;

$$u_{-N} = \left\{ \begin{array}{l} T_\tau \left[-\alpha\tau A \sum_{s=-N+1}^0 R^{-s+1}g_s + \alpha g_0 + \varphi \right] \quad \text{if } 0 < \mu < 2\tau, \\ T_\tau \left[\alpha \left[\frac{1}{2}(I - \tau A) (R(\tau A^{1/2}) + R(-\tau A^{1/2})) + \frac{1}{2i}A^{1/2} (R(\tau A^{1/2}) + R(-\tau A^{1/2})) \right] \right. \\ \times \tau \sum_{s=-N+1}^0 R^{-s+1}g_s + 1/(2i)A^{-1/2}(I + \tau^2 A) [R^2(-\tau A^{1/2}) - R^2(\tau A^{1/2})] g_0 \\ \left. + \sum_{s=1}^2 \frac{\tau}{2i}A^{-1/2} [R^{2-s}(-\tau A^{1/2}) - R^{2-s}(\tau A^{1/2})] f_s - (I - \tau A)\tau \right. \\ \left. \times \sum_{s=-N+1}^0 R^{-s+1}g_s \right] \tau^{-1} - \alpha g_0 + \varphi \quad \text{if } 2\tau \leq \mu < 3\tau, \\ T_\tau \left[\alpha \left\{ \frac{1}{2}(I - \tau A) (R^{[\mu/\tau]-1}(\tau A^{1/2}) + R^{[\mu/\tau]-1}(-\tau A^{1/2})) \right. \right. \\ \left. \left. + \frac{1}{2i}A^{1/2} (R^{[\mu/\tau]-1}(\tau A^{1/2}) + R^{[\mu/\tau]-1}(-\tau A^{1/2})) \right\} \tau \sum_{s=-N+1}^0 R^{-s+1}g_s \right. \\ \left. + \sum_{s=1}^{[\mu/\tau]-1} \frac{\tau}{2i}A^{-1/2} [R^{[\mu/\tau]-s}(-\tau A^{1/2}) - R^{[\mu/\tau]-s}(\tau A^{1/2})] f_s \right. \\ \left. + 1/(2i)A^{-1/2}(I + \tau^2 A) [R^{[\mu/\tau]}(-\tau A^{1/2}) - R^{[\mu/\tau]}(\tau A^{1/2})] g_0 \right. \\ \left. - \left\{ \frac{1}{2}(I - \tau A) (R^{[\mu/\tau]-2}(\tau A^{1/2}) + R^{[\mu/\tau]-2}(-\tau A^{1/2})) \right. \right. \\ \left. \left. - \frac{1}{2i}A^{1/2} (R^{[\mu/\tau]-2}(\tau A^{1/2}) + R^{[\mu/\tau]-2}(-\tau A^{1/2})) \right\} \tau \sum_{s=-N+1}^0 R^{-s+1}g_s \right. \\ \left. - 1/(2i)A^{-1/2}(I + \tau^2 A) [R^{[\mu/\tau]-1}(-\tau A^{1/2}) - R^{[\mu/\tau]-1}(\tau A^{1/2})] g_0 \right. \\ \left. - \sum_{s=1}^{[\mu/\tau]-2} \frac{\tau}{2i}A^{-1/2} [R^{[\mu/\tau]-1-s}(-\tau A^{1/2}) - R^{[\mu/\tau]-1-s}(\tau A^{1/2})] f_s \right] \tau^{-1} \\ \left. + \varphi \right] \quad \text{if } 3\tau \leq \mu, \end{array} \right.$$

where

$$T_\tau = \left\{ \begin{array}{l} (I + \alpha R^N)^{-1} \quad \text{if } 0 < \mu < 2\tau, \\ (I - \alpha \left[\frac{1}{2}(I - \tau A) (R(\tau A^{1/2}) + R(-\tau A^{1/2})) \right. \right. \\ \left. \left. + \frac{1}{2i}A^{1/2} (R(\tau A^{1/2}) + R(-\tau A^{1/2})) - (I - \tau A) \right] R^N)^{-1} \quad \text{if } 2\tau \leq \mu < 3\tau, \\ \left\{ I - \alpha \left\{ \frac{1}{2}(I - \tau A) (R^{[\mu/\tau]-1}(\tau A^{1/2}) + R^{[\mu/\tau]-1}(-\tau A^{1/2})) \right. \right. \right. \\ \left. \left. + \frac{1}{2i}A^{1/2} (R^{[\mu/\tau]-1}(\tau A^{1/2}) + R^{[\mu/\tau]-1}(-\tau A^{1/2})) \right\} \tau^{-1} R^N \right. \\ \left. - \left\{ \frac{1}{2}(I - \tau A) (R^{[\mu/\tau]-2}(\tau A^{1/2}) + R^{[\mu/\tau]-2}(-\tau A^{1/2})) \right. \right. \\ \left. \left. - \frac{1}{2i}A^{1/2} (R^{[\mu/\tau]-2}(\tau A^{1/2}) + R^{[\mu/\tau]-2}(-\tau A^{1/2})) \right\} \tau^{-1} R^N \right\}^{-1} \quad \text{if } 3\tau \leq \mu, \end{array} \right.$$

and on the estimates

$$\|R(\pm\tau A^{1/2})\|_{H \rightarrow H} \leq 1, \quad \|\tau A^{1/2}R(\pm\tau A^{1/2})\|_{H \rightarrow H} \leq 1, \quad (6)$$

$$\|R^k\|_{H \rightarrow H} \leq M(1 + \delta\tau)^{-k}, \quad \|AR^k\|_{H \rightarrow H} \leq M(k\tau)^{-1}, \quad k \geq 1, \quad (7)$$

and on the following lemma.

Lemma 2 *The operator*

$$Q_\tau = \begin{cases} I + \alpha R^N & \text{if } 0 < \mu < 2\tau, \\ I - \alpha \left[\frac{1}{2}(I - \tau A) (R(\tau A^{1/2}) + R(-\tau A^{1/2})) \right. \\ \quad \left. + \frac{1}{2i} A^{1/2} (R(\tau A^{1/2}) + R(-\tau A^{1/2})) - (I - \tau A) \right] R^N \\ I - \alpha \left\{ \left[\frac{1}{2}(I - \tau A) (R^{[\mu/\tau]-1}(\tau A^{1/2}) + R^{[\mu/\tau]-1}(-\tau A^{1/2})) \right. \right. \\ \quad \left. \left. + \frac{1}{2i} A^{1/2} (R^{[\mu/\tau]-1}(\tau A^{1/2}) + R^{[\mu/\tau]-1}(-\tau A^{1/2})) \right] \tau^{-1} R^N \right. \\ \quad \left. - \left[\frac{1}{2}(I - \tau A) (R^{[\mu/\tau]-2}(\tau A^{1/2}) + R^{[\mu/\tau]-2}(-\tau A^{1/2})) \right. \right. \\ \quad \left. \left. - \frac{1}{2i} A^{1/2} (R^{[\mu/\tau]-2}(\tau A^{1/2}) + R^{[\mu/\tau]-2}(-\tau A^{1/2})) \right] \tau^{-1} R^N \right\} & \text{if } 2\tau \leq \mu < 3\tau, \\ \left. - \left[\frac{1}{2}(I - \tau A) (R^{[\mu/\tau]-2}(\tau A^{1/2}) + R^{[\mu/\tau]-2}(-\tau A^{1/2})) \right. \right. \\ \quad \left. \left. - \frac{1}{2i} A^{1/2} (R^{[\mu/\tau]-2}(\tau A^{1/2}) + R^{[\mu/\tau]-2}(-\tau A^{1/2})) \right] \tau^{-1} R^N \right\} & \text{if } 3\tau \leq \mu, \end{cases}$$

has an inverse

$$T_\tau = \begin{cases} (I + \alpha R^N)^{-1} & \text{if } 0 < \mu < 2\tau, \\ (I - \alpha \left[\frac{1}{2}(I - \tau A) (R(\tau A^{1/2}) + R(-\tau A^{1/2})) \right. \\ \quad \left. + \frac{1}{2i} A^{1/2} (R(\tau A^{1/2}) + R(-\tau A^{1/2})) - (I - \tau A) \right] R^N)^{-1} & \text{if } 2\tau \leq \mu < 3\tau, \\ \left\{ I - \alpha \left\{ \left[\frac{1}{2}(I - \tau A) (R^{[\mu/\tau]-1}(\tau A^{1/2}) + R^{[\mu/\tau]-1}(-\tau A^{1/2})) \right. \right. \right. \\ \quad \left. \left. + \frac{1}{2i} A^{1/2} (R^{[\mu/\tau]-1}(\tau A^{1/2}) + R^{[\mu/\tau]-1}(-\tau A^{1/2})) \right] \tau^{-1} R^N \right. \right. \\ \quad \left. \left. - \left[\frac{1}{2}(I - \tau A) (R^{[\mu/\tau]-2}(\tau A^{1/2}) + R^{[\mu/\tau]-2}(-\tau A^{1/2})) \right. \right. \right. \\ \quad \left. \left. - \frac{1}{2i} A^{1/2} (R^{[\mu/\tau]-2}(\tau A^{1/2}) + R^{[\mu/\tau]-2}(-\tau A^{1/2})) \right] \tau^{-1} R^N \right\}^{-1} & \text{if } 3\tau \leq \mu, \end{cases}$$

and the estimate holds:

$$\|T_\tau\|_{H \rightarrow H} \leq M, \quad (8)$$

where M does not depend on τ .

Proof. Note that if $0 < \mu < 2\tau$, then

$$T_\tau - (I + \alpha A e^{-A})^{-1} = T_\tau (I + \alpha e^{-A})^{-1} \alpha [AR^N - A e^{-A}].$$

If $2\tau \leq \mu < 3\tau$, then

$$\begin{aligned} T_\tau - (I + \alpha [Ac(\tau) + As(\tau)] e^{-A})^{-1} &= T_\tau (I + \alpha [c(\tau) + As(\tau)] e^{-A})^{-1} \\ &\times \alpha \left\{ [Ac(\tau) + As(\tau)] e^{-A} + \left[\frac{1}{2}(I - \tau A) (R(\tau A^{1/2}) + R(-\tau A^{1/2})) \right] \right\} \end{aligned}$$

$$+\frac{1}{2i}A^{1/2}\left(R(\tau A^{1/2})+R(-\tau A^{1/2})\right)-(I-\tau A)R^N\}.$$

If $3\tau \leq \mu$, then

$$\begin{aligned} T_\tau-(I+\alpha\left[Ac\left(\left[\frac{\mu}{\tau}\right]\tau\right)+As\left(\left[\frac{\mu}{\tau}\right]\tau\right)\right]e^{-A})^{-1} &= \alpha T_\tau\left(I+\alpha\left[Ac\left(\left[\frac{\mu}{\tau}\right]\tau\right)+As\left(\left[\frac{\mu}{\tau}\right]\tau\right)\right]e^{-A}\right)^{-1} \\ \times\left\{\alpha\left[Ac\left(\left[\frac{\mu}{\tau}\right]\tau\right)+As\left(\left[\frac{\mu}{\tau}\right]\tau\right)\right]e^{-A}+\left[\frac{1}{2}(I-\tau A)\left(R^{[\mu/\tau]-1}(\tau A^{1/2})+R^{[\mu/\tau]-1}(-\tau A^{1/2})\right)\right.\right. \\ &\quad \left.+\frac{1}{2i}A^{1/2}\left(R^{[\mu/\tau]-1}(\tau A^{1/2})+R^{[\mu/\tau]-1}(-\tau A^{1/2})\right)\right]\tau^{-1}R^N \\ &\quad -\left[\frac{1}{2}(I-\tau A)\left(R^{[\mu/\tau]-2}(\tau A^{1/2})+R^{[\mu/\tau]-2}(-\tau A^{1/2})\right)\right. \\ &\quad \left.-\frac{1}{2i}A^{1/2}\left(R^{[\mu/\tau]-2}(\tau A^{1/2})+R^{[\mu/\tau]-2}(-\tau A^{1/2})\right)\right]\tau^{-1}R^N\}. \end{aligned}$$

Using the last formulas and the estimates

$$\begin{aligned} \|(I+\alpha Ae^{-A})^{-1}\|_{H\rightarrow H} &\leq M, \quad \|(I+\alpha[Ac(\tau)+As(\tau)]e^{-A})^{-1}\|_{H\rightarrow H} \leq M, \\ \|(I+\alpha\left[Ac\left(\left[\frac{\mu}{\tau}\right]\tau\right)+As\left(\left[\frac{\mu}{\tau}\right]\tau\right)\right]e^{-A})^{-1}\|_{H\rightarrow H} &\leq M, \\ \|AR^N-Ae^{-A}\|_{H\rightarrow H} &\leq M\tau, \\ \|[Ac(\tau)+As(\tau)]e^{-A}+\left[\frac{1}{2}(I-\tau A)\left(R(\tau A^{1/2})+R(-\tau A^{1/2})\right)\right. \\ &\quad \left.+\frac{1}{2i}A^{1/2}\left(R(\tau A^{1/2})+R(-\tau A^{1/2})\right)-(I-\tau A)R^N\right]\|_{H\rightarrow H} \leq M\tau, \\ \|Ac\left(\left[\frac{\mu}{\tau}\right]\tau\right)+As\left(\left[\frac{\mu}{\tau}\right]\tau\right)e^{-A}+\left\{\frac{1}{2}(I-\tau A)\left(R^{[\mu/\tau]-1}(\tau A^{1/2})+R^{[\mu/\tau]-1}(-\tau A^{1/2})\right)\right. \\ &\quad \left.+\frac{1}{2i}A^{1/2}\left(R^{[\mu/\tau]-1}(\tau A^{1/2})+R^{[\mu/\tau]-1}(-\tau A^{1/2})\right)\right\}\tau^{-1}R^N \\ &\quad -\left\{\frac{1}{2}(I-\tau A)\left(R^{[\mu/\tau]-2}(\tau A^{1/2})+R^{[\mu/\tau]-2}(-\tau A^{1/2})\right)\right. \\ &\quad \left.-\frac{1}{2i}A^{1/2}\left(R^{[\mu/\tau]-2}(\tau A^{1/2})+R^{[\mu/\tau]-2}(-\tau A^{1/2})\right)\right\}\tau^{-1}R^N\|_{H\rightarrow H} \leq M\tau, \end{aligned}$$

we can obtain the estimate (8). The proof of these estimates are based on estimates (2), (3), (6) and (7). Lemma 2 is proved.

We can obtain the same results for the solution of the following difference schemes of second order of convergence for approximately solving problem (1):

$$\left\{ \begin{array}{l} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k + \frac{\tau^2}{4}A^2u_{k+1} = f_k, \\ f_k = f(t_k), \quad t_k = k\tau, \quad 1 \leq k \leq N-1, \\ \tau^{-1}(I + \tau^2A)(u_1 - u_0) = Z_1, \\ Z_1 = \frac{\tau}{2}(f(0) - Au_0) + (g(0) - Au_0), \\ \tau^{-1}(u_k - u_{k-1}) + A(I + \frac{\tau}{2}A)u_k = (I + \frac{\tau}{2}A)g_k, \\ g_k = g(t_k - \frac{\tau}{2}), \quad t_k = k\tau, \quad -(N-1) \leq k \leq 0, \\ u_{-N} = \alpha \left[\frac{u_{[\frac{\mu}{\tau}] - u_{[\frac{\mu}{\tau}] - 1}}{\tau} + (\mu - [\frac{\mu}{\tau}]\tau)(f_{[\frac{\mu}{\tau}]} - Au_{[\frac{\mu}{\tau}]}) \right] + \varphi; \end{array} \right. \quad (9)$$

$$\left\{ \begin{array}{l} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{1}{2}Au_k + \frac{1}{4}A(u_{k+1} + u_{k-1}) = f_k, \\ f_k = f(t_k), \quad t_k = k\tau, \quad 1 \leq k \leq N-1, \\ \tau^{-1}(I + \tau^2A)(u_1 - u_0) = Z_1, \\ Z_1 = \frac{\tau}{2}(f(0) - Au_0) + (g(0) - Au_0), \\ \tau^{-1}(u_k - u_{k-1}) + A(I + \frac{\tau}{2}A)u_k = (I + \frac{\tau}{2}A)g_k, \\ g_k = g(t_k - \frac{\tau}{2}), \quad t_k = k\tau, \quad -(N-1) \leq k \leq 0, \\ u_{-N} = \alpha \left[\frac{u_{[\frac{\mu}{\tau}] - u_{[\frac{\mu}{\tau}] - 1}}{\tau} + (\mu - [\frac{\mu}{\tau}]\tau)(f_{[\frac{\mu}{\tau}]} - Au_{[\frac{\mu}{\tau}]}) \right] + \varphi. \end{array} \right. \quad (10)$$

3 Applications

First, for an application of Theorems 1 and 2 we consider the mixed problem for hyperbolic-parabolic equation

$$\left\{ \begin{array}{l} v_{yy} - (a(x)v_x)_x + \delta v = f(y, x), \quad 0 < y < 1, \quad 0 < x < 1, \\ v_y - (a(x)v_x)_x + \delta v = g(y, x), \quad -1 < y < 0, \quad 0 < x < 1, \\ v(-1, x) = v_y(1, x) + \varphi(x), \quad 0 \leq x \leq 1, \\ v(y, 0) = v(y, 1), \quad v_x(y, 0) = v_x(y, 1), \quad -1 \leq y \leq 1, \\ v(0+, x) = v(0-, x), \quad v_y(0+, x) = v_y(0-, x), \quad 0 \leq x \leq 1. \end{array} \right. \quad (11)$$

The problem (11) has a unique smooth solution $v(y, x)$ for the smooth $a(x) > 0$ ($x \in (0, 1)$), $\varphi(x)$ ($x \in [0, 1]$) and $f(y, x)$ ($y \in [0, 1]$, $x \in [0, 1]$), $g(y, x)$ ($y \in [-1, 0]$, $x \in [0, 1]$) functions and $\delta = \text{const} > 0$. This allows us to reduce the mixed problem (11) to the nonlocal boundary value problem (1) in a Hilbert space H with a self-adjoint positive definite operator A defined by (11). Let us give a number of corollaries of the abstract Theorem 1.

Theorem 3 *The solutions of the nonlocal boundary value problem (11) satisfy the stability estimates*

$$\begin{aligned} \max_{-1 \leq y \leq 1} \|v(y)\|_{L_2[0,1]} &\leq M \left[\|f(0)\|_{L_2[0,1]} + \max_{0 \leq y \leq 1} \|f_y(y)\|_{L_2[0,1]} \right. \\ &\quad \left. + \|g(0)\|_{L_2[0,1]} + \max_{-1 \leq y \leq 0} \|g_y(y)\|_{L_2[0,1]} + \|\varphi\|_{L_2[0,1]} \right], \\ \max_{-1 \leq y \leq 1} \|v(y)\|_{W_2^1[0,1]} &\leq M \left[\|f(0)\|_{L_2[0,1]} + \max_{0 \leq y \leq 1} \|f_y(y)\|_{L_2[0,1]} \right. \\ &\quad \left. + \|g(0)\|_{L_2[0,1]} + \max_{-1 \leq y \leq 0} \|g_y(y)\|_{L_2[0,1]} + \|\varphi\|_{W_2^1[0,1]} \right], \\ \max_{-1 \leq y \leq 1} \|v(y)\|_{W_2^2[0,1]} + \max_{-1 \leq y \leq 0} \|v_y(y)\|_{L_2[0,1]} + \max_{0 \leq y \leq 1} \|v_{yy}(y)\|_{L_2[0,1]} \\ &\leq M \left[\|\varphi\|_{W_2^1[0,1]} + \|f(0)\|_{W_2^1[0,1]} + \|f_y(0)\|_{L_2[0,1]} + \max_{0 \leq y \leq 1} \|f_{yy}(y)\|_{L_2[0,1]} \right. \\ &\quad \left. + \|g(0)\|_{W_2^1[0,1]} + \|g_y(0)\|_{L_2[0,1]} + \max_{-1 \leq y \leq 0} \|g_{yy}(y)\|_{L_2[0,1]} \right], \end{aligned}$$

where M does not depend on $f(y, x)$ ($y \in [0, 1]$, $x \in [0, 1]$), $g(y, x)$ ($y \in [-1, 0]$, $x \in [0, 1]$) and $\varphi(x)$ ($x \in [0, 1]$).

The proof of this theorem is based on the abstract Theorem 1 and the symmetry properties of the space operator generated by the problem (11).

Now, the abstract Theorem 2 is applied in the investigation of difference scheme of the first order of accuracy with respect to one variable for approximate solutions of the mixed boundary value problem (11). The discretization of problem (11) is carried out in two steps. In the first step let us define the grid space

$$[0, 1]_h = \{x : x_n = nh, 0 \leq n \leq M, Mh = 1\}.$$

We introduce the Hilbert space $L_{2h} = L_2([0, 1]_h)$ of the grid functions $\varphi^h(x)$ defined on $[0, 1]_h$, equipped with the norm

$$\|\varphi^h\|_{L_{2h}} = \left(\sum_{n=1}^{M-1} |\varphi^h(x)|^2 h \right)^{1/2}.$$

To the differential operator A generated by the problem (11) we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \left\{ -(a(x)\varphi_x^-)_{x,n} + \delta\varphi_n \right\}_1^{M-1}, \quad (12)$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi^n\}_0^M$ satisfying the conditions $\varphi^0 = \varphi^M$, $\varphi^1 - \varphi^0 = \varphi^M - \varphi^{M-1}$. It is known that A_h^x is a self-adjoint positive definite operator in L_{2h} . With the help of A_h^x we arrive at the nonlocal boundary-value problem

$$\begin{cases} \frac{d^2 v^h(t,x)}{dy^2} + A_h^x v^h(y,x) = f^h(y,x), & 0 \leq y \leq 1, x \in [0, 1]_h, \\ \frac{dv^h(t,x)}{dy} + A_h^x v^h(y,x) = f^h(y,x), & -1 \leq y \leq 0, x \in [0, 1]_h, \\ v^h(-1, x) = \frac{dv^h(1,x)}{dy} + \varphi^h(x), & x \in [0, 1]_h, \\ v^h(0+, x) = v^h(0-, x), \frac{dv^h(0+,x)}{dy} = \frac{dv^h(0-,x)}{dy}, & x \in [0, 1]_h \end{cases} \quad (13)$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (13) by the difference scheme (5)

$$\begin{cases} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_{k+1}^h = f_k^h(x), & x \in [0, 1]_h, \\ f_{k+1}^h(x) = \{f(y_{k+1}, x_n)\}_1^{M-1}, & y_{k+1} = (k+1)\tau, 1 \leq k \leq N-1, N\tau = 1, \\ \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h = g_k^h(x), & x \in [0, 1]_h, \\ g_k^h(x) = \{g(y_k, x_n)\}_1^{M-1}, & y_k = k\tau, -N+1 \leq k \leq 0, \\ u_0^h(x) = \frac{u_N^h(x) - u_{N-1}^h(x)}{\tau} + \varphi^h(x), & x \in [0, 1]_h, \\ \frac{u_1^h(x) - u_0^h(x)}{\tau} = -A_h^x u_0^h(x) + g_0^h(x), & g_0^h(x) = g^h(0, x), x \in [0, 1]_h. \end{cases} \quad (14)$$

Theorem 4 *Let τ and h be sufficiently small numbers. Then the solutions of difference scheme (14) satisfy the following stability estimates:*

$$\begin{aligned} & \max_{-N \leq k \leq N} \|u_k^h\|_{L_{2h}} \leq M_1 \left[\|f_1^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|(f_k^h - f_{k-1}^h)\tau^{-1}\|_{L_{2h}} \right. \\ & \quad \left. + \|g_0^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|(g_k^h - g_{k-1}^h)\tau^{-1}\|_{L_{2h}} + \|\varphi^h\|_{L_{2h}} \right], \\ & \max_{-N+1 \leq k \leq N} \|\tau^{-1}(u_k^h - u_{k-1}^h)\|_{L_{2h}} + \max_{-N \leq k \leq N} \|(u_k^h)_x\|_{L_{2h}} \\ & \leq M_1 \left[\|f_1^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|(f_k^h - f_{k-1}^h)\tau^{-1}\|_{L_{2h}} \right. \\ & \quad \left. + \|g_0^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|(g_k^h - g_{k-1}^h)\tau^{-1}\|_{L_{2h}} + \|\varphi_x^h\|_{L_{2h}} \right], \\ & \max_{1 \leq k \leq N-1} \|\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}} \end{aligned}$$

$$\begin{aligned}
 & + \max_{-N \leq k \leq N} \| (u_{-}^k)_x \|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \| \tau^{-1}(u_k^h - u_{k-1}^h) \|_{L_{2h}} \\
 & \leq M_1 \left[\| f_{1x}^h \|_{L_{2h}} + \| \tau^{-1}(f_2^h - f_1^h) \|_{L_{2h}} \right. \\
 & + \max_{2 \leq k \leq N-1} \| \tau^{-2}(f_{k+1}^h - 2f_k^h + f_{k-1}^h) \|_{L_{2h}} + \| g_{0x}^h \|_{L_{2h}} + \| \tau^{-1}(g_0^h - g_{-1}^h) \|_{L_{2h}} \\
 & \left. + \max_{-N+1 \leq k \leq -1} \| \tau^{-2}(g_{k+1}^h - 2g_k^h + g_{k-1}^h) \|_{L_{2h}} + \| (\varphi_{-}^h)_x \|_{L_{2h}} \right].
 \end{aligned}$$

Here M_1 does not depend on $\tau, h, \varphi^h(x)$ and $f_k^h(x), 1 \leq k \leq N-1, g_k^h, -N+1 \leq k \leq 0$.

The proof of Theorem 4 is based on the abstract Theorem 2, and the symmetry properties of the difference operator A_h^x defined by the formula (12).

Second, let Ω be the unit open cube in the n -dimensional Euclidean space \mathbb{R}^n ($0 < x_k < 1, 1 \leq k \leq n$) with boundary $S, \bar{\Omega} = \Omega \cup S$. In $[0, 1] \times \Omega$ we consider the mixed boundary value problem for the multidimensional hyperbolic-parabolic equation

$$\begin{cases}
 v_{yy} - \sum_{r=1}^n (a_r(x)v_{x_r})_{x_r} = f(y, x), & 0 \leq y \leq 1, x = (x_1, \dots, x_n) \in \Omega, \\
 v_y - \sum_{r=1}^n (a_r(x)v_{x_r})_{x_r} = g(y, x), & -1 \leq y \leq 0, x = (x_1, \dots, x_n) \in \Omega, \\
 v(-1, x) = v_y(1, x) + \varphi(x), & x \in \bar{\Omega}, \\
 u(y, x) = 0, & x \in S, -1 \leq y \leq 1,
 \end{cases} \quad (15)$$

where $a_r(x)$ ($x \in \Omega$), $\varphi(x)$ ($x \in \bar{\Omega}$) and $f(y, x)$ ($y \in (0, 1), x \in \Omega$), $g(y, x)$ ($y \in (-1, 0), x \in \Omega$) are given smooth functions and $a_r(x) > 0$.

We introduce the Hilbert space $L_2(\bar{\Omega})$ – the space of the all integrable functions defined on $\bar{\Omega}$, equipped with the norm

$$\| f \|_{L_2(\bar{\Omega})} = \left\{ \int \dots \int_{x \in \bar{\Omega}} |f(x)|^2 dx_1 \dots dx_n \right\}^{1/2}.$$

The problem (15) has a unique smooth solution $v(y, x)$ for the smooth $a_r(x) > 0$ and $f(y, x), g(y, x)$ functions. This allows us to reduce the mixed problem (15) to the nonlocal boundary value problem (1) in Hilbert space H with a self-adjoint positive definite operator A defined by (15). Let us give a number of corollaries of the abstract Theorem 1.

Theorem 5 *The solutions of the nonlocal boundary value problem (15) satisfy the stability estimates*

$$\begin{aligned}
& \max_{-1 \leq y \leq 1} \|v(y)\|_{L_2(\bar{\Omega})} \leq M \left[\|f(0)\|_{L_2(\bar{\Omega})} + \max_{0 \leq y \leq 1} \|f_y(y)\|_{L_2(\bar{\Omega})} \right. \\
& \quad \left. + \|g(0)\|_{L_2(\bar{\Omega})} + \max_{-1 \leq y \leq 0} \|g_y(y)\|_{L_2(\bar{\Omega})} + \|\varphi\|_{L_2(\bar{\Omega})} \right], \\
& \max_{-1 \leq y \leq 1} \|v(y)\|_{W_2^1(\bar{\Omega})} \leq M \left[\|f(0)\|_{L_2(\bar{\Omega})} + \max_{0 \leq y \leq 1} \|f_y(y)\|_{L_2(\bar{\Omega})} \right. \\
& \quad \left. + \|g(0)\|_{L_2(\bar{\Omega})} + \max_{-1 \leq y \leq 0} \|g_y(y)\|_{L_2(\bar{\Omega})} + \|\varphi\|_{W_2^1(\bar{\Omega})} \right], \\
& \max_{-1 \leq y \leq 1} \|v(y)\|_{W_2^2(\bar{\Omega})} + \max_{-1 \leq y \leq 0} \|v_y(y)\|_{L_2(\bar{\Omega})} + \max_{0 \leq y \leq 1} \|v_{yy}(y)\|_{L_2(\bar{\Omega})} \\
& \leq M \left[\|\varphi\|_{W_2^1(\bar{\Omega})} + \|f(0)\|_{W_2^1(\bar{\Omega})} + \|f_y(0)\|_{L_2(\bar{\Omega})} + \max_{0 \leq y \leq 1} \|f_{yy}(y)\|_{L_2(\bar{\Omega})} \right. \\
& \quad \left. + \|g(0)\|_{W_2^1(\bar{\Omega})} + \|g_y(0)\|_{L_2(\bar{\Omega})} + \max_{-1 \leq y \leq 0} \|g_{yy}(y)\|_{L_2(\bar{\Omega})} \right],
\end{aligned}$$

where M does not depend on $f(y, x)$ ($y \in [0, 1]$, $x \in [0, 1]$), $g(y, x)$ ($y \in [-1, 0]$, $x \in [0, 1]$) and $\varphi(x)$ ($x \in [0, 1]$).

The proof of this theorem is based on the abstract Theorem 1 and the symmetry properties of the space operator generated by the problem (15).

Now, the abstract Theorem 2 is applied in the investigation of difference schemes of the second order of accuracy with respect to one variable for approximate solutions of the mixed boundary value problem (15). The discretization of problem (15) is carried out in two steps. In the first step let us define the grid sets

$$\begin{aligned}
\tilde{\Omega}_h &= \{x = x_m = (h_1 m_1, \dots, h_n m_n), m = (m_1, \dots, m_n), \\
& 0 \leq m_r \leq N_r, h_r N_r = L, r = 1, \dots, n\}, \\
\Omega_h &= \tilde{\Omega}_h \cap \Omega, S_h = \tilde{\Omega}_h \cap S.
\end{aligned}$$

We introduce the Banach space $L_{2h} = L_2(\tilde{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \dots, h_n m_n)\}$ defined on $\tilde{\Omega}_h$, equipped with the norm

$$\|\varphi^h\|_{L_2(\tilde{\Omega}_h)} = \left(\sum_{x \in \tilde{\Omega}_h} |\varphi^h(x)|^2 h_1 \cdots h_n \right)^{1/2}.$$

To the differential operator A generated by the problem (15) we assign the difference operator A_h^x by the formula

$$A_h^x u_x^h = - \sum_{r=1}^n (a_r(x) u_{x_r}^h)_{x_r, j_r} \quad (16)$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\tilde{\Omega}_h)$. With the help of A_h^x we arrive at the nonlocal boundary-value problem

$$\begin{cases} \frac{d^2 v^h(y, x)}{dy^2} + A_h^x v^h(y, x) = f^h(y, x), & 0 \leq y \leq 1, x \in \tilde{\Omega}_h, \\ \frac{dv^h(y, x)}{dy} + A_h^x v^h(y, x) = f^h(y, x), & -1 \leq y \leq 0, x \in \tilde{\Omega}_h, \\ v^h(-1, x) = \frac{dv^h(1, x)}{dy} + \varphi^h(x), & x \in \tilde{\Omega}_h, \\ v^h(0+, x) = v^h(0-, x), \quad \frac{dv^h(0+, x)}{dy} = \frac{dv^h(0-, x)}{dy}, & x \in \tilde{\Omega}_h \end{cases} \quad (17)$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (13) by the difference scheme (5)

$$\begin{cases} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_{k+1}^h = f_k^h(x), & x \in \tilde{\Omega}_h, \\ f_{k+1}^h(x) = \{f(y_{k+1}, x_n)\}_1^{M-1}, & y_{k+1} = (k+1)\tau, 1 \leq k \leq N-1, N\tau = 1, \\ \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h = g_k^h(x), & x \in \tilde{\Omega}_h, \\ g_k^h(x) = \{g(y_k, x_n)\}_1^{M-1}, & y_k = k\tau, -N+1 \leq k \leq -1, \\ u_0^h(x) = \frac{u_N^h(x) - u_{N-1}^h(x)}{\tau} + \varphi^h(x), & x \in \tilde{\Omega}_h, \\ \frac{u_1^h(x) - u_0^h(x)}{\tau} = -A_h^x u_0^h(x) + g_0^h(x), & g_0^h(x) = g^h(0, x), x \in \tilde{\Omega}_h. \end{cases} \quad (18)$$

Theorem 6 *Let τ and $|h|$ be sufficiently small numbers. Then the solutions of the difference scheme (18) satisfy the following stability estimates:*

$$\begin{aligned} & \max_{-N \leq k \leq N} \|u_k^h\|_{L_{2h}} \leq M_1 \left[\|f_1^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|(f_k^h - f_{k-1}^h)\tau^{-1}\|_{L_{2h}} \right. \\ & \quad \left. + \|g_0^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|(g_k^h - g_{k-1}^h)\tau^{-1}\|_{L_{2h}} + \|\varphi^h\|_{L_{2h}} \right], \\ & \max_{-N+1 \leq k \leq N} \|\tau^{-1}(u_k^h - u_{k-1}^h)\|_{L_{2h}} + \max_{-N \leq k \leq N} \sum_{r=1}^n \|(u_k^h)_{x_r, j_r}\|_{L_{2h}} \\ & \leq M_1 \left[\|f_1^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|(f_k^h - f_{k-1}^h)\tau^{-1}\|_{L_{2h}} \right] \end{aligned}$$

$$\begin{aligned}
& + \|g_0^h\|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|(g_k^h - g_{k-1}^h)\tau^{-1}\|_{L_{2h}} + \sum_{r=1}^n \|(\varphi^h)_{\bar{x}_r, j_r}\|_{L_{2h}}, \\
& \quad \max_{1 \leq k \leq N-1} \|\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}} \\
& + \max_{-N \leq k \leq N} \sum_{r=1}^n \| (u_k^h)_{\bar{x}_r, j_r} \|_{L_{2h}} + \max_{-N+1 \leq k \leq 0} \|\tau^{-1}(u_k^h - u_{k-1}^h)\|_{L_{2h}} \\
& \leq M_1 \left[\sum_{r=1}^n \| (f_1^h)_{\bar{x}_r, j_r} \|_{L_{2h}} + \|\tau^{-1}(f_2^h - f_1^h)\|_{L_{2h}} \right. \\
& + \max_{2 \leq k \leq N-1} \|\tau^{-2}(f_{k+1}^h - 2f_k^h + f_{k-1}^h)\|_{L_{2h}} + \sum_{r=1}^n \| (g_0^h)_{\bar{x}_r, j_r} \|_{L_{2h}} + \|\tau^{-1}(g_0^h - g_{-1}^h)\|_{L_{2h}} \\
& \quad \left. + \max_{-N+1 \leq k \leq -1} \|\tau^{-2}(g_{k+1}^h - 2g_k^h + g_{k-1}^h)\|_{L_{2h}} + \sum_{r=1}^n \| (\varphi^h)_{\bar{x}_r, j_r} \|_{L_{2h}} \right].
\end{aligned}$$

Here M_1 does not depend on τ , h , $\varphi^h(x)$ and $f_k^h(x)$, $1 \leq k \leq N-1$, g_k^h , $-N+1 \leq k \leq 0$.

The proof of Theorem 6 is based on the abstract Theorem 2, and the symmetry properties of the difference operator A_h^x defined by the formula (16).

Note that applying the second order of accuracy difference schemes (9) and (10), we can construct the second order of accuracy difference schemes with respect to one variable for approximate solutions of the boundary value problems (11) and (15). This approach permits us to obtain stability estimates for the solutions of these difference schemes.

4 Numerical analysis

We consider the nonlocal boundary value problem

$$\left\{ \begin{array}{l} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = (-2 + \pi^2(1-t^2)) \sin \pi x, \quad 0 < t, x < 1, \\ \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = (-2t + \pi^2(1-t^2)) \sin \pi x, \\ \quad -1 < t < 0, \quad 0 < x < 1, \\ u(0+, x) = u(0-, x), \quad u_t(0+, x) = u_t(0-, x), \quad 0 \leq x \leq 1, \\ u(-1, x) = u_t(1, x) + 2 \sin \pi x, \quad 0 \leq x \leq 1, \\ u(t, 0) = u(t, 1) = 0, \quad 0 \leq t \leq 1, \end{array} \right. \quad (19)$$

for a hyperbolic-parabolic equation with the exact solution

$$u(t, x) = (1 - t^2) \sin \pi x.$$

For approximate solutions of the nonlocal boundary value problem (19), we will use the first order of accuracy and a second order of accuracy difference schemes. We have the second order or fourth order difference equations with respect to n with matrix coefficients. To solve the difference equations we have applied a procedure of modified Gauss elimination method for difference equations with respect to n with matrix coefficients. The results of numerical experiments permit us to show that the second order of accuracy difference schemes were more accurate in comparison with the first order of accuracy difference scheme.

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