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On Difference Schemes for Hyperbolic-Elliptic Equations

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Abstract

The nonlocal boundary value problem for the hyperbolic-elliptic equation

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t) & (0 \le t \le 1), \\ -\frac{d^2u(t)}{dt^2} + Au(t) = g(t) & (-1 \le t \le 0), \\ u(0) = \varphi, & u(1) = u(-1) \end{cases}$$

in a Hilbert space H is considered. The difference schemes approximately solving this boundary value problem are presented. The stability estimates for the solution of these difference schemes are established. In applications, the stability estimates for the solutions of the difference schemes of the mixed type boundary value problems for hyperbolic-elliptic equations are obtained.

1 Introduction

It is known (see [1]–[4]) that various boundary value problems for hyperbolic-elliptic equations can be reduced to the nonlocal boundary value problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t) & (0 \le t \le 1), \\ -\frac{d^2u(t)}{dt^2} + Au(t) = g(t) & (-1 \le t \le 0), \\ u(0) = \varphi, & u(1) = u(-1) \end{cases} \tag{1}$$

for differential equations in a Hilbert space H, with the self-adjoint positively definite operator A.

A function $u(t)$ is called a solution of problem (1) if the following conditions are satisfied:
i. $u(t)$ is twice continuously differentiable in the region $[-1,0)\bigcup(0,1]$ and continuously differentiable on the segment $[-1,1]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.
ii. The element $u(t)$ belongs to $D(A)$ for all $t\in[-1,1]$, and the function $Au(t)$ is continuous on $[-1,1]$.
iii. $u(t)$ satisfies the equation and boundary value conditions (1).

Theorem 1 [5] *Suppose that $\varphi\in D(A)$, and let $f(t)$ be continuously differentiable on $[0,1]$ and $g(t)$ be continuously differentiable on $[-1,0]$ functions. Then there is a unique solution of the problem (1) and the stability inequalities*

$$\max_{-1\le t\le 1}\parallel u(t)\parallel_H\le M\left[\parallel\varphi\parallel_H+\max_{-1\le t\le 0}\parallel A^{-1/2}g(t)\parallel_H+\max_{0\le t\le 1}\parallel A^{-1/2}f(t)\parallel_H\right],$$

$$\max_{-1\le t\le 1}\parallel\frac{du}{dt}\parallel_H+\max_{-1\le t\le 1}\parallel A^{1/2}u(t)\parallel_H\le M\Big[\parallel A^{1/2}\varphi\parallel_H$$
$$+\int_{-1}^{0}\parallel g(t)\parallel_H\,dt+\int_{0}^{1}\parallel f(t)\parallel_H\,dt\Big],$$

$$\max_{-1\le t\le 1}\parallel\frac{d^2u}{dt^2}\parallel_H+\max_{-1\le t\le 1}\parallel Au(t)\parallel_H\le M\Big[\parallel A\varphi\parallel_H+\parallel g(0)\parallel_H$$
$$+\parallel f(0)\parallel_H+\int_{-1}^{0}\parallel g'(t)\parallel_H\,dt+\int_{0}^{1}\parallel f'(t)\parallel_H\,dt\Big]$$

hold, where M does not depend on $f(t)$, $t\in[0,1]$, $g(t)$, $t\in[-1,0]$ and φ.

In the paper [5] the first order of accuracy difference scheme for approximately solving the boundary value problem (1)

$$\begin{cases}\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_{k+1}=f_k,\ f_k=f(t_{k+1}),\\ t_{k+1}=(k+1)\tau,\ 1\le k\le N-1,\quad N\tau=1,\\ -\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=g_k,\ g_k=g(t_k),\\ t_k=k\tau,\ -N+1\le k\le -1,\quad u_0=\varphi,\\ u_N=u_{-N},\quad u_1-u_0=u_0-u_{-1}\end{cases}\tag{2}$$

was investigated.

A study of discretization, over time only, of the nonlocal boundary value problem also permits one to include general difference schemes in applications, if the differential operator in space variables, A, is replaced by the difference operators A_h that act in the Hilbert spaces H_h and are uniformly self-adjoint positively definite in h for $0 < h \leq h_0$.

Theorem 2 [6] *Let $\varphi \in D(A)$. Then the solution of the difference scheme* (2) *obeys the stability inequalities*

$$\max_{-N\leq k\leq N} \| u_k \|_H \leq M \left[\| \varphi \|_H + \max_{-N+1\leq k\leq -1} \| A^{-1/2} g_k \|_H + \max_{1\leq k\leq N-1} \| A^{-1/2} f_k \|_H \right],$$

$$\max_{-N+1\leq k\leq N} \left\| \frac{u_k - u_{k-1}}{\tau} \right\|_H + \max_{-N\leq k\leq N} \| A^{1/2} u_k \|_H \leq M \Big[\| A^{1/2} \varphi \|_H$$

$$+ \sum_{k=-N+1}^{-1} \tau \| g_k \|_H + \sum_{k=1}^{N-1} \tau \| f_k \|_H \Big],$$

$$\max_{-N+1\leq k\leq N-1} \left\| \frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} \right\|_H + \max_{-N\leq k\leq N} \| A u_k \|_H \leq M \Big[\| A\varphi \|_H$$

$$+ \| g_{-1} \|_H + \| f_1 \|_H + \sum_{k=-N+1}^{-1} \| g_k - g_{k-1} \|_H + \sum_{k=2}^{N-1} \| f_k - f_{k-1} \|_H \Big],$$

where M does not depend on τ, φ, and f_k, $1 \leq k \leq N-1$, $g_k, -N+1 \leq k \leq -1$.

Methods for numerical solutions of the nonlocal boundary value problems for partial differential equations have been studied extensively by many researches (see [8–17], [22–26] and the references therein).

In the present paper the second order of accuracy difference schemes approximately solving the boundary value problem (1) are presented. The stability estimates for the solution of these difference schemes are established. In applications, the stability estimates for the solutions of the difference schemes of the mixed type boundary value problems for hyperbolic-elliptic equations are obtained. The theoretical statements for the solution of this difference scheme are supported by the results of numerical experiments.

2 The second order of accuracy difference schemes

Applying the second order of accuracy difference schemes of paper [7] for hyperbolic equations and the second order of accuracy difference scheme for elliptic equations, we will construct the following second order of accuracy difference schemes for approximately solving the boundary value problem (1):

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + Au_k + \frac{\tau^2}{4}A^2u_{k+1} = f_k, \\ f_k = f(t_k),\ t_k = k\tau, \quad 1 \le k \le N-1,\ N\tau = 1, \\ -\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + Au_k = g_k, \\ g_k = g(t_k),\ t_k = k\tau,\ -N+1 \le k \le -1, \quad u_0 = \varphi, \\ u_N = u_{-N},\ u_1 - u_0 - \frac{\tau^2}{2}(f_0 - Au_0) = u_0 - u_{-1} - \frac{\tau^2}{2}(g_0 - Au_0), \\ g_0 = g(0), \quad f_0 = f(0), \end{cases} \tag{3}$$

and

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + \frac{1}{2}Au_k + \frac{1}{4}(Au_{k+1} + Au_{k-1}) = f_k, \\ f_k = f(t_k),\ t_k = k\tau,\ 1 \le k \le N-1,\ N\tau = 1, \\ -\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + Au_k = g_k, \\ g_k = g(t_k),\ t_k = k\tau,\ -N+1 \le k \le -1, \quad u_0 = \varphi, \\ u_N = u_{-N}, \quad (I + \frac{\tau^2 A}{4})(u_1 - u_0) - \frac{\tau^2}{2}(f_0 - Au_0) \\ = u_0 - u_{-1} - \frac{\tau^2}{2}(g_0 - Au_0), \\ g_0 = g(0), \quad f_0 = f(0). \end{cases} \tag{4}$$

Theorem 3 *Let $\varphi \in D(A)$. Then the solution of the difference scheme* (3) *obeys the stability inequalities*

$$\max_{-N \le k \le N} \| u_k \|_H \le M \left[\| \varphi \|_H + \max_{-N+1 \le k \le 0} \| A^{-1/2} g_k \|_H + \max_{0 \le k \le N-1} \| A^{-1/2} f_k \|_H \right],$$

$$\max_{-N+1 \le k \le N} \| \frac{u_k - u_{k-1}}{\tau} \|_H + \max_{-N \le k \le N} \| A^{1/2} u_k \|_H \le M \Big[\| A^{1/2} \varphi \|_H$$

$$+ \sum_{k=-N+1}^{0} \tau \| g_k \|_H + \sum_{k=0}^{N-1} \tau \| f_k \|_H \Big],$$

$$\max_{-N+1 \le k \le N-1} \| \frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} \|_H + \max_{-N \le k \le N} \| Au_k \|_H \le M \Big[\| A\varphi \|_H$$

$$+\parallel g_0 \parallel_H + \parallel f_0 \parallel_H + \sum_{k=-N+1}^{0} \parallel g_k - g_{k-1} \parallel_H + \sum_{k=1}^{N-1} \parallel f_k - f_{k-1} \parallel_H \Big],$$

where M does not depend on τ, φ, and f_k, $0 \le k \le N-1$, g_k, $-N+1 \le k \le 0$.

The proof of Theorem 3 follows the scheme of the proof of Theorem 2 and is based on the formulas

$$u_k = (D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}[(D(-\tau A^{1/2}) - I)D^{k-1}(\tau A^{1/2})$$
$$+(I - D(\tau A^{1/2}))D^{k-1}(-\tau A^{1/2})]u_0$$
$$+(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^k(\tau A^{1/2}) - D^k(-\tau A^{1/2}))(u_0 - u_{-1})$$
$$+\frac{\tau^2}{2}(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^k(\tau A^{1/2}) - D^k(-\tau A^{1/2}))(f_0 - g_0)$$
$$-\sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2}[D^{k-s}(\tau A^{1/2}) - D^{k-s}(-\tau A^{1/2})]f_s,$$
$$1 \le k \le N-1, \quad D(\pm\tau A^{1/2}) = (1 \pm i\tau A^{1/2} - \frac{\tau^2 A}{2})^{-1},$$

$$u_k = R^{-k}u_0 + (I - R^{2N})^{-1}(R^{N-k} - R^{N+k})[R^N u_0 - u_{-N}]$$
$$+(I - R^{2N})^{-1}(R^{N-k} - R^{N+k}) \sum_{s=-N+1}^{-1} B^{-1}[R^{N-s} - R^{N+s}]R^{-1}(2 + \tau B)^{-1} g_s \tau$$
$$+\sum_{s=-N+1}^{-1} B^{-1}(R^{-(k+s)} - R^{|s-k|})(2 + \tau B)^{-1} R^{-1} g_s \tau,$$
$$-N+1 \le k \le -1, \quad R = (1 + \tau B)^{-1}, \quad B = \frac{A\tau + A^{1/2}\sqrt{\tau^2 A + 4}}{2},$$

$$u_{-N} = T\{(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}[(D(-\tau A^{1/2}) - I)D^{N-1}(\tau A^{1/2})$$
$$+(I - D(\tau A^{1/2}))D^{N-1}(-\tau A^{1/2})]u_0 + (D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}$$
$$\times(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))u_0 - (D(\tau A^{1/2})$$
$$-D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))$$
$$\times\{Ru_0 + (I - R^{2N})^{-1}(R^{N+1} - R^{N-1})R^N u_0 + (I - R^{2N})^{-1}(R^{N+1} - R^{N-1})$$

$$\times \sum_{s=-N+1}^{-1} B^{-1}[R^{N-s} - R^{N+s}]R^{-1}(2+\tau B)^{-1} g_s \tau$$

$$+ \sum_{s=-N+1}^{-1} B^{-1}(R^{1-s} - R^{1+s})(2+\tau B)^{-1}R^{-1} g_s \tau \}$$

$$+\frac{\tau^2}{2}(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^k(\tau A^{1/2}) - D^k(-\tau A^{1/2}))(f_0 - g_0)$$

$$-\sum_{s=1}^{N-1} \frac{\tau}{2i} A^{-1/2}[D^{N-s}(\tau A^{1/2}) - D^{N-s}(-\tau A^{1/2})] f_s \},$$

$$\begin{aligned} T &= (I - (I - R^{2N})^{-1}(R^{N+1} - R^{N-1})(D(\tau A^{1/2}) \\ &\quad -D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})))^{-1} \end{aligned}$$

and on the estimates

$$\| D(\pm\tau A^{1/2}) \|_{H\to H} \le 1, \quad \tau \| A^{1/2} D(\pm\tau A^{1/2}) \|_{H\to H} \le 2, \tag{5}$$

$$\| (k\tau B)^{\alpha} R^k \|_{H\to H} \le M(1+\delta\tau)^{-k}, \quad k \ge 1,\ 0 \le \alpha \le 1,\ \delta > 0,\ M > 0, \tag{6}$$

and on the following lemmas.

Lemma 1 *The estimate holds:*

$$\|[D^N\left(\pm\tau A^{1/2}\right) - \exp\{\mp i A^{1/2}\}]A^{-1}\|_{H\to H} \le \frac{\tau}{2}. \tag{7}$$

Proof. We use the identity

$$D^N\left(\pm\tau A^{1/2}\right) - \exp\{\mp i A^{1/2}\} = \int_0^1 \Psi'(s\tau A^{1/2})\, ds,$$

where

$$\Psi(s\tau A^{1/2}) = D^N\left(\pm s\tau A^{1/2}\right) \exp\{\mp i(1-s)A^{1/2}\}.$$

The derivative $\Psi'(s\tau A^{1/2})$ is given by

$$\Psi'(s\tau A^{1/2}) = D^{N+1}\left(\mp s\tau A^{1/2}\right)\left(\mp i \frac{\tau^2 s^2 A^{3/2}}{2}\right) \exp\{\mp i(1-s)A^{1/2}\}.$$

Thus,

$$D^N\left(\pm\tau A^{1/2}\right)-\exp\{\mp iA^{1/2}\}$$
$$=\mp\int\limits_0^1 D^{N+1}\left(\pm s\tau A^{1/2}\right)\left(iA^{3/2}\right)\frac{1}{2}\tau^2 s^2\exp\{\mp i(1-s)A^{1/2}\}\,ds.$$

Using the last identity and estimates (6) and

$$\|\exp\{\mp i(1-s)A^{1/2}\}\|\leq 1, \tag{8}$$

we obtain

$$\|[D^N\left(\pm\tau A^{1/2}\right)-\exp\{\mp iA^{1/2}\}]A^{-1}\|_{H\to H}$$
$$\leq \frac{1}{2}\int\limits_0^1 \|D^N\left(\pm s\tau A^{1/2}\right)\|_{H\to H}\tau s$$
$$\times\|\tau sA^{1/2}D\left(\pm s\tau A^{1/2}\right)\|_{H\to H}\|\exp\{\mp i(1-s)A^{1/2}\}\|_{H\to H}\,ds\leq\tau\int\limits_0^1 s\,ds=\frac{\tau}{2}.$$

Lemma 2 *The following estimate holds:*

$$\|\,T\,\|_{H\to H}\leq M, \tag{9}$$

where M does not depend on τ.

Proof. Since

$$\begin{aligned} T &= (I-R^{2N})(I-R^{2N}+(R^{N+1}-R^{N-1})(D(\tau A^{1/2})\\ &\quad -D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2})-D^N(-\tau A^{1/2})))^{-1}\end{aligned}$$

and

$$\widetilde{T}-\{I-\exp\{-2A^{1/2}\}+2A^{1/2}s(1)\exp\{-A^{1/2}\}\}^{-1}$$
$$=\widetilde{T}\{I-\exp\{-2A^{1/2}\}+2A^{1/2}s(1)\exp\{-A^{1/2}\}\}^{-1}$$
$$\times\quad\{R^{2N}-\exp\{-2A^{1/2}\}+2A^{1/2}s(1)\exp\{-A^{1/2}\}-(R^{N+1}-R^{N-1})$$
$$\times\quad(D(\tau A^{1/2})-D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2})-D^N(-\tau A^{1/2}))\}$$

and

$$\|\,\{I-\exp\{-2A^{1/2}\}+2A^{1/2}s(1)\exp\{-A^{1/2}\}\}^{-1}\,\|_{H\to H}\leq M, \tag{10}$$

to prove (9) it suffices to establish the estimate

$$\|R^{2N} - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\} - (R^{N+1} - R^{N-1}) \tag{11}$$
$$\times(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))\|_{H\to H} \le M\tau.$$

Here

$$\widetilde{T} = (I - R^{2N} + (R^{N+1} - R^{N-1})(D(\tau A^{1/2})$$
$$-D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})))^{-1},$$
$$s(1) = A^{-1/2}\frac{e^{iA^{1/2}} - e^{-iA^{1/2}}}{2i}.$$

The estimate (10) was proved in [4]. Finally, using the identity

$$R^{2N} - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\} - (R^{N+1} - R^{N-1})$$
$$\times(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))$$
$$= R^{2N} - \exp\{-2A^{1/2}\} + [2A^{1/2}s(1) - \frac{1}{i}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))]\exp\{-A^{1/2}\}$$
$$+\frac{1}{i}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))[\exp\{-A^{1/2}\} - R^N]$$
$$+\frac{1}{i}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))[R^N - (R^{N+1} - R^{N-1})(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}]$$

and the estimates (5), (6) and (7), we obtain the estimate (11).

Theorem 4 *Let $\varphi \in D(A^{3/2})$. Then the solution of the difference scheme* (4) *obeys the stability inequalities*

$$\max_{-N\le k\le N} \| u_k \|_H \le M\Big[\| \left(I \pm \frac{1}{2}i\tau A^{1/2}\right)\varphi \|_H$$
$$+ \max_{-N+1\le k\le 0} \| A^{-1/2}g_k \|_H + \max_{0\le k\le N-1} \| A^{-1/2}f_k \|_H\Big],$$
$$\max_{-N+1\le k\le N} \| \frac{u_k - u_{k-1}}{\tau} \|_H + \max_{-N\le k\le N} \| A^{1/2}u_k \|_H$$
$$\le M\Big[\| A^{1/2}\left(I \pm \frac{1}{2}i\tau A^{1/2}\right)\varphi \|_H$$
$$+ \sum_{k=-N+1}^{0} \tau \| g_k \|_H + \sum_{k=0}^{N-1} \tau \| f_k \|_H\Big],$$

$$\max_{-N+1\leq k\leq N-1} \| \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} \|_H + \max_{-N\leq k\leq N} \| Au_k \|_H$$
$$\leq M\Big[\| A\left(I\pm\frac{1}{2}i\tau A^{1/2}\right)\varphi \|_H + \| g_0 \|_H + \| f_0 \|_H$$
$$+\sum_{k=-N+1}^{0} \| g_k - g_{k-1} \|_H + \sum_{k=1}^{N-1} \| f_k - f_{k-1} \|_H\Big]$$

where M *does not depend on* τ, φ, *and* f_k, $0\leq k\leq N-1$, g_k, $-N+1\leq k\leq 0$.

The proof of Theorem 4 follows the scheme of the proof of Theorem 2 and is based on the formulas

$$u_k = (D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}[(I - D(-\tau A^{1/2}))D^{k-1}(-\tau A^{1/2})$$
$$+(D(\tau A^{1/2}) - I)D^{k-1}(\tau A^{1/2})]u_0$$
$$+(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^k(\tau A^{1/2}) - D^k(-\tau A^{1/2}))(I+\frac{\tau^2 A}{4})^{-1}(u_0 - u_{-1})$$
$$+\frac{\tau^2}{2}(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^k(\tau A^{1/2}) - D^k(-\tau A^{1/2}))(I+\frac{\tau^2 A}{4})^{-1}(f_0 - g_0)$$
$$+\sum_{s=1}^{k-1}(I+\frac{\tau^2 A}{4})^{-1}(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}[D^{k-s}(\tau A^{1/2}) - D^{k-s}(-\tau A^{1/2})]f_s,$$
$$1\leq k\leq N-1,\quad D(\pm\tau A^{1/2}) = (1\mp\frac{i\tau A^{1/2}}{2})(I\pm\frac{i\tau A^{1/2}}{2})^{-1},$$

$$u_k = R^{-k}u_0 + (I-R^{2N})^{-1}(R^{N-k} - R^{N+k})[R^N u_0 - u_{-N}]$$
$$+(I-R^{2N})^{-1}(R^{N-k} - R^{N+k})\sum_{s=-N+1}^{-1} B^{-1}[R^{N-s} - R^{N+s}]R^{-1}(2+\tau B)^{-1}g_s\tau$$
$$+\sum_{s=-N+1}^{-1} B^{-1}(R^{-(k+s)} - R^{|s-k|})(2+\tau B)^{-1}R^{-1}g_s\tau,$$
$$-N+1\leq k\leq -1,\quad R=(1+\tau B)^{-1},\quad B=\frac{A\tau + A^{1/2}\sqrt{\tau^2 A+4}}{2},$$

$$u_{-N} = T\{(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}[(I - D(-\tau A^{1/2}))D^{N-1}(-\tau A^{1/2})$$

$$+(D(\tau A^{1/2})-I)D^{N-1}(\tau A^{1/2})]u_0+(D(\tau A^{1/2})-D(-\tau A^{1/2}))^{-1}$$

$$\begin{aligned}&\times\quad (D^N(\tau A^{1/2})-D^N(-\tau A^{1/2}))(I+\frac{\tau^2 A}{4})^{-1}u_0\\ &-\quad (D(\tau A^{1/2})-D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2})-D^N(-\tau A^{1/2}))\end{aligned}$$

$$\times\{Ru_0+(I-R^{2N})^{-1}(R^{N+1}-R^{N-1})R^N u_0+(I-R^{2N})^{-1}(R^{N+1}-R^{N-1})$$

$$\times\sum_{s=-N+1}^{-1}B^{-1}[R^{N-s}-R^{N+s}]R^{-1}(2+\tau B)^{-1}g_s\tau$$

$$+\sum_{s=-N+1}^{-1}B^{-1}(R^{1-s}-R^{1+s})(2+\tau B)^{-1}R^{-1}g_s\tau\}$$

$$\begin{aligned}&+\quad \frac{\tau^2}{2}(D(\tau A^{1/2})-D(-\tau A^{1/2}))^{-1}(D^k(\tau A^{1/2})\\ &-\quad D^k(-\tau A^{1/2}))(I+\frac{\tau^2 A}{4})^{-1}(f_0-g_0)\end{aligned}$$

$$-\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}[D^{N-s}(\tau A^{1/2})-D^{N-s}(-\tau A^{1/2})]f_s\},$$

$$\begin{aligned}T\quad &=\quad (I-(I-R^{2N})^{-1}(R^{N+1}-R^{N-1})(D(\tau A^{1/2})\\ &-\quad D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2})-D^N(-\tau A^{1/2}))(I+\frac{\tau^2 A}{4})^{-1})^{-1}\end{aligned}$$

and on the estimates (6) and

$$\| D(\pm\tau A^{1/2})\|_{H\to H}\leq 1,\quad \tau\| A^{1/2}(I\pm\frac{i\tau A^{1/2}}{2})^{-1}\|_{H\to H}\leq 2, \tag{12}$$

and on the following lemmas.

Lemma 3 *The estimate holds:*

$$||[D^N\left(\pm\tau A^{1/2}\right)-\exp\{\mp iA^{1/2}\}]A^{-1}||_{H\to H}\leq\frac{\tau}{4}. \tag{13}$$

Proof. We use the identity

$$D^N\left(\pm\tau A^{1/2}\right) - \exp\{\mp iA^{1/2}\} = \int\limits_0^1 \Psi'(s\tau A^{1/2})\,ds,$$

where

$$\Psi(s\tau A^{1/2}) = D^N\left(\pm s\tau A^{1/2}\right)\exp\{\mp i(1-s)A^{1/2}\}.$$

The derivative $\Psi'(s\tau A^{1/2})$ is given by

$$\Psi'(s\tau A^{1/2}) = D^{N-1}\left(\pm s\tau A^{1/2}\right)\left(\pm iA^{1/2}\right)$$

$$\times\left(-\frac{1}{4}\tau^2 s^2 A\right)\left(I \pm \frac{1}{2}i\tau A^{1/2}\right)^{-2}\exp\{\mp i(1-s)A^{1/2}\}.$$

Thus,

$$D^N\left(\pm\tau A^{1/2}\right) - \exp\{\mp iA^{1/2}\}$$

$$= \mp\int\limits_0^1 D^{N-1}\left(\pm s\tau A^{1/2}\right)\left(iA^{3/2}\right)\frac{1}{4}\tau^2 s^2\left(I \pm \frac{1}{2}i\tau A^{1/2}\right)^{-2}\exp\{\mp i(1-s)A^{1/2}\}\,ds.$$

Using the last identity and the estimates (12) and (8), we obtain

$$\|[D^N\left(\pm\tau A^{1/2}\right) - \exp\{\mp iA^{1/2}\}]A^{-1}\|_{H\to H} \leq \frac{\tau}{2}\int\limits_0^1 \|D^{N-1}\left(\pm s\tau A^{1/2}\right)\|_{H\to H}s$$

$$\times\parallel isA^{1/2}\frac{1}{2}\tau\left(I \pm \frac{1}{2}i\tau A^{1/2}\right)^{-2}\parallel_{H\to H}\parallel\exp\{\mp i(1-s)A^{1/2}\}\|_{H\to H}\,ds$$

$$\leq \frac{\tau}{2}\int\limits_0^1 s\,ds = \frac{\tau}{4}.$$

Lemma 4 *The following estimate holds:*

$$\parallel T \parallel_{H\to H} \leq M, \tag{14}$$

where M does not depend on τ.

Proof. Since

$$T = (I - R^{2N})(I - R^{2N} + (R^{N+1} - R^{N-1})(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))(I + \frac{\tau^2 A}{4})^{-1})^{-1}$$

and

$$\widetilde{T} - \{I - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\}\}^{-1}$$
$$= \widetilde{T}\{I - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\}\}^{-1}$$

$$\times \{R^{2N} - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\} - (R^{N+1} - R^{N-1})$$
$$\times (D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))(I + \frac{\tau^2 A}{4})^{-1}\}$$

and (10), to prove (14) it suffices to establish the estimate

$$\|R^{2N} - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\} - (R^{N+1} - R^{N-1}) \tag{15}$$
$$\times(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))(I + \frac{\tau^2 A}{4})^{-1}\|_{H\to H} \le M\sqrt{\tau}.$$

Here

$$\widetilde{T} = (I - R^{2N} + (R^{N+1} - R^{N-1})(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}$$
$$\times(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))(I + \frac{\tau^2 A}{4})^{-1})^{-1}.$$

Finally, using the identity

$$R^{2N} - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\} - (R^{N+1} - R^{N-1})$$
$$\times(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))(I + \frac{\tau^2 A}{4})^{-1}$$
$$= R^{2N} - \exp\{-2A^{1/2}\} + [2A^{1/2}s(1) - \frac{1}{i}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))]\exp\{-A^{1/2}\}$$
$$+\frac{1}{i}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))[\exp\{-A^{1/2}\} - R^N]$$
$$+\frac{1}{i}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))[R^N - (R^{N+1} - R^{N-1})$$
$$\times(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(I + \frac{\tau^2 A}{4})^{-1}]$$

and the estimates (6), (12) and (13), we obtain the estimate (15).

3 Applications

First, for application of Theorem 1 and Theorems 3, 4 we consider the mixed problem for hyperbolic-elliptic equation

$$\begin{cases} v_{yy}-(a(x)v_x)_x+\delta v=f(y,x), \quad 0<y<1,\ 0<x<1, \\ -v_{yy}-(a(x)v_x)_x+\delta v=g(y,x), \quad -1<y<0,\ 0<x<1, \\ v(1,x)=v(-1,x), \quad v(0,x)=\varphi(x), \quad 0\le x\le 1, \\ v(y,0)=v(y,1), \quad v_x(y,0)=v_x(y,1), \quad -1\le y\le 1, \\ v(0+,x)=v(0-,x), \quad v_y(0+,x)=v_y(0-,x), \quad 0\le x\le 1. \end{cases} \tag{16}$$

The problem (16) has a unique smooth solution $v(y,x)$ for the smooth $a(x)>0$ $(x\in(0,1))$, $\varphi(x)$ $(x\in[0,1])$ and $f(y,x)$ $(y\in[0,1],\ x\in[0,1])$, $g(y,x)$ $(y\in[-1,0],\ x\in[0,1])$ functions and $\delta=\text{const}>0$. This allows us to reduce the mixed problem (16) to the nonlocal boundary value problem (1) in a Hilbert space H with a self-adjoint positive definite operator A defined by (16). Let us give a number of corollaries of the abstract Theorem 1.

Theorem 5 *The solutions of the nonlocal boundary value problem* (16) *satisfy the stability estimates*

$$\max_{-1\le y\le 1}\|v(y)\|_{L_2[0,1]}\le M\Big[\max_{0\le y\le 1}\|f(y)\|_{L_2[0,1]}$$

$$+\max_{-1\le y\le 0}\|g(y)\|_{L_2[0,1]}+\|\varphi\|_{L_2[0,1]}\Big],$$

$$\max_{-1\le y\le 1}\|v(y)\|_{W_2^1[0,1]}\le M\Big[\max_{0\le y\le 1}\|f(y)\|_{L_2[0,1]}$$

$$+\max_{-1\le y\le 0}\|g(y)\|_{L_2[0,1]}+\|\varphi\|_{W_2^1[0,1]}\Big],$$

$$\max_{-1\le y\le 1}\|v(y)\|_{W_2^2[0,1]}+\max_{-1\le y\le 1}\|v_{yy}(y)\|_{L_2^2[0,1]}\le M\Big[\|\varphi\|_{W_2^2[0,1]}+\|f(0)\|_{L_2[0,1]}$$

$$+\|g(0)\|_{L_2[0,1]}+\max_{0\le y\le 1}\|f_y(y)\|_{L_2[0,1]}+\max_{-1\le y\le 0}\|g_y(y)\|_{L_2[0,1]}\Big]$$

hold, where M does not depend on $f(y,x)$ $(y\in[0,1], x\in[0,1])$, $g(y,x)$ $(y\in[-1,0],\ x\in[0,1])$ and $\varphi(x)$ $(x\in[0,1])$.

The proof of this theorem is based on the abstract Theorem 1 and the symmetry properties of the space operator generated by the problem (16).

Now, the abstract Theorems 3 and 4 are applied in the investigation of difference schemes of second order of accuracy with respect to one variable for approximate solutions of the mixed boundary value problem (16). The discretization of problem (16) is carried out in two steps. In the first step let us define the grid space

$$[0,1]_h=\{x:\ x_n=nh,\ 0\leq n\leq M,\ Mh=1\}.$$

We introduce the Hilbert space $L_{2h}=L_2([0,1]_h)$ of the grid functions $\varphi^h(x)$ defined on $[0,1]_h$, equipped with the norm

$$\parallel \varphi^h \parallel_{L_{2h}}=\Big(\sum_{n=1}^{M-1}|\varphi^h(x)|^2h\Big)^{1/2}.$$

To the differential operator A generated by the problem (16) we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x)=\{-(a(x)\varphi_{\bar{x}})_{x,n}+\delta\varphi_n\}_1^{M-1}, \tag{17}$$

acting in the space of grid functions $\varphi^h(x)=\{\varphi^n\}_0^M$ satisfying the conditions $\varphi^0=\varphi^M,\quad \varphi^1-\varphi^0=\varphi^M-\varphi^{M-1}$. It is known that A_h^x is a self-adjoint positively definite operator in L_{2h} . With the help of A_h^x we arrive at the nonlocal boundary-value problem

$$\begin{cases}\frac{d^2v^h(t,x)}{dy^2}+A_h^xv^h(y,x)=f^h(y,x),\quad 0\leq y\leq 1,\ x\in[0,1]_h,\\ -\frac{d^2v^h(t,x)}{dy^2}+A_h^xv^h(y,x)=f^h(y,x),\quad -1\leq y\leq 0,\ x\in[0,1]_h,\\ v^h(-1,x)=v^h(1,x),\quad v^h(0,x)=\varphi^h(x),\quad x\in[0,1]_h,\\ v^h(0+,x)=v^h(0-,x),\quad v_y^h(0+,x)=v_y^h(0-,x),\quad x\in[0,1]_h\end{cases} \tag{18}$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (18) by the difference scheme (3)

$$
\begin{cases}
\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_k^h+\frac{\tau^2}{4}(A_h^x)^2 u_{k+1}^h=f_k^h(x), \quad x\in[0,1]_h,\\
f_k^h(x)=\{f(y_k,x_n)\}_1^{M-1}, \quad y_k=k\tau,\ 1\le k\le N-1,\ N\tau=1,\\
-\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_k^h=g_k^h(x), \quad x\in[0,1]_h,\\
g_k^h(x)=\{g(y_k,x_n)\}_1^{M-1}, \quad y_k=k\tau,\ -N+1\le k\le -1,\\
u_0^h(x)=\varphi^h(x), \quad u_N^h(x)=u_{-N}^h(x), \quad x\in[0,1]_h,\\
u_1^h(x)-u_0^h(x)-\frac{\tau^2}{2}(f_0^h(x)-A_h^x u_0^h(x))\\
=u_0^h(x)-u_{-1}^h(x)-\frac{\tau^2}{2}(g_0^h(x)-A_h^x u_0^h(x)), \quad x\in[0,1]_h,\\
g_0^h(x)=g^h(0,x), \quad f_0^h(x)=f^h(0,x), \quad x\in[0,1]_h.
\end{cases}
\tag{19}
$$

Theorem 6 *Let τ and h be sufficiently small numbers. Then the solutions of the difference scheme* (19) *satisfy the following stability estimates:*

$$
\max_{-N\le k\le N}\|u_k^h\|_{L_{2h}}\le M_1\Big[\max_{0\le k\le N-1}\parallel f_k^h\parallel_{L_{2h}}
+\max_{-N+1\le k\le 0}\parallel g_k^h\parallel_{L_{2h}}+\|\varphi^h\|_{L_{2h}}\Big],
$$

$$
\max_{-N+1\le k\le N}\|\tau^{-1}(u_k^h-u_{k-1}^h)\|_{L_{2h}}+\max_{-N\le k\le N}\parallel (u_k^h)_x\parallel_{L_{2h}}\le M_1\Big[\max_{0\le k\le N-1}\parallel f_k^h\parallel_{L_{2h}}
+\max_{-N+1\le k\le 0}\parallel g_k^h\parallel_{L_{2h}}+\|\varphi_{\bar{x}}^h\|_{L_{2h}}\Big],
$$

$$
\max_{1\le k\le N-1}\|\tau^{-2}(u_{k+1}^h-2u_k^h+u_{k-1}^h)\|_{L_{2h}}+\max_{-N\le k\le N}\parallel (u_{\bar{x}}^k)_x\parallel_{L_{2h}}
\le M_1\Big[\max_{1\le k\le N-1}\parallel \tau^{-1}(f_k^h-f_{k-1}^h)\parallel_{L_{2h}}+\parallel f_0^h\parallel_{L_{2h}}
+\max_{-N+1\le k\le 0}\parallel \tau^{-1}(g_k^h-g_{k-1}^h)\parallel_{L_{2h}}+\parallel g_0^h\parallel_{L_{2h}}+\|(\varphi_{\bar{x}}^h)_x\parallel_{L_{2h}}\Big].
$$

Here M_1 does not depend on τ, h, $\varphi^h(x)$ and $f_k^h(x)$, $0\le k\le N-1$, g_k^h, $-N+1\le k\le 0$.

The proof of Theorem 6 is based on the abstract Theorem 3, and the symmetry properties of the difference operator A_h^x defined by the formula (17).

Now we replace problem (18) by the difference scheme (4)

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+\frac{1}{2}A_h^x u_k^h+\frac{1}{4}(A_h^x u_{k+1}^h+A_h^x u_{k-1}^h)=f_k^h(x), \\ x\in[0,1]_h,\ f_k^h(x)=\{f(y_k,x_n)\}_1^{M-1},\ y_k=k\tau,\ 1\le k\le N-1,\ N\tau=1, \\ -\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_k^h=g_k^h(x), \quad x\in[0,1]_h, \\ g_k^h(x)=\{g(y_k,x_n)\}_1^{M-1},\ y_k=k\tau,\ -N+1\le k\le -1, \\ u_0^h(x)=\varphi^h(x),\ u_N^h(x)=u_{-N}^h(x),\ x\in[0,1]_h, \\ (I+\frac{\tau^2}{4}A_h^x)(u_1^h(x)-u_0^h(x))-\frac{\tau^2}{2}(f_0^h(x)-A_h^x u_0^h(x)) \\ =u_0^h(x)-u_{-1}^h(x)-\frac{\tau^2}{2}(g_0^h(x)-A_h^x u_0^h(x)), \quad x\in[0,1]_h, \\ g_0^h(x)=g^h(0,x), \quad f_0^h(x)=f^h(0,x), \quad x\in[0,1]_h. \end{cases} \tag{20}$$

Theorem 7 *Let τ and h be sufficiently small numbers.Then the solutions of the difference scheme* (20) *satisfy the following stability estimates:*

$$\max_{-N\le k\le N}\|u_k^h\|_{L_{2h}}\le M_1\Big[\max_{0\le k\le N-1}\parallel f_k^h\parallel_{L_{2h}}$$

$$+\max_{-N+1\le k\le 0}\parallel g_k^h\parallel_{L_{2h}}+\|\varphi^h\|_{L_{2h}}+\tau\|\varphi_{\bar{x}}^h\|_{L_{2h}}\Big],$$

$$\max_{-N+1\le k\le N}\|\tau^{-1}(u_k^h-u_{k-1}^h)\|_{L_{2h}}+\max_{-N\le k\le N}\parallel (u_k^h)_x\parallel_{L_{2h}}\le M_1\Big[\max_{0\le k\le N-1}\parallel f_k^h\parallel_{L_{2h}}$$

$$+\max_{-N+1\le k\le 0}\parallel g_k^h\parallel_{L_{2h}}+\|\varphi_{\bar{x}}^h\|_{L_{2h}}+\tau\|(\varphi_{\bar{x}}^h)_x\|_{L_{2h}}\Big],$$

$$\max_{1\le k\le N-1}\|\tau^{-2}(u_{k+1}^h-2u_k^h+u_{k-1}^h)\|_{L_{2h}}+\max_{-N\le k\le N}\parallel \{(u_{\bar{x}}^k)_{x,n}\}\parallel_{L_{2h}}$$

$$\le M_1\Big[\max_{1\le k\le N-1}\parallel \tau^{-1}(f_k^h-f_{k-1}^h)\parallel_{L_{2h}}+\parallel f_0^h\parallel_{L_{2h}}+\parallel g_0^h\parallel_{L_{2h}}$$

$$+\max_{-N+1\le k\le 0}\parallel \tau^{-1}(g_k^h-g_{k-1}^h)\parallel_{L_{2h}}+\|(\varphi_{\bar{x}}^h)_x\}\parallel_{L_{2h}}+\tau\|(\varphi_{\bar{x}}^h)_{xx}\|_{L_{2h}}\Big].$$

Here M_1 does not depend on τ, h, $\varphi^h(x)$ and $f_k^h(x)$, $0\le k\le N-1$, g_k^h, $-N+1\le k\le 0$.

The proof of Theorem 7 is based on the abstract Theorem 4, and the symmetry properties of the difference operator A_h^x defined by the formula (17).

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1,\ 1 \le k \le n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$ we consider the mixed boundary value problem for the multidimensional hyperbolic-elliptic equation

$$\begin{cases} v_{yy} - \sum\limits_{r=1}^{n} (a_r(x) v_{x_r})_{x_r} = f(y,x), \quad y > 0,\ x = (x_1, \ldots, x_n) \in \Omega, \\ -v_{yy} - \sum\limits_{r=1}^{n} (a_r(x) v_{x_r})_{x_r} = g(y,x), \quad y < 0,\ x = (x_1, \ldots, x_n) \in \Omega, \\ v(1,x) = v(-1,x), \quad v(0,x) = \varphi(x), \quad x \in \overline{\Omega}, \\ u(y,x) = 0, \quad x \in S, \quad -1 \le y \le 1, \end{cases} \tag{21}$$

where $a_r(x)$ $(x \in \Omega)$, $\varphi(x)$ $(x \in \overline{\Omega})$ and $f(y,x)$ $(y \in (0,1),\ x \in \Omega)$, $g(y,x)$ $(y \in (-1,0),\ x \in \Omega)$ are given smooth functions and $a_r(x) > 0$.

We introduce the Hilbert spaces $L_2(\overline{\Omega})$ — the space of all integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{L_2(\overline{\Omega})} = \Big\{ \int \cdots \int\limits_{x \in \overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_n \Big\}^{1/2}.$$

The problem (21) has a unique smooth solution $v(y,x)$ for the smooth $a_r(x) > 0$ and $f(y,x)$, $g(y,x)$ functions. This allows us to reduce the mixed problem (21) to the nonlocal boundary value problem (1) in a Hilbert space H with a self-adjoint positively definite operator A defined by (21). Let us give a number of corollaries of the abstract Theorem 1.

Theorem 8 *The solutions of the nonlocal boundary value problem* (21) *satisfy the stability estimates*

$$\begin{aligned} \max_{1 \le y \le 1} \|v(y)\|_{L_2(\overline{\Omega})} \ &\le\ M_1 \Big[\max_{0 \le y \le 1} \|f(y)\|_{L_2(\overline{\Omega})} \\ + \| \varphi \|_{L_2(\overline{\Omega})} \ &+\ \max_{-1 \le y \le 0} \|g(y)\|_{L_2(\overline{\Omega})} \Big], \end{aligned}$$

$$\begin{aligned} \max_{-1 \le y \le 1} \|v(y)\|_{W_2^1(\overline{\Omega})} \ &\le\ M_1 \Big[\max_{0 \le y \le 1} \| f(y) \|_{L_2(\overline{\Omega})} \\ + \| \varphi \|_{W_2^1(\overline{\Omega})} \ &+\ \max_{-1 \le y \le 0} \|g(y)\|_{L_2(\overline{\Omega})} \Big], \end{aligned}$$

$$\begin{aligned}&\max_{-1\le y\le 1}\|v(y)\|_{W_2^2(\overline{\Omega})}+\max_{1\le y\le 1}\|v_{yy}(y)\|_{L_2(\overline{\Omega})}\\ \le\ & M_1\Big[\max_{0\le y\le 1}\|f_y(y)\|_{L_2(\overline{\Omega})}+\parallel f(0)\parallel_{L_2(\overline{\Omega})}\\ +\ & \parallel\varphi\parallel_{W_2^2(\overline{\Omega})}+\parallel g(0)\parallel_{L_2(\overline{\Omega})}+\max_{-1\le y\le 0}\parallel g_y(y)\parallel_{L_2(\overline{\Omega})}\Big]\end{aligned}$$

hold, where M_1 does not depend on $f(y,x)$ $(y\in(0,1),\ x\in\Omega)$, $g(y,x)$ $(y\in(-1,0),\ x\in\Omega)$ and $\varphi(x)$ $(x\in\overline{\Omega})$.

The proof of this theorem is based on the abstract Theorem 1 and the symmetry properties of the space operator generated by the problem (21).

Now, the abstract Theorems 3 and 4 are applied in the investigation of difference schemes of the second order of accuracy with respect to one variable for approximate solutions of the mixed boundary value problem (21). The discretization of problem (21) is carried out in two steps. In the first step let us define the grid sets

$$\begin{gathered}\widetilde{\Omega}_h=\{x=x_m=(h_1m_1,\dots,h_nm_n),\quad m=(m_1,\dots,m_n),\\ 0\le m_r\le N_r,\quad h_rN_r=L,\quad r=1,\dots,n\},\\ \Omega_h=\widetilde{\Omega}_h\cap\Omega,\qquad S_h=\widetilde{\Omega}_h\cap S.\end{gathered}$$

We introduce the Banach space $L_{2h}=L_2(\widetilde{\Omega}_h)$ of the grid functions $\varphi^h(x)=\{\varphi(h_1m_1,\dots,h_nm_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norm

$$\parallel\varphi^h\parallel_{L_2(\widetilde{\Omega}_h)}=\Big(\sum_{x\in\widetilde{\Omega}_h}|\varphi^h(x)|^2h_1\cdots h_n\Big)^{1/2}.$$

To the differential operator A generated by the problem (21) we assign the difference operator A_h^x by the formula

$$A_h^xu_x^h=-\sum_{r=1}^n(a_r(x)u_{\bar{x}_r}^h)_{x_r,j_r} \tag{22}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x)=0$ for all $x\in S_h$. It is known that A_h^x is a self-adjoint positively definite operator in $L_2(\widetilde{\Omega}_h)$. With the help of A_h^x we arrive at the nonlocal boundary-value problem

$$\begin{cases}\frac{d^2v^h(t,x)}{dy^2}+A_h^xv^h(y,x)=f^h(y,x),\qquad 0\le y\le 1,\ x\in\widetilde{\Omega}_h,\\ -\frac{d^2v^h(t,x)}{dy^2}+A_h^xv^h(y,x)=f^h(y,x),\quad -1\le y\le 0,\ x\in\widetilde{\Omega}_h,\\ v^h(-1,x)=v^h(1,x),\quad v^h(0,x)=\varphi^h(x),\quad x\in\widetilde{\Omega}_h,\\ v^h(0+,x)=v^h(0-,x),\quad v_y^h(0+,x)=v_y^h(0-,x),\quad x\in\widetilde{\Omega}_h\end{cases} \tag{23}$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (18) by the difference scheme (3)

$$\begin{cases} \frac{u^h_{k+1}(x)-2u^h_k(x)+u^h_{k-1}(x)}{\tau^2}+A^x_h u^h_k+\frac{\tau^2}{4}(A^x_h)^2u^h_{k+1}=f^h_k(x),\ x\in\widetilde{\Omega}_h, \\ f^h_k(x)=\{f(y_k,x_n)\}_1^{M-1}, \quad y_k=k\tau,\ 1\le k\le N-1,\ N\tau=1, \\ -\frac{u^h_{k+1}(x)-2u^h_k(x)+u^h_{k-1}(x)}{\tau^2}+A^x_h u^h_k=g^h_k(x), \qquad x\in\widetilde{\Omega}_h, \\ g^h_k(x)=\{g(y_k,x_n)\}_1^{M-1}, \quad y_k=k\tau, \quad -N+1\le k\le -1, \\ u^h_0(x)=\varphi^h(x), \quad u^h_N(x)=u^h_{-N}(x), \quad x\in\widetilde{\Omega}_h, \\ u^h_1(x)-u^h_0(x)-\frac{\tau^2}{2}(f^h_0(x)-A^x_h u^h_0(x)) \\ =u^h_0(x)-u^h_{-1}(x)-\frac{\tau^2}{2}(g^h_0(x)-A^x_h u^h_0(x)), \quad x\in\widetilde{\Omega}_h, \\ g^h_0(x)=g^h(0,x), \quad f^h_0(x)=f^h(0,x), \quad x\in\widetilde{\Omega}_h. \end{cases} \tag{24}$$

Theorem 9 *Let τ and $|h|$ be sufficiently small numbers.Then the solutions of the difference scheme* (19) *satisfy the following stability estimates:*

$$\max_{-N\le k\le N}\|u^h_k\|_{L_{2h}}\le M_1\Big[\max_{0\le k\le N-1}\|\ f^h_k\ \|_{L_{2h}} + \max_{-N+1\le k\le 0}\|\ g^h_k\ \|_{L_{2h}}+\|\varphi^h\|_{L_{2h}}\Big],$$

$$\max_{-N+1\le k\le N}\|\tau^{-1}(u^h_k-u^h_{k-1})\|_{L_{2h}}+\max_{-N\le k\le N}\sum_{r=1}^n\|(u^h_k)_{x_r,j_r}\|_{L_{2h}}$$

$$\le M_1\Big[\max_{0\le k\le N-1}\|\ f^h_k\ \|_{L_{2h}}+\max_{-N+1\le k\le 0}\|\ g^h_k\ \|_{L_{2h}}+\sum_{r=1}^n\|(\varphi^h)_{\bar{x}_r,j_r}\|_{L_{2h}} Bigr],$$

$$\max_{1\le k\le N-1}\|\tau^{-2}(u^h_{k+1}-2u^h_k+u^h_{k-1})\|_{L_{2h}}+\max_{-N\le k\le N}\sum_{r=1}^n\|(u^h_k)_{\bar{x}_r x_r,j_r})\|_{L_{2h}}$$

$$\le M_1\Big[\max_{1\le k\le N-1}\|\ \tau^{-1}(f^h_k-f^h_{k-1})\ \|_{L_{2h}}+\|\ f^h_0\ \|_{L_{2h}}$$

$$+\max_{-N+1\le k\le 0}\|\ \tau^{-1}(g^h_k-g^h_{k-1})\ \|_{L_{2h}}+\|\ g^h_0\ \|_{L_{2h}}+\sum_{r=1}^n\||(\varphi^h)_{\bar{x}_r x_r,j_r}\|_{L_{2h}}\Big].$$

Here M_1 does not depend on $\tau, h,\ \varphi^h(x)$ and $f^h_k(x),\ 0\le k\le N-1,\ g^h_k,\ -N+1\le k\le 0$.

The proof of Theorem 9 is based on the abstract Theorem 3, and the symmetry properties of the difference operator A_h^x defined by the formula (22).

Now we replace problem (18) by the difference scheme (4)

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+\frac{1}{2}A_h^x u_k^h+\frac{1}{4}(A_h^x u_{k+1}^h+A_h^x u_{k-1}^h)=f_k^h(x), \\ x\in\widetilde{\Omega}_h,\ f_k^h(x)=\{f(y_k,x_n)\}_1^{M-1},\ y_k=k\tau,\ 1\le k\le N-1,\ N\tau=1, \\ -\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_k^h=g_k^h(x),\quad x\in\widetilde{\Omega}_h, \\ g_k^h(x)=\{g(y_k,x_n)\}_1^{M-1},\quad y_k=k\tau,\quad -N+1\le k\le -1, \\ u_0^h(x)=\varphi^h(x),\quad u_N^h(x)=u_{-N}^h(x),\quad x\in\widetilde{\Omega}_h, \\ (I+\frac{\tau^2}{4}A_h^x)(u_1^h(x)-u_0^h(x))-\frac{\tau^2}{2}(f_0^h(x)-A_h^x u_0^h(x)) \\ =u_0^h(x)-u_{-1}^h(x)-\frac{\tau^2}{2}(g_0^h(x)-A_h^x u_0^h(x)),\ x\in\widetilde{\Omega}_h, \\ g_0^h(x)=g^h(0,x),\quad f_0^h(x)=f^h(0,x),\quad x\in\widetilde{\Omega}_h. \end{cases} \tag{25}$$

Theorem 10 *Let τ and $|h|$ be sufficiently small numbers. Then the solutions of the difference scheme* (25) *satisfy the following stability estimates:*

$$\max_{-N\le k\le N}\|u_k^h\|_{L_{2h}}\le M_1\Big[\max_{0\le k\le N-1}\|\ f_k^h\ \|_{L_{2h}}$$
$$+\max_{-N+1\le k\le 0}\|\ g_k^h\ \|_{L_{2h}}+\|\varphi^h\|_{L_{2h}}+\tau\sum_{r=1}^n\|(\varphi^h)_{\bar{x}_r,j_r}\|_{L_{2h}}\Big],$$

$$\max_{-N+1\le k\le N}\|\tau^{-1}(u_k^h-u_{k-1}^h)\|_{L_{2h}}+\max_{-N\le k\le N}\sum_{r=1}^n\|(u_k^h)_{x_r,j_r}\|_{L_{2h}}$$
$$\le M_1\Big[\max_{0\le k\le N-1}\|\ f_k^h\ \|_{L_{2h}}+\max_{-N+1\le k\le 0}\|\ g_k^h\ \|_{L_{2h}}$$
$$+\sum_{r=1}^n\|(\varphi^h)_{\bar{x}_r,j_r}\|_{L_{2h}}+\tau\sum_{r=1}^n\|(\varphi^h)_{\bar{x}_r x_r,j_r}\|_{L_{2h}}\Big],$$

$$\max_{1\le k\le N-1}\|\tau^{-2}(u_{k+1}^h-2u_k^h+u_{k-1}^h)\|_{L_{2h}}+\max_{-N\le k\le N}\sum_{r=1}^n\|(u_k^h)_{\bar{x}_r x_r,j_r}\|_{L_{2h}}$$
$$\le M_1\Big[\max_{1\le k\le N-1}\|\ \tau^{-1}(f_k^h-f_{k-1}^h)\ \|_{L_{2h}}+\|\ f_0^h\ \|_{L_{2h}}+\|\ g_0^h\ \|_{L_{2h}}$$

$$+\max_{-N+1\le k\le 0}\parallel \tau^{-1}(g_k^h-g_{k-1}^h)\parallel_{L_{2h}}$$

$$+\sum_{r=1}^{n}\|(\varphi^h)_{\bar{x}_r x_r,j_r}\|_{L_{2h}}+\tau\sum_{r=1}^{n}\|(\varphi^h)_{\bar{x}_r x_r x_r,j_r}\|_{L_{2h}}\Big].$$

Here M_1 does not depend on τ, h, $\varphi^h(x)$ and $f_k^h(x)$, $0\le k\le N-1$, g_k^h, $-N+1\le k\le 0$.

The proof of Theorem 10 is based on the abstract Theorem 4, and the symmetry properties of the difference operator A_h^x defined by the formula (22).

References

[1] Salahatdinov M. S., *Equations of Mixed-Composite Type*, FAN, Tashkent, 1974. (in Russian)

[2] Djuraev T. D., *Boundary Value Problems for Equations of Mixed and Mixed-Composite Types*, FAN, Tashkent, 1979. (in Russian)

[3] Bazarov D. and Soltanov H., *Some Local and Nonlocal Boundary Value Problems for Equations of Mixed and Mixed-Composite Types*, Ylym, Ashgabat, 1995. (in Russian)

[4] Krein S. G., *Linear Differential Equations in a Banach Space*, Nauka, Moscow, 1966. (in Russian).

[5] Ashyralyev A., Yurtsever A. and Judakova G., *On hyperbolic-elliptic equations in a Hilbert space*, in: Some Problems of Applied Mathematics, Fatih University, Istanbul, Turkey, 2000, pp. 70–79.

[6] Ashyralyev A. and Judakova G., *A note on the nonlocal boundary value problem for hyperbolic-elliptic equations in a Hilbert space*, in: Proceeding of the International Conference "Modern Mathematical Problems", Astana, October 3-5, 2002, pp. 66-70.

[7] Ashyralyev A. and Sobolevskii P. E., *A note on the difference schemes for hyperbolic equations*, Abstract and Applied Analysis, **16** (2001), No. 2, 63–70.

[8] Eidelman Yu. S, *Well-posedness of the direct and inverse problems for a differential equation in a Hilbert space*, Dokl. Akad. Nauk Ukraini, Ser. A Fiz.-Mat. and Tech. Sciences, No. 12 (1993), 17–21. (in Russian)

[9] Tikhonov I. V. and Eidelman Yu. S, *On well-posed problems for a special kind of evolution equation*, Mat. Zametki, **56** (1994), No. 2, 99–113. (in Russian)

[10] Ashyralyev A. and Soltanov H., *On elliptic-parabolic equations in a Hilbert space*, in: Proceeding of the IMM and CS of Turkmenistan, Ashgabat, No. 4, 1995, pp. 101–104. (in Russian)

[11] Ashyralyev A. and Kendirli B., *Well-posedness of the nonlocal boundary value problem for elliptic equations*, Functional Differential Equations, **9** (2002), No. 1–2, 35–55.

[12] Gorbachuk V. I. and Gorbachuk M. L., *Boundary Value Problems for Differential-Operator Equations*, Kiev, Naukova Dumka, 1984. (in Russian)

[13] Ashyralyev A. and Yurtsever H. A., *On a nonlocal boundary value problem for semilinear hyperbolic-parabolic equations*, Nonlinear Analysis. Theory, Methods, and Applications, **47** (2001), 3585–3592.

[14] Ashyralyev A. and Yurtsever H. A., *Difference schemes for hyperbolic-parabolic equations*, Functional Differential Equations, **7** (2000), No. 3–4, 189–203.

[15] Ashyralyev A. and Muradov I., *On hyperbolic-parabolic equations in a Hilbert space*, in: Probl.of Math. and of Econ. Mod. of Turkm., Ashgabat, 1995, pp. 26–28. (in Russian)

[16] Ashyralyev A., *The stability of difference schemes for partial differential equations of mixed types*, in: Proc. of the Sec. Int. Symp. on Math. and Comp. Appl., Baku, 1999, pp. 314–316.

[17] Ashyralyev A. and Karatay I., *On the second order of accuracy difference schemes of the nonlocal boundary value problem for parabolic equations*, Functional Differential Equations, **10** (2003), No. 1–2, 45–63.

[18] Wang Y. G. and Oberguggenberger M., *Nonlinear parabolic equations with regularized derivatives*, Journal of Mathematical Analysis and Applications, **233** (1999), No. 2, 644–658.

[19] Beyn W.-J. and Garay B. M., *Estimates of variable stepsize Runge–Kutta methods for sectorial evolution equations with nonsmooth data*, Applied Numerical Mathematics, **41** (2002), No. 3, 369–400.

[20] Rautmann R., *$H^{2,r}$-convergent approximation schemes to the Navier-Stokes equations*, Nonlinear Analysis. Theory, Methods, and Applications, **30** (1997), No. 4, 1915–1926.

[21] Ashyralyev A., *Method of Positive Operators of Investigations of the High Order of Accuracy Difference Schemes for Parabolic and Elliptic Equations*, Doctor of Sciences Thesis, Kiev, 1992. (in Russian).

[22] Ashyralyev A., *On well-posedness of the nonlocal boundary value problem for elliptic equations*, Numerical Functional Analysis and Optimization, **24** (2003), No. 1–2, 1–15.

[23] Ashyralyev A. and Hanalyev A., *Coercive stability of nonlocal boundary value problem for parabolic equations in spaces of smooth functions*, Izv. Akad. Nauk Turkmenistan. Ser. Fiz.-Tekhn. Khim. Geol. Nauk, No. 3 (1996), 3–10. (in Turkmen)

[24] Ashyralyev A. and Hanalyev A., *Coercive estimate in Hölder norms for parabolic equations with a dependent operator*, in: Modeling the Processes in Exploration of Gas Deposits and Applied Problems of Theoretical Gas Hydrodynamics, Ilim, Ashgabat, 1998, pp.154-162. (in Russian)

[25] Ashyralyev A., Hanalyev A. and Sobolevskii P. E., *Coercive solvability of nonlocal boundary value problem for parabolic equations*, Abstract and Applied Analysis, **6** (2001), No. 1, 53–61.

[26] Skubachevskii A. L., *Elliptic Functional Differential Equations and Applications*, Operator Theory – Advances and Applications, vol. 91, Birkhäuser Verlag, 1997.