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Positivity of Difference Operators Generated by the Nonlocal Boundary Conditions

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Abstract

In the present paper a second order difference operator A_h^x of a second order approximation of the differential operator A^x defined by the formula

$$A^x u = -a(x)\frac{d^2u}{dx^2} + \delta u$$

with domain $\mathcal{D}(A^x) = \{u \in C^{(2)}[0,1] : \ u(0) = u(1), u'(0) = u'(1)\}$ is presented. Here $a(x)$ is a smooth function defined on the segment $[0,1]$ and $a(x) > 0,\ \delta > 0$. The positivity of A_h^x in C_h and Hölder spaces is established.

Key words: Positive operator; Nonlocal boundary conditions; Hölder space.

1 Introduction

Let us consider a differential operator A^x defined by the formula

$$A^x u = -a(x)\frac{d^2u}{dx^2} + \delta u, \tag{1}$$

with domain $D(A^x) = \{u \in C^{(2)}[0,1] : \ u(0) = u(1), u'(0) = u'(1)\}$. Here $a(x)$ is a smooth function defined on the segment $[0,1]$ and $a(x) > 0,\ \delta > 0$.

In [4, 5] the difference operator A_h^x of a first-order of approximation for the differential operator (1) was considered. The positivity of this operator in C_h and Hölder spaces was established.

Let us define the grid space $[0,1]_h = \{x_k = kh,\ 0 \le k \le N,\ Nh = 1\}$, N is a fixed positive integer. The number h is called the step of the grid space. A function $\varphi^h = \{\varphi_k\}_0^N$ defined on $[0,1]_h$ will be called a grid function. To the operator A^x defined by the formula (1) we assign the difference operator A_h^x of a second order of approximation defined by the formula

$$A_h^x u^h = \left\{ -a(x_k)\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \delta u_k \right\}_1^{N-1},\ u_h = \{u_k\}_0^N, \tag{2}$$

which acts on grid functions defined on $[0,1]_h$ with $u_0 = u_N$ and $-u_2 + 4u_1 - 3u_0 = u_{N-2} - 4u_{N-1} + 3u_N$.

We denote by $C_h = C[0,1]_h$ and $C_h^\alpha = C^\alpha[0,1]_h$ the Banach spaces of all grid functions $v^h = \{v_k\}_1^{N-1}$ defined on $[0,1]_h$ and equipped with the norms

$$\left\| v^h \right\|_{C_h} = \max_{1 \le k \le N-1} |v_k|,$$

$$\left\| v^h \right\|_{C_h^\alpha} = \max_{1 \le k \le N-1} |v_k| + \max_{1 \le k < k+r \le N-1} \frac{|v_{k+r} - v_k|}{(r\tau)^\alpha}.$$

In the present paper we will investigate the resolvent of the operator $-A_h^x$, *i.e.*, in solving the equation

$$A_h^x u^h + \lambda u^h = f^h \tag{3}$$

or

$$-a_k \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \delta u_k + \lambda u_k = f_k,$$

$$a_k = a(x_k),\ f_k = f(x_k),\ 1 \le k \le N-1,$$

$$u_0 = u_N,\ \ - u_2 + 4u_1 - 3u_0 = u_{N-2} - 4u_{N-1} + 3u_N.$$

The positivity of the difference operator A_h^x defined by the formula (2) in C_h and the Hölder spaces C_h^α is established.

2 Green's function

In this section we will study the strong positivity in C_h of the operator A_h^x defined by formula (2) in the case $a(x) \equiv 1$.

Lemma 1 *Let $\lambda \geq 0$. Then the equation (3) is uniquely solvable, and the following formula holds*

$$u^h = (A_h^x + \lambda)^{-1} f^h = \left\{ \sum_{j=1}^{N-1} J(k,j;\lambda+\delta) f_j h \right\}_0^N , \tag{4}$$

where

$$J(k,1;\lambda+\delta) = J(k,N-1;\lambda+\delta)$$
$$= \frac{1+\mu h}{2+3\mu h} \frac{(R^{N-3}+1)(4R-1)}{2\mu}(I - \frac{2-\mu h}{2+3\mu h} R^{N-2})^{-1}$$

for $k=0$ and $k=N$;

$$J(k,j;\lambda+\delta) = -\frac{1+\mu h}{2+3\mu h} \frac{(R^2-4R+1)(R^{j-2}+R^{N-j-2})}{2\mu}(I - \frac{2-\mu h}{2+3\mu h} R^{N-2})^{-1}$$

for $2 \leq j \leq N-2$ and $k=0$, $k=N$;

$$J(k,1;\lambda+\delta) = \frac{1+\mu h}{2+3\mu h} \frac{1+\mu h}{2+\mu h}(2\mu)^{-1}\{R^{k-1}(2(R+3)+R^2(R-3))$$
$$+R^{N-k}(4-R)(1+R) + R^{N+k-3}(1-4R)(1+R)$$
$$+R^{2N-k-3}(3R-1-2R^2(3R+1))\}(1-R^N)^{-1}(I - \frac{2-\mu h}{2+3\mu h} R^{N-2})^{-1},$$

$$J(k,N-1;\lambda+\delta) = -\frac{1+\mu h}{2+3\mu h} \frac{1+\mu h}{2+\mu h}(2\mu)^{-1}\{R^k(R-4)(R+1)$$
$$+R^{N-k-1}(-2(R+3)+R^2(3-R)) + R^{N+k-3}(1-3R+2R^2(3R+1))$$
$$+R^{2N-k-3}(4R-1)(R+1)\}(1-R^N)^{-1}(I - \frac{2-\mu h}{2+3\mu h} R^{N-2})^{-1},$$

$$J(k,j;\lambda+\delta) = \frac{1+\mu h}{2+3\mu h} \frac{1+\mu h}{2+\mu h}(2\mu)^{-1}\{(R-1)^3(R^{j+k-2}+R^{2N-2-j-k})$$
$$+(-1+3R+R^2(3-R))(R^{N-k+j-2}+R^{N+k-j-2}) + 2(1-3R)(R^{2N-2+j-k}$$
$$+R^{2N-2-j+k}) + 2R^{|j-k|}(R^N-1)\left(R-3+R^{N-2}(-1+3R)\right)\}$$
$$\times(1-R^N)^{-1}(1 - \frac{2-\mu h}{2+3\mu h} R^{N-2})^{-1}$$

for $2 \leq j \leq N-2$ *and* $1 \leq k \leq N-1$. *Here*

$$R = (1+\mu h)^{-1}, \quad \mu = \frac{1}{2}\left(h(\lambda+\delta) + \sqrt{(\lambda+\delta)(4+h^2(\lambda+\delta))}\right).$$

Proof. We see that the problem (3) can be obviously rewritten as the equivalent nonlocal boundary value problem for the first order linear difference equations

$$\begin{cases} \frac{u_k - u_{k-1}}{h} + \mu u_k = z_k, \qquad 1 \leq k \leq N, \\ u_0 = u_N, \quad -u_2 + 4u_1 - 3u_0 = u_{N-2} - 4u_{N-1} + 3u_N, \\ -\frac{z_{k+1} - z_k}{h} + \mu z_k = (1+\mu h) f_k, \qquad 1 \leq k \leq N-1. \end{cases}$$

From that there follows the system of recursion formulas

$$\begin{cases} u_k = Ru_{k-1} + hRz_k, \quad 1 \leq k \leq N, \\ z_k = Rz_{k-1} + hf_k, \qquad 1 \leq k \leq N-1. \end{cases}$$

Hence

$$\begin{cases} u_k = R^k u_0 + \sum_{i=1}^{k} R^{k-i+1} h z_i, \ 1 \leq k \leq N, \\ z_k = R^{N-k} z_N + \sum_{j=k}^{N-1} R^{j-k} h f_j, \ \ 1 \leq k \leq N-1. \end{cases}$$

From the first formula and the condition $u_N = u_0$ it follows that

$$u_N = R^N u_0 + \sum_{i=1}^{N} R^{N-i+1} h z_i.$$

Since $1 - R^N \neq 0$, it follows that

$$u_N = u_0 = \frac{1}{1-R^N} \sum_{i=1}^{N} R^{N-i+1} h z_i = \frac{1}{1-R^N}\left\{ hRz_N + \sum_{i=1}^{N} R^{N-i+1} h z_i \right\}$$

$$= \frac{1}{1-R^N}\left\{ \left(hR + \sum_{i=1}^{N} R^{2N-2i+1} h \right) z_N + \sum_{i=1}^{N-1} hR^{N-i+1} \sum_{j=i}^{N-1} R^{j-i} h f_j \right\}$$

$$= \frac{1}{1-R^N}\left\{ \frac{\left(R - R^{2N+1}\right)}{1-R^2} h z_N + \sum_{j=1}^{N-1} h^2 \sum_{i=1}^{j} R^{N+j-2i+1} f_j \right\}$$

$$= \frac{1}{(1-R^N)(1-R^2)}\left[R(1-R^{2N})hz_N + \sum_{j=1}^{N-1} h^2\left[R^{N-j+1}-R^{N+j+1}\right]f_j\right],$$

and for $k,\ 1 \le k \le N-1$,

$$u_k = \frac{1}{1-R^N}\left\{hR^{k+1}z_N + \sum_{i=1}^{N-1} R^{k+N-i+1}hz_i\right\} + \sum_{i=1}^{k} R^{k-i+1}hz_i$$

$$= \frac{R^k}{(1-R^N)}\left\{\frac{(R-R^{2N+1})}{1-R^2}hz_N + \sum_{j=1}^{N-1} h^2\left[R^{N-j+1}-R^{N+j+1}\right]f_j\right\}$$

$$+\sum_{i=1}^{k} R^{N+k-2i+1}hz_N + \sum_{i=1}^{k}\sum_{j=i}^{N-1} h^2 R^{k+j-2i+1}f_j$$

$$= \frac{1}{1-R^2}\left[R^{k+1}+R^{N-k+1}\right]hz_N$$

$$+\frac{1}{(1-R^N)(1-R^{N-1})}\sum_{j=1}^{N-1} h^2\left[R^{N-j+1}-R^{N+j+1}\right]f_j$$

$$+\sum_{j=1}^{k} h^2 \sum_{i=1}^{j} R^{k+j-2i+1}f_j + \sum_{j=k+1}^{N-1} h^2 \sum_{i=1}^{k} R^{k+j-2i+1}f_j$$

$$= \frac{1}{1-R^2}\left[R^{k+1}+R^{N-k+1}\right]hz_N$$

$$+\frac{R^k}{(1-R^N)(1-R^{N-1})}\sum_{j=1}^{N-1} h^2\left[R^{N-j+1}-R^{N+j+1}\right]f_j$$

$$+\frac{1}{1-R^2}\sum_{j=1}^{N-1} h^2(R^{|k-j|+1}-R^{k+j+1})f_j.$$

Now by using the formulas for u_N, u_0, u_k and the condition

$$-u_2+4u_1-3u_0 = u_{N-2}-4u_{N-1}+3u_N$$

we can write

$$u_0+u_N = 2\frac{R+R^{N+1}}{1-R^2}hz_N$$

$$+\frac{2}{(1-R^N)(1-R^2)}\sum_{j=1}^{N-1}h^2\left(R^{N-j+1}-R^{N+j+1}\right)f_j,$$

$$u_1+u_{N-1}=\frac{2}{1-R^2}(R^2+R^N)hz_N$$

$$+\frac{(R+R^{N-1})}{(1-R^N)(1-R^2)}\sum_{j=1}^{N-1}h^2\left(R^{N-j+1}-R^{N+j+1}\right)f_j$$

$$+\frac{1}{1-R^2}\sum_{j=1}^{N-1}h^2\left(R^{|1-j|+1}+R^{|N-1-j|+1}-R^{2+j}-R^{N+j}\right)f_j,$$

$$u_2+u_{N-2}=\frac{2}{1-R^2}(R^3+R^{N-1})hz_N$$

$$+\frac{(R^2+R^{N-2})}{(1-R^N)(1-R^2)}\sum_{j=1}^{N-1}h^2\left(R^{N-j+1}-R^{N+j+1}\right)f_j$$

$$+\frac{1}{1-R^2}\sum_{j=1}^{N-1}h^2\left(R^{|2-j|+1}+R^{|N-2-j|+1}-R^{j+3}-R^{N-1+j}\right)f_j$$

$$=\frac{2}{1-R^2}(R^3+R^{N-1})hz_N$$

$$+\frac{(R^2+R^{N-2})}{(1-R^N)(1-R^2)}\sum_{j=1}^{N-1}h^2\left(R^{N-j+1}-R^{N+j+1}\right)f_j$$

$$+\frac{1}{1-R^2}\sum_{j=1}^{N-1}h^2\left(R^{j-1}+R^{N-1-j}-R^{j+3}-R^{N-1+j}\right)f_j$$

$$+\frac{1}{1-R^2}\left(\left(R^2-1\right)f_1h^2+(R^2-1)f_{N-1}h^2\right).$$

Since

$$u_2+u_{N-2}+3\left(u_0+u_N\right)=4(u_1+u_{N-1}),$$

we have that

$$\frac{2}{1-R^2}(R^3+R^{N-1})hz_N+\frac{(R^2+R^{N-2})}{(1-R^N)(1-R^2)}$$

$$\times\sum_{j=1}^{N-1}h^2\left(R^{N-j+1}-R^{N+j+1}\right)f_j$$

$$+\frac{1}{1-R^2}\sum_{j=1}^{N-1} h^2\left(R^{j-1}+R^{N-1-j}-R^{j+3}-R^{N-1+j}\right)f_j$$

$$-h^2(f_1+f_{N-1})+6\frac{R+R^{N+1}}{1-R^2}hz_N$$

$$+\frac{6}{(1-R^N)(1-R^2)}\sum_{j=1}^{N-1} h^2\left(R^{N-j+1}-R^{N+j+1}\right)f_j$$

$$=\frac{8}{1-R^2}(R^2+R^N)hz_N+\frac{4(R+R^{N-1})}{(1-R^N)(1-R^2)}$$

$$\times\sum_{j=1}^{N-1} h^2(R^{N-j+1}-R^{N+j+1})f_j$$

$$+\frac{4}{1-R^2}\sum_{j=1}^{N-1} h^2(R^j+R^{N-j}-R^{j+2}-R^{N+j})f_j.$$

Hence from here z_N can be found as

$$z_N=\frac{-hR^2(-1+R^2)(-1+R^N)(f_1+f_{N-1})}{2(-1+R)R(-1+R^N)(-3R^2+R^3-R^N+3R^{N+1})}$$

$$+\frac{(6R^2-4R^3+R^4+R^N-4R^{1+N})\sum_{j=1}^{N-1} h(R^{N-j+1}-R^{N+j+1})f_j}{2(-1+R)R(-1+R^N)(-3R^2+R^3-R^N+3R^{N+1})}$$

$$-\frac{R^2(-1+R^N)\sum_{j=1}^{N-1} h\left(R^{j-1}+R^{N-1-j}-R^{j+3}-R^{N-1+j}\right)f_j}{2(-1+R)R(-1+R^N)(-3R^2+R^3-R^N+3R^{N+1})}$$

$$-\frac{4R^2(-1+R^N)\sum_{j=1}^{N-1} h(R^j+R^{N-j}-R^{j+2}-R^{N+j})f_j}{2(-1+R)R(-1+R^N)(-3R^2+R^3-R^N+3R^{N+1})}$$

Now using the formulas for z_N and $u_0=u_N$ we obtain

$$u_N=u_0=\frac{h^2(R^N+R^3)(4R-1)(f_1+f_{N-1})}{2(R-1)(-3R^2+R^3-R^N+3R^{N+1})}$$

$$-\sum_{j=2}^{N-2}\frac{h^2R^{1-j}(R^2-4R+1)(R^{2j}+R^N)}{2(R-1)(-3R^2+R^3-R^N+3R^{N+1})}f_j$$

$$= \frac{1+\mu h}{2+3\mu h}\frac{(R^{N-3}+1)(4R-1)}{2\mu}(I-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}(f_1+f_{N-1})$$

$$-\sum_{j=2}^{N-2}\frac{1+\mu h}{2+3\mu h}\frac{(R^2-4R+1)(R^{j-2}+R^{N-j-2})}{2\mu}(I-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}f_jh.$$

The formula for u_k in the case $k = 0$ and $k = N$ is proved. Now, consider $1 \leq k \leq N-1$. We have that

$$u_k = \frac{1}{1-R^2}\left[R^{k+1}+R^{N-k+1}\right]hz_N$$

$$+\frac{R^k}{(1-R^N)(1-R^{N-1})}\sum_{j=1}^{N-1}h^2\left[R^{N-j+1}-R^{N+j+1}\right]f_j$$

$$+\frac{1}{1-R^2}\sum_{j=1}^{N-1}h^2(R^{|k-j|+1}-R^{k+j+1})f_j$$

$$=(-h^2(-1+R)(R^k(3R^5-2R^4-R^6-6R^3+12R^{N+2}-R^{N+1}+4R^{N+3}$$

$$-4R^{N+4})+R^{N-k}(R^{N+1}+2R^{N+3}-4R^4-3R^5+R^6-3R^{N+2}+6R^{N+4}))f_1$$

$$+h^2(-1+R)(R^k(3R^5+R^4-3R^6-R^{N+1}+3R^{N+2}-2R^{N+3}-6R^{N+4}$$

$$+6R^{2N+2})+R^{N-k}(R^{N+1}-9R^{N+2}-4R^{N+3}+2R^4+6R^3+R^6-3R^5))f_{N-1})$$

$$\times(2(R-1)R(1-R^2)\left(-1+R^N\right)(-3R^2+R^3-R^N+3R^{N+1}))^{-1}$$

$$+\sum_{j=2}^{N-2}h^2(R^{2+j+k}(R-1)(-1+3R-3R^2+R^3+6R^{N+1})$$

$$-R^{N+2+j-k}(1+4R-4R^3+R^4-8R^{N+1}+6R^{N+2})-R^{N+2+k-j}(1+3R+3R^2$$

$$-6R^3+3R^4-8R^{N+1}+6R^{N+2})-R^{2N+2-j-k}(R-1)^4$$

$$+R^{|j-k|}(2(R-1)R^2(3R^2-R^3+R^N-R^{2N}-6R^{N+1}-3R^{N+2}+R^{N+3}$$

$$+3R^{2N+1}))f_j(\ 2(R-1)R(1-R^2)\left(-1+R^N\right)(-3R^2+R^3-R^N+3R^{N+1}))^{-1}$$

$$=\frac{1+\mu h}{2+3\mu h}\frac{1+\mu h}{2+\mu h}(2\mu)^{-1}\{R^{k-1}(2(R+3)+R^2(R-3))$$

$$+R^{N-k}(4-R)(1+R)+R^{N+k-3}(1-4R)(1+R)$$

$$+R^{2N-k-3}(3R-1-2R^2(3R+1))\}(1-R^N)^{-1}(I-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}f_1$$

$$-\frac{1+\mu h}{2+3\mu h}\frac{1+\mu h}{2+\mu h}(2\mu)^{-1}\{R^k(R-4)(R+1)$$

$$+R^{N-k-1}(-2(R+3)+R^2(3-R))+R^{N+k-3}(1-3R+2R^2(3R+1))$$

$$+R^{2N-k-3}(4R-1)(R+1)\}(1-R^N)^{-1}(I-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}f_{N-1}$$

$$+\frac{1+\mu h}{2+3\mu h}\frac{1+\mu h}{2+\mu h}(2\mu)^{-1}\sum_{j=2}^{N-2}\{(R-1)^3(R^{j+k-2}+R^{2N-2-j-k})$$

$$+(-1+3R+R^2(3-R))(R^{N-k+j-2}+R^{N+k-j-2})+2(1-3R)(R^{2N-2+j-k}$$

$$+R^{2N-2-j+k})+2R^{|j-k|}(R^N-1)\left(R-3+R^{N-2}(-1+3R)\right)\}$$

$$\times(1-R^N)^{-1}(1-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}\ f_jh.$$

Lemma 1 is proved.

The grid function $J(k,j;\lambda+\delta)$ is called the Green's function of the resolvent equation (3). Notice that

$$J(k,j;\lambda+\delta)=J(j,k;\lambda+\delta)\geq 0,$$

$$\sum_{j=1}^{N-1}J(k,j;\lambda+\delta)h=\frac{1}{\lambda+\delta},\quad 1\leq j<k\leq N.$$

Thus, we obtain the formula for the resolvent $(\lambda I+A_h^x)^{-1}$ in the case $\lambda\geq 0$. In the same way we can obtain a formula as (4) for the resolvent $(\lambda I+A_h^x)^{-1}$ in the case of complex λ. But we need to obtain that $1+2\mu h, 2+3\mu h, 1-R^N$, and $1-\frac{2-\mu h}{2+3\mu h}R^{N-2}$ are not equal to zero.

3 Positivity of difference operators in C_h

Theorem 1 *For all $\lambda,\ \lambda\in R_\varphi=\{\lambda:\ |\arg\lambda|\leq\varphi,\ \varphi\leq\pi/2\}$ the resolvent $(\lambda I+A_h^x)^{-1}$ defined by the formula (4) is subject to the bound*

$$\left\|(\lambda I+A_h^x)^{-1}\right\|_{C_h\to C_h}\leq M(\varphi,\delta)(1+|\lambda|)^{-1},$$

where $M(\varphi,\delta)$ does not depend on h.

The proof of this theorem is based on the following lemmas:

Lemma 2 [4] *If* $\operatorname{Re}\lambda \geq 0,$ *then* $\operatorname{Re}\mu > 0.$

Lemma 3 [4] *The following estimate holds*

$$|\mu| \geq \sqrt{|\lambda+\delta|}.$$

Lemma 4 [4] *The following estimate holds:*

$$|R| \leq \frac{1}{1+\sqrt{|\lambda+\delta|}h\cos\varphi} < 1,$$

where $|\varphi| < \pi/2.$

Lemma 5 *The following inequality holds:*

$$\left|\frac{2-\mu h}{2+3\mu h}\right| \leq 1,$$

where h *is sufficiently small.*

Proof. Let $\mu = \rho e^{i\beta}$ and h be sufficiently small, then μh is also small since

$$\arg\mu = 2^{-1}\arg(\lambda+\delta),$$

$$\mu = \rho(\cos\beta + i\sin\beta).$$

Then

$$|\arg\mu| = |\beta| < \pi/2.$$

Now

$$\left|\frac{2-\rho h(\cos\beta+i\sin\beta)}{2+3\rho h(\cos\beta+i\sin\beta)}\right| = \frac{\sqrt{(2-\rho h\cos\beta)^2+(\rho h\sin\beta)^2}}{\sqrt{(2+3\rho h\cos\beta)^2+(3\rho h\sin\beta)^2}}$$

$$= \sqrt{\frac{4-4\rho h\cos\beta+\rho^2h^2\cos^2\beta+\rho^2h^2\sin^2\beta}{4+9\rho^2h^2\cos^2\beta+12\rho h\cos\beta+9\rho^2h^2\sin^2\beta}}$$

$$= \sqrt{\frac{4-4\rho h\cos\beta+\rho^2h^2}{4+9\rho^2h^2+12\rho h\cos\beta}} \leq 1.$$

Lemma 6 *The following estimate holds:*

$$\left|\frac{1+\mu h}{2+3\mu h}\right| \leq 1.$$

Proof. Let $\mu = \rho e^{i\beta} = \rho(\cos\beta + i\sin\beta)$. Then

$$\left|\frac{1+\rho h(\cos\beta + i\sin\beta)}{2+3\rho h(\cos\beta + i\sin\beta)}\right| = \sqrt{\frac{(1+\rho h\cos\beta)^2 + (\rho h\sin\beta)^2}{(2+3\rho h\cos\beta)^2 + (3\rho h\sin\beta)^2}}$$

$$= \sqrt{\frac{1+2\rho h\cos\beta + \rho^2 h^2}{4+12\rho h\cos\beta + 9\rho^2 h^2}} \leq 1.$$

Lemma 7 *The following inequalities hold:*

$$|(1 - \frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}| > 0,$$

$$|(1-R^N)^{-1}| > 0,$$

where h is sufficiently small.

The proof of this lemma is based on the triangle inequality and on the estimates of Lemmas 4 and 6.

In the sequel for the proof of strong positivity of the difference operator in C_h we will need to consider the following nonlocal boundary value problem

$$\begin{cases} -\frac{u_{k+1}-2u_k+u_{k-1}}{h^2} + (\delta+\lambda)u_k = f_k, \quad 1 \leq k \leq N-1, \\ u_0 = u_N, \ -u_2 + 4u_1 - 3u_0 = u_{N-2} - 4u_{N-1} + 3u_N + 2h\phi. \end{cases} \tag{5}$$

Theorem 2 *Let $\lambda \in R_\varphi$. Then for the solution of the nonlocal boundary value problem the following inequality holds:*

$$\max_{0\leq k\leq N} |u_k| \leq M(\delta,\varphi)\left(\frac{1}{1+|\lambda|}\left\|f^h\right\|_{C_h} + M(\delta,\varphi)|\phi|\right),$$

where $M(\delta,\varphi)$ does not depend on f, ϕ and h.

Proof. Let u_k be a solution of the general nonlocal boundary value problem (5) and w_k be a solution of the nonlocal boundary value problem (3) in the case $a_k = 1$. Then we can write

$$u_k = w_k + v_k,$$

where v_k is the solution of the following nonlocal boundary value problem

$$\begin{cases} -\frac{v_{k+1}-2v_k+v_{k-1}}{h^2} + (\delta+\lambda)v_k = 0, \ 1 \leq k \leq N-1, \ v_0 = v_N, \\ -v_2 + 4v_1 - 3v_0 = v_{N-2} - 4v_{N-1} + 3v_N + 2h\phi. \end{cases}$$

Using the formula

$$v_k = -\frac{2}{\mu}\frac{1+\mu h}{2+3\mu h}\frac{(R^{k-1}+R^{N-1-k})\phi}{(1-\frac{2-\mu h}{2+3\mu h}R^{N-2})},\ 1 \leq k \leq N-1,$$

for the solution of v_k and by Lemmas 6 and 7, we obtain

$$\max_{1\leq k\leq N-1} |v_k| \leq \frac{M(\delta,\varphi)}{|\mu|}|\phi|.$$

Theorem 2 is proved.

4 Positivity of the difference operator A_h^x in C_h

Now we will investigate the strong positivity of the difference operator (2) in C_h. In the sequel we will need the following difference analogue of Nirenberg's inequality which was obtained by Sobolevskii and Neginskii [3]:

$$\max_{0\leq k\leq N-1}\left|\frac{u_{k+1}-u_k}{h}\right| \tag{6}$$

$$\leq K\left[\alpha \max_{1\leq k\leq N-1}\frac{|u_{k+1}-2u_k+u_{k-1}|}{h^2}+\alpha^{-1}\max_{0\leq k\leq N}|u_k|\right],$$

where K is a constant, $\alpha>0$ is a small number.

We consider the difference operator A_h^x defined by the formula (2). If $a_k=a=$ const, then using the substitution $\lambda+\delta=a\lambda_1$ and the results of Section 2, we can obtain the estimate

$$\left\|(\lambda I + A_h^x)^{-1}\right\|_{C_h\to C_h} \leq M(\varphi,\delta)(1+|\lambda|)^{-1}$$

or

$$\max_{0\leq k\leq N}|u_k| \leq M(\delta,\varphi)\left(\frac{1}{1+|\lambda|}\|f\|_{C_h}+|\phi|\right)$$

and the coercive estimate

$$\max_{1\leq k\leq N-1}\frac{|u_{k+1}-2u_k+u_{k-1}|}{h^2} \leq M(\varphi,\delta)\max_{1\leq k\leq N-1}|f_k| \tag{7}$$

for the solutions of the difference equation with constant coefficients. Here $M(\varphi,\delta)$ does not depend on h and λ.

Now, let $a(x)$ be a continuous function on $[0,1]=\Omega$. Similarly to [1], using the method of frozen coefficients and the coercive estimate for the solutions of the difference equation with constant coefficients, we obtain the following theorem.

Theorem 3 *Let h be a sufficiently small number. Then for all $\lambda \in R_\varphi$ and $|\lambda| \geq K_0(\delta,\varphi) > 0$ the resolvent $(\lambda I + A_h^x)^{-1}$ is subject to the bound*

$$\left\|(\lambda I + A_h^x)^{-1}\right\|_{C_h \to C_h} \leq M(\varphi,\delta)(1+|\lambda|)^{-1}, \tag{8}$$

where $M(\varphi,\delta)$ does not depend on h.

Proof. Given $\varepsilon > 0$, there exists a system $\{Q_j\}$, $j = 1,\ldots,r$, of intervals and two half-intervals (containing 0 and 1, respectively) that covers the segment $[0,1]$ and such that $|a(x_1) - a(x_2)| < \varepsilon$, $x_1, x_2 \in Q_j$, because of the compactness of $[0,1]$. For this system we construct a partition of unity, that is, a system of smooth nonnegative functions $\xi_j(x)$ $(i = 1,\ldots,r)$ with $\operatorname{supp}\xi_j(x) \subset Q_j$, $\xi_j(0) = \xi_j(1)$, $\xi_j'(0) = \xi_j'(1)$ and $\xi_1(x) + \cdots + \xi_r(x) = 1$ in $\bar{\Omega} = [0,1]$.

It is clear that for positivity of the difference operator (2) it suffices to establish the estimate

$$\max_{0\leq k\leq N} |u_k| \leq \frac{M}{|\lambda|+1} \max_{1\leq k\leq N-1} |f_k| \tag{9}$$

for the solutions of difference equation (3).

Using $w_k = \xi_j(x_k)u_k$, we obtain

$$w_0 = w_N, \quad -w_2 + 4w_1 - 3w_0 = w_{N-2} - 4w_{N-1} + 3w_N + \phi,$$

where

$$\phi = -(\xi_j(2h) - \xi_j(0))u_2 + 4(\xi_j(h) - \xi_j(0))u_1$$
$$-(\xi_j(1-2h) - \xi_j(1))u_{N-2} + 4(\xi_j(h) - \xi_j(1))u_{N-1}$$

and

$$(\delta+\lambda)\,w_k - a_k\frac{w_{k+1} - 2w_k + w_{k-1}}{h^2}$$
$$= \xi_j(x_k)f_k - a_k\left\{\frac{\xi_j(x_k) - \xi_j(x_{k-1})}{h}\cdot\frac{u_k - u_{k-1}}{h}\right.$$
$$\left. + \frac{\xi_j(x_{k+1}) - 2\xi_j(x_k) + \xi_j(x_{k-1})}{h^2}u_k + \frac{\xi_j(x_{k+1}) - \xi_j(x_k)}{h}\cdot\frac{u_{k+1} - u_k}{h}\right\}.$$

Then we have the following nonlocal boundary value problem

$$(\delta+\lambda)\,w_k - a^j\,\frac{w_{k+1} - 2w_k + w_{k-1}}{h^2} = F_k^j, \quad j = 1,\ldots,r, \tag{10}$$

$$w_0 = w_N, \quad -w_2 + 4w_1 - 3w_0 = w_{N-2} - 4w_{N-1} + 3w_N + \phi,$$

where $a^j = a(x^j)$ and

$$F_k^j = \xi_j(x_k)f_k - a_k \left\{ \frac{\xi_j(x_k) - \xi_j(x_{k-1})}{h} \cdot \frac{u_k - u_{k-1}}{h} \right.$$

$$\left. + \frac{\xi_j(x_{k+1}) - 2\xi_j(x_k) + \xi_j(x_{k-1})}{h^2} u_k + \frac{\xi_j(x_{k+1}) - \xi_j(x_k)}{h} \cdot \frac{u_{k+1} - u_k}{h} \right\}$$

$$+ \left[a_k - a^j\right] \frac{w_{k+1} - 2w_k + w_{k-1}}{h^2}.$$

Since (10) is a difference equation with constant coefficients, we have the estimates

$$(1 + |\lambda|) \max_{0 \le k \le N} |w_k| \le K(\varphi, \delta) \left[\max_{1 \le k \le N-1} \left| F_k^j \right| + (1 + |\lambda|) |\phi| \right], \quad \lambda \in R_\varphi, \tag{11}$$

$$\max_{1 \le k \le N-1} \left| \frac{w_{k+1} - 2w_k + w_{k-1}}{h^2} \right| \le M(\varphi, \delta) \left[\max_{1 \le k \le N-1} \left| F_k^j \right| + (1 + |\lambda|) |\phi| \right]. \tag{12}$$

Using the definition of Q_j and the continuity of $a(x)$ as well as the smoothness of $\xi_i(x)$, we obtain

$$\max_{1 \le k \le N-1} \left| F_k^j \right| \le M(\varphi, \delta) \left[\max_{1 \le k \le N-1} |f_k| + \max_{0 \le k \le N} |u_k| + \max_{0 \le k \le N-1} \frac{|u_{k+1} - u_k|}{h} \right]$$

$$+\varepsilon \max_{1 \le k \le N-1} \left| \frac{w_{k+1} - 2w_k + w_{k-1}}{h^2} \right|$$

and

$$|\phi_2| \le M(\varphi, \delta) h \max_{0 \le k \le N} |u_k| .$$

Assume that $0 < \varepsilon < \frac{1}{M(\varphi,\delta)}$, then from the last estimate it follows that

$$\max_{1 \le k \le N-1} \left| \frac{\xi_j(x_{k+1})u_{k+1} - 2\xi_j(x_k)u_k + \xi_j(x_{k-1})u_{k-1}}{h^2} \right|$$

$$\le \frac{M(\varphi, \delta)}{1 - \varepsilon M(\varphi, \delta)} \left\{ \max_{1 \le k \le N-1} |f_k| + \max_{0 \le k \le N} |u_k| \right.$$

$$\left. + \max_{0 \le k \le N-1} \frac{|u_{k+1} - u_k|}{h} + (1 + |\lambda|) h\phi \right\},$$

$$\max_{1 \le k \le N-1} \left| F_k^j \right| \le \frac{M(\varphi, \delta)}{1 - \varepsilon M(\varphi, \delta)} \left[\max_{1 \le k \le N-1} |f_k| + \max_{0 \le k \le N} |u_k| \right.$$

$$+\max_{0\le k\le N-1}\frac{|u_{k+1}-u_k|}{h}\Bigg]+(1+|\lambda|)h\max_{0\le k\le N}|u_k|\,.$$

From this and the estimate (11) it follows that, for any $j=1,\dots,r$,

$$(1+|\lambda|)\max_{0\le k\le N}\left|\xi_j(x_k)u_k\right|$$

$$\le K(\varphi,\delta)\frac{M(\varphi,\delta)}{1-\varepsilon M(\varphi,\delta)}\Bigg[\max_{1\le k\le N-1}|f_k|$$

$$+\max_{0\le k\le N-1}\frac{|u_{k+1}-u_k|}{h}+(1+(1+|\lambda|)h)\max_{0\le k\le N}|u_k|\Bigg].$$

With the triangle inequality, we have

$$\max_{1\le k\le N-1}\left|\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}\right|\le K_1(\varphi,\delta)\Bigg[\max_{1\le k\le N-1}|f_k| \tag{13}$$

$$+\max_{0\le k\le N-1}\frac{|u_{k+1}-u_k|}{h}+(1+(1+|\lambda|)h)\max_{0\le k\le N}|u_k|\Bigg],$$

$$(1+|\lambda|)\max_{0\le k\le N}|u_k|\le M_1(\varphi,\delta)\Bigg[\max_{1\le k\le N-1}|f_k|+\max_{0\le k\le N-1}\frac{|u_{k+1}-u_k|}{h} \tag{14}$$

$$+(1+(1+|\lambda|)h)\max_{0\le k\le N}|u_k|\Bigg].$$

Now using the inequality (6) we obtain

$$F=\max_{1\le k\le N-1}|f_k|+(1+(1+|\lambda|)h)\max_{0\le k\le N}|u_k|+\max_{0\le k\le N-1}\frac{|u_{k+1}-u_k|}{h}$$

$$\le K_2(\varphi,\delta)\Bigg[\max_{1\le k\le N-1}|f_k|+\alpha^{-1}(1+(1+|\lambda|)h)\max_{0\le k\le N}|u_k|$$

$$+\alpha\max_{1\le k\le N-1}\left|\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}\right|\Bigg].$$

Hence for small α from the last inequality and the inequality (13) it follows that

$$F\le M_2(\varphi,\delta)\Bigg[\alpha^{-1}(1+(1+|\lambda|)h)\max_{0\le k\le N}|u_k|+\max_{1\le k\le N-1}|f_k|\Bigg].$$

Therefore from (14) it follows

$$(1+|\lambda|)\max_{0\le k\le N}|u_k|\le M_2(\varphi,\delta)\Bigg[\alpha^{-1}(1+(1+|\lambda|)h)\max_{0\le k\le N}|u_k|+\max_{1\le k\le N-1}|f_k|\Bigg].$$

Hence for all λ,

$$|\lambda|>\frac{M_2(\varphi,\delta)}{2\alpha-M_2(\varphi,\delta)h}-1=K_0(\varphi,\delta)\,,$$

we have the estimate (9). Theorem 3 is proved.

5 Structure of the fractional spaces and positivity of difference operators in C_h^α

The operator A_h^x commutes with its resolvent $(\lambda + A_h^x)^{-1}$. Therefore, by Theorem 3 we obtain that the operator A_h^x is positive in the fractional spaces $E_\alpha(C_h, A_h^x)$ generated by the difference operator A_h^x. Recall that $E_\alpha(C_h, A_h^x)$ is the set of all grid functions u^h for which the following norm

$$\left\| u^h \right\|_{E_\alpha(C_h, A_h^x)} = \sup_{\lambda > 0} \lambda^\alpha \left\| A_h^x (\lambda + A_h^x)^{-1} u^h \right\|_{C_h} + \left\| u^h \right\|_{C_h}$$

is finite. Since for fixed h the operators A_h^x are bounded, this norm is finite for all grid functions.

Let C_h^β $(0 \leq \beta \leq 1)$ denote the Banach space of all grid functions $f^h = \{f_k\}_1^{N-1}$ with $f_1 = f_{N-1}$ equipped with the norm

$$\left\| f^h \right\|_{C_h^\beta} = \max_{1 \leq k < k+j \leq N-1} \frac{|f_k - f_{k+j}|}{(j\tau)^\beta} + \left\| u^h \right\|_{C_h}.$$

The main result of this paper is the following theorem on the structure of the fractional spaces $E_\alpha(C_h, A_h^x)$.

Theorem 4 *For $0 < \alpha < 1/2$ the norms of the spaces $E_\alpha(C_h, A_h^x)$ and $C_h^{2\alpha}$ are equivalent uniformly in h, $0 < h < h_0$.*

The results of Theorems 3 and 4 permit us to obtain the positivity in $C_h^{2\alpha}$ norms of the operators A_h^x.

Theorem 5 *Let h be sufficiently small number. Then for all $\lambda \in R_\varphi$, $|\lambda| \geq K_0(\delta, \varphi) > 0$ and $0 < \alpha < 1/2$ the resolvent $(\lambda + A_h^x)^{-1}$ is subject to the bound*

$$\left\| (\lambda + A_h^x)^{-1} \right\|_{C_h^{2\alpha} \to C_h^{2\alpha}} \leq \frac{M(\varphi, \delta)}{\alpha(1 - 2\alpha)} (1 + |\lambda|)^{-1}, \tag{15}$$

where $M(\varphi, \delta)$ does not depend on h and α.

The proof of Theorem 4 relies on certain properties of Green's function $J(k, j; \lambda + \delta)$ of the resolvent equation (3). In the case $a(x) \equiv a^2$ we have that

$$(A_h^x + \lambda)^{-1} f^h = \left\{ \sum_{j=1}^{N-1} J(k, j; \lambda + \delta) f_j h_1 \right\}_0^N, \tag{16}$$

where

$$J(k,1;\lambda+\delta) = J(k,N-1;\lambda+\delta)$$
$$= \frac{1+\mu h_1}{2+3\mu h_1}\frac{(R^{N-3}+1)(4R-1)}{2\mu}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}$$

for $k=0$ and $k=N$;

$$J(k,j;\lambda+\delta) = -\frac{1+\mu h_1}{2+3\mu h_1}\frac{(R^2-4R+1)(R^{j-2}+R^{N-j-2})}{2\mu}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}$$

for $2 \leq j \leq N-2$ and $k=0$, $k=N$;

$$J(k,1;\lambda+\delta) = \frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\{R^{k-1}(2(R+3)+R^2(R-3))$$
$$+R^{N-k}(4-R)(1+R)+R^{N+k-3}(1-4R)(1+R)$$
$$+R^{2N-k-3}(3R-1-2R^2(3R+1))\}(1-R^N)^{-1}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1},$$
$$J(k,N-1;\lambda+\delta) = -\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\{R^k(R-4)(R+1)$$
$$+R^{N-k-1}(-2(R+3)+R^2(3-R))+R^{N+k-3}(1-3R+2R^2(3R+1))$$
$$+R^{2N-k-3}(4R-1)(R+1)\}(1-R^N)^{-1}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1},$$
$$J(k,j;\lambda+\delta) = \frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\{(R-1)^3(R^{j+k-2}+R^{2N-2-j-k})$$
$$+(-1+3R+R^2(3-R))(R^{N-k+j-2}+R^{N+k-j-2})+2(1-3R)(R^{2N-2+j-k}$$
$$+R^{2N-2-j+k})+2R^{|j-k|}(R^N-1)\left(R-3+R^{N-2}(-1+3R)\right)\}$$
$$\times(1-R^N)^{-1}(1-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}$$

for $2 \leq j \leq N-2$ and $1 \leq k \leq N-1$. Here

$$R = (1+\mu h_1)^{-1},\ h_1 = a^{-1}h,$$
$$\mu = \frac{1}{2}\left(h_1(\lambda+\delta)+\sqrt{(\lambda+\delta)(4+h_1^2(\lambda+\delta))}\right).$$

A direct consequence of the last formulas is

$$\sum_{j=1}^{N-1} J(k,j;\lambda+\delta)h_1 = \frac{1}{\lambda+\delta}. \tag{17}$$

Now, we will give the proof of Theorem 4. First we consider the case $a(x)=a^2$. Let $a>0$. For any $\lambda>0$ we have the obvious identity

$$A_h^x(\lambda+A_h^x)^{-1}f^h = \lambda\left[\frac{1}{\lambda+\delta}-(\lambda+A_h^x)^{-1}\right]f^h+\frac{\delta}{\lambda+\delta}f^h. \tag{18}$$

By formulas (16), (17) and the identity (18) we can write

$$\{A_h^x\,(\lambda+A_h^x)^{-1}\,f^h\}_k = \lambda\sum_{j=1}^{N-1} J(k,j;\lambda+\delta)\,[f_m-f_j]\,h_1+\frac{\delta}{\lambda+\delta}f_m. \tag{19}$$

Let $k=0$. Then using (19) for $m=1$, we obtain

$$\{A_h^x\,(\lambda+A_h^x)^{-1}\,f^h\}_0 = \lambda\sum_{j=1}^{N-1} J(0,j;\lambda+\delta)\,[f_1-f_j]\,h_1+\frac{\delta}{\lambda+\delta}f_1$$

$$= -\lambda\sum_{j=2}^{N-2}\frac{1+\mu h_1}{2+3\mu h_1}\frac{(R^2-4R+1)(R^{j-2}+R^{N-j-2})}{2\mu}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\,[f_1-f_j]\,h_1$$

$$+\lambda\frac{1+\mu h_1}{2+3\mu h_1}\frac{(R^{N-3}+1)(4R-1)}{2\mu}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\,[f_1-f_{N-1}]\,h_1+\frac{\delta}{\lambda+\delta}f_1$$

$$= -\lambda\sum_{j=3}^{N-3}\frac{1+\mu h_1}{2+3\mu h_1}\frac{(R^2-4R+1)R^{j-2}}{2\mu}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\,[f_1-f_j]\,h_1$$

$$= -\lambda\sum_{j=3}^{N-3}\frac{1+\mu h_1}{2+3\mu h_1}\frac{(R^2-4R+1)R^{N-j-2}}{2\mu}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\,[f_{N-1}-f_j]\,h_1$$

$$-\lambda\frac{1+\mu h_1}{2+3\mu h_1}\frac{(R^2-4R+1)(1+R^{N-4})}{2\mu}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\,[f_1-f_2]\,h_1$$

$$-\lambda\frac{1+\mu h_1}{2+3\mu h_1}\frac{(R^2-4R+1)(R^{N-4}+1)}{2\mu}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\,[f_{N-1}-f_{N-2}]\,h_1+\frac{\delta}{\lambda+\delta}f_1.$$

We have that

$$\lambda^{\alpha}\left|\{A_h^x(\lambda+A_h^x)^{-1}f^h\}_0\right|\leq M[\lambda^{1+\alpha}\sum_{j=3}^{N-3}\frac{\frac{1}{(1+\sqrt{|\lambda+\delta|}h_1\cos\varphi)^{j-2}}}{|\mu|}|f_1-f_j|h_1$$

$$+\lambda^{1+\alpha}\sum_{j=3}^{N-3}\frac{\frac{1}{(1+\sqrt{|\lambda+\delta|}h_1\cos\varphi)^{N-j-2}}}{|\mu|}|f_{N-1}-f_j|h_1$$

$$+\lambda^{1+\alpha}\frac{1}{|\mu|}|f_1-f_2|h_1+\lambda^{1+\alpha}\frac{1}{|\mu|}|f_{N-1}-f_{N-2}|h_1+\frac{\lambda^{\alpha}\delta}{\lambda+\delta}|f_1|]$$

$$\leq M(\varphi,\delta)[\sum_{j=3}^{N-3}\frac{1}{((j-2)h_1)^{\frac{1}{2}+\alpha}}((j-1)h_1)^{2\alpha}h_1$$

$$+\lambda^{1+\alpha}\frac{1}{\sqrt{\lambda+\delta}}h_1^{1+2\alpha}+1]\left\|f^h\right\|_{C_h^{2\alpha}}\leq M_1(\varphi,\delta)\left\|f^h\right\|_{C_h^{2\alpha}}.$$

Thus,

$$\lambda^{\alpha}\left|\{A_h^x(\lambda+A_h^x)^{-1}f^h\}_0\right|\leq M_1(\varphi,\delta)\left\|f^h\right\|_{C_h^{2\alpha}}. \tag{20}$$

The proof of the estimate

$$\lambda^{\alpha}\left|\{A_h^x(\lambda+A_h^x)^{-1}f^h\}_N\right|\leq M_1(\varphi,\delta)\left\|f^h\right\|_{C_h^{2\alpha}}$$

follows the scheme of the proof of the estimate (20) and is based on the formula (19) for $m=N-1$. Let $1\leq k\leq N-1$. Then using (19) for $m=k$, we obtain

$$\{A_h^x(\lambda+A_h^x)^{-1}f^h\}_k=\lambda\sum_{j=1}^{N-1}J(k,j;\lambda+\delta)\left[f_k-f_j\right]h_1+\frac{\delta}{\lambda+\delta}f_k$$

$$=\lambda\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\{R^{k-1}(2(R+3)+R^2(R-3))$$

$$+R^{N+k-3}(1-4R)(1+R)\}(1-R^N)^{-1}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\left[f_k-f_1\right]h_1$$

$$+\lambda\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\{+R^{N-k}(4-R)(1+R)$$

$$+R^{2N-k-3}(3R-1-2R^2(3R+1))\}(1-R^N)^{-1}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\left[f_k-f_{N-1}\right]h_1$$

$$-\lambda\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\{+R^{N-k-1}(-2(R+3)+R^2(3-R))$$

$$+R^{2N-k-3}(4R-1)(R+1)\}(1-R^N)^{-1}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\left[f_k-f_{N-1}\right]h_1$$

$$-\lambda\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\{R^k(R-4)(R+1)$$

$$+R^{N+k-3}(1-3R+2R^2(3R+1))\}(1-R^N)^{-1}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\left[f_k-f_1\right]h_1$$

$$+\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\lambda\sum_{j=2}^{N-2}\{(R-1)^3R^{j+k-2}$$

$$+(-1+3R+R^2(3-R))R^{N-k+j-2}+2(1-3R)R^{2N-2+j-k}\}$$

$$\times(1-R^N)^{-1}(1-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\ \left[f_1-f_j\right]h_1$$

$$+\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\lambda\sum_{j=2}^{N-2}\{(R-1)^3R^{j+k-2}$$

$$+(-1+3R+R^2(3-R))R^{N-k+j-2}+2(1-3R)R^{2N-2+j-k}\}$$

$$\times(1-R^N)^{-1}(1-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\ \left[f_k-f_1\right]h_1$$

$$+\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\lambda\sum_{j=2}^{N-2}\{(R-1)^3R^{2N-2-j-k}$$

$$+(-1+3R+R^2(3-R))R^{N+k-j-2}+2(1-3R)(R^{2N-2-j+k}$$

$$\times(1-R^N)^{-1}(1-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\ \left[f_{N-1}-f_j\right]h_1$$

$$+\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\lambda\sum_{j=2}^{N-2}\{(R-1)^3R^{2N-2-j-k}$$

$$+(-1+3R+R^2(3-R))R^{N+k-j-2}+2(1-3R)(R^{2N-2-j+k}$$

$$\times(1-R^N)^{-1}(1-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\ \left[f_k-f_{N-1}\right]h_1$$

$$+\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\lambda\sum_{j=2}^{N-2}2R^{|j-k|}(R^N-1)\left(R-3+R^{N-2}(-1+3R)\right)$$

$$\times(1-R^N)^{-1}(1-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\ \ [f_k-f_j]\,h_1+\frac{\delta}{\lambda+\delta}f_k.$$

The proof of the estimate

$$\lambda^\alpha\left|\{A_h^x\,(\lambda+A_h^x)^{-1}\,f^h\}_k\right|\le M_1(\varphi,\delta)\left\|f^h\right\|_{C_h^{2\alpha}}$$

follows the scheme of the proof of the estimate (20) and is based on the last formula. Thus, for any $\lambda\ge 0$ and $k=0,\dots,N$ we establish the validity of the inequality

$$\left|\lambda^\alpha\{A_h^x\,(\lambda+A_h^x)^{-1}\,f^h\}_k\right|\le M_2(\varphi,\delta)\left\|f^h\right\|_{C_h^{2\alpha}}.$$

This means that

$$\left\|f^h\right\|_{E_\alpha\left(C_h,A_h^x\right)}\le M_2(\varphi,\delta)\left\|f^h\right\|_{C_h^{2\alpha}}.$$

Now let us prove the opposite inequality. For any positive operator A_h^x we can write

$$v=\int_0^\infty\sum_{j=1}^{N-1}J\,(k,j;t+\delta)\,A_h^x\,(t+A_h^x)^{-1}\,f_jh_1\,dt.$$

Consequently,

$$f_k-f_{k+r}=\int_0^\infty\sum_{j=1}^{N-1}t^{-\alpha}\,[J\,(k,j;t+\delta)-J\,(k+r,j;t+\delta)]\,t^\alpha A_h^x\,(t+A_h^x)^{-1}\,f_jh_1\,dt,$$

whence

$$|f_k-f_{k+r}|\le\int_0^\infty t^{-\alpha}\sum_{j=1}^{N-1}|J\,(k,j;t+\delta)-J\,(k+r,j;t+\delta)|\,h_1\,dt\left\|f^h\right\|_{E_\alpha\left(C_h,A_h^x\right)}.$$

Let

$$T_h=|rh_1|^{-2\alpha}\int_0^\infty t^{-\alpha}\sum_{j=1}^{N-1}|J\,(k,j;t+\delta)-J\,(k+r,j;t+\delta)|\,h_1\,dt.$$

The proof of the estimate

$$\frac{|f_k-f_{k+r}|}{|rh_1|^{2a}}\le T_h\left\|f^h\right\|_{E_\alpha\left(C_h,A_h^x\right)}$$

follows the scheme of the paper [2] and is based on the Lemmas 2, 3, 4 and 5. Thus, for any $1 \leq k < k+r \leq N-1$ we have established the inequality

$$\frac{|f_k - f_{k+r}|}{|rh_1|^{2\alpha}} \leq \frac{M}{\alpha(1-2\alpha)} \left\| f^h \right\|_{E_\alpha(C_h, A_h^x)}.$$

This means that the following inequality holds:

$$\left\| f^h \right\|_{C_h^{2\alpha}} \leq \frac{M}{\alpha(1-2\alpha)} \left\| f^h \right\|_{E_\alpha(C_h A_h^x)}.$$

Theorem 2 in the case $a(x) = a^2$ is proved. Now, let $a(x)$ be a continuous function and let $x, x_0 \in [0,1]$ be arbitrary fixed points. It is easy to show that

$$\left| \left(A_h^x - A_h^{x_0} \right) \left(A_h^{x_0} \right)^{-1} \right| \leq M.$$

Therefore, using the formula

$$A_h^x (A_h^x + \lambda)^{-1} f^h = A_h^{x_0} \left(A_h^{x_0} + \lambda \right)^{-1} f^h$$

$$+\lambda \left(\lambda + A_h^x \right)^{-1} \left[A_h^x - A_h^{x_0} \right] \left(A_h^{x_0} \right)^{-1} A_h^{x_0} \left(A_h^{x_0} + \lambda \right)^{-1} f^h,$$

we derive

$$\left| \lambda^\alpha A_h^x (A_h^x + \lambda)^{-1} f^h \right| \leq \left\| f^h \right\|_{E_\alpha(C_h, A_h^{x_0})}$$

$$+M\lambda \left\| (\lambda + A_h^x)^{-1} \right\|_{C_h \to C_h} \left\| f^h \right\|_{E_\alpha(C_h, A_h^{x_0})} \leq M_1 \left\| f^h \right\|_{E_\alpha(C_h, A_h^{x_0})}.$$

From that it follows

$$\left\| \left\| f^h \right\|_{E_\alpha(C_h, A_h^{x_0})} \right\| \leq M_1 \left\| f^h \right\|_{E_\alpha(C_h, A_h^{x_0})}.$$

Theorem 4 is proved.

The results of this paper and the abstract results of papers [6, 7, 8, 9] permit us to investigate the well-posedness of the nonlocal boundary-value problems for elliptic differential and difference equations in the Banach spaces.

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