

Interaction of Species in Patchy Environment with Cross Diffusion

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Abstract

The aim of the present paper is to investigate the dynamics of two species living in a habitat of two identical patches linked by migration, in which the per capita migration rate of each species is influenced not only by its own but also by the other one's density, *i.e.*, there is cross diffusion present. Numerical studies show that at a critical value of the bifurcation parameter the system undergoes a Turing bifurcation, *i.e.*, the stable constant steady state loses its stability and spatially nonconstant stationary solutions, patterns emerge and the cross migration response is an important factor that should not be ignored when pattern emerges.

Key words: Cross diffusion, Diffusive instability, Pattern formation.

1 Introduction

The theory of spatial pattern formation via Turing bifurcations (see [8]), wherein an equilibrium of a nonlinear system is asymptotically stable in the absence of diffusion but unstable in the presence of diffusion, plays an important role in ecology, embryology and elsewhere in biology and chemistry (see [1, 2, 5, 6]). Since the relation between the organisms and the space seems to be essential to the stability of an ecological system, the effect of diffusion on the possibility of species coexistence in an ecological community has been an important subject in population biology (see [4, 7, 8]). We consider two species living in a habitat of two identical patches linked by migration and we show that at a critical value of the bifurcation parameter the system undergoes a Turing bifurcation, *i. e.*, the stable constant steady state loses its stability and spatially nonconstant stationary solutions, and patterns emerge.

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This paper is organized as follows: In Section 2 the model is built, in Section 3 its linearization is treated and the conditions for the Turing bifurcation are established (these are the main results of this paper), in Section 4 we consider examples to illustrate what can be expected, in Section 5 we summarize the main conclusions of the study.

2 The model

We consider two species living in a habitat of two identical patches linked by migration.

Let $u_i(t, j) :=$ density of species i in patch j at time t , for $i, j = 1, 2; t \in \mathbb{R}$. The interaction is described as a system of differential equations as follows:

$$\begin{aligned}
 \dot{u}_1(t, 1) &= u_1(t, 1) F_1(u_1(t, 1), u_2(t, 1)) \\
 &\quad + d_1(\rho_1(u_2(t, 2))u_1(t, 2) - \rho_1(u_2(t, 1))u_1(t, 1)), \\
 \dot{u}_2(t, 1) &= u_2(t, 1) F_2(u_1(t, 1), u_2(t, 1)) \\
 &\quad + d_2(\rho_2(u_1(t, 2))u_2(t, 2) - \rho_2(u_1(t, 1))u_2(t, 1)), \\
 \dot{u}_1(t, 2) &= u_1(t, 2) F_1(u_1(t, 2), u_2(t, 2)) \\
 &\quad + d_1(\rho_1(u_2(t, 1))u_1(t, 1) - \rho_1(u_2(t, 2))u_1(t, 2)), \\
 \dot{u}_2(t, 2) &= u_2(t, 2) F_2(u_1(t, 2), u_2(t, 2)) \\
 &\quad + d_2(\rho_2(u_1(t, 1))u_2(t, 1) - \rho_2(u_1(t, 2))u_2(t, 2)),
 \end{aligned} \tag{1}$$

where $d_i > 0$ ($i = 1, 2$) are the diffusion coefficients and $\rho_i \in C^1$ ($i = 1, 2$) are positive functions modeling the cross-diffusion effect. We say that the cross diffusion is strong if $|\rho'_{iu_k}|$ ($i \neq k$) is large. If $\rho_i = 1$, $i = 1, 2$, then we have mere “self-diffusion”.

First we consider the kinetic system without migration, *i.e.*, $d_1 = d_2 = 0$:

$$\begin{aligned}
 \dot{u}_1(t, 1) &= u_1(t, 1) F_1(u_1(t, 1), u_2(t, 1)), \\
 \dot{u}_2(t, 1) &= u_2(t, 1) F_2(u_1(t, 1), u_2(t, 1)), \\
 \dot{u}_1(t, 2) &= u_1(t, 2) F_1(u_1(t, 2), u_2(t, 2)), \\
 \dot{u}_2(t, 2) &= u_2(t, 2) F_2(u_1(t, 2), u_2(t, 2)).
 \end{aligned} \tag{2}$$

Suppose that $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ is a positive homogeneous equilibrium of (2). The

Jacobian matrix of the system without diffusion linearized at $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ is

$$J_k = \begin{pmatrix} \bar{u}_1 F'_{1u_1} & \bar{u}_1 F'_{1u_2} & 0 & 0 \\ \bar{u}_2 F'_{2u_1} & \bar{u}_2 F'_{2u_2} & 0 & 0 \\ 0 & 0 & \bar{u}_1 F'_{1u_1} & \bar{u}_1 F'_{1u_2} \\ 0 & 0 & \bar{u}_2 F'_{2u_1} & \bar{u}_2 F'_{2u_2} \end{pmatrix}, \quad (3)$$

$$\det(J_k - \lambda I) = \begin{vmatrix} \bar{u}_1 F'_{1u_1} - \lambda & \bar{u}_1 F'_{1u_2} & 0 & 0 \\ \bar{u}_2 F'_{2u_1} & \bar{u}_2 F'_{2u_2} - \lambda & 0 & 0 \\ 0 & 0 & \bar{u}_1 F'_{1u_1} - \lambda & \bar{u}_1 F'_{1u_2} \\ 0 & 0 & \bar{u}_2 F'_{2u_1} & \bar{u}_2 F'_{2u_2} - \lambda \end{vmatrix}. \quad (4)$$

The characteristic polynomial is

$$\begin{aligned} D_4(\lambda) &= (D_2(\lambda))^2, & D_2(\lambda) &= \lambda^2 - \lambda(\bar{u}_1 F'_{1u_1} + \bar{u}_2 F'_{2u_2}) \\ & & &+ (F'_{1u_1} F'_{2u_2} - F'_{2u_1} F'_{1u_2}) \bar{u}_1 \bar{u}_2. \end{aligned} \quad (5)$$

Assume that

$$\bar{u}_1 F'_{1u_1} + \bar{u}_2 F'_{2u_2} < 0 \quad \text{and} \quad (F'_{1u_1} F'_{2u_2} - F'_{2u_1} F'_{1u_2}) > 0, \quad (6)$$

then the coexistence equilibrium point $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ is linearly asymptotically stable.

3 The linearized problem

Returning to system (1), we see that $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ is also a spatially homogeneous equilibrium of the system with diffusion. The Jacobian matrix of the system with diffusion at $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ can be written as

$$J_D = \begin{pmatrix} \bar{u}_1 F'_{1u_1} - d_1 \rho_1 & \bar{u}_1 F'_{1u_2} - d_1 \rho'_1 \bar{u}_1 & d_1 \rho_1 & d_1 \rho'_1 \bar{u}_1 \\ \bar{u}_2 F'_{2u_1} - d_2 \rho'_2 \bar{u}_2 & \bar{u}_2 F'_{2u_2} - d_2 \rho_2 & d_2 \rho'_2 \bar{u}_2 & d_2 \rho_2 \\ d_1 \rho_1 & d_1 \rho'_1 \bar{u}_1 & \bar{u}_1 F'_{1u_1} - d_1 \rho_1 & \bar{u}_1 F'_{1u_2} - d_1 \rho'_1 \bar{u}_1 \\ d_2 \rho'_2 \bar{u}_2 & d_2 \rho_2 & \bar{u}_2 F'_{2u_1} - d_2 \rho'_2 \bar{u}_2 & \bar{u}_2 F'_{2u_2} - d_2 \rho_2 \end{pmatrix}, \quad (7)$$

where ρ_1 and ρ'_1 are to be taken at \bar{u}_2 and ρ_2, ρ'_2 at \bar{u}_1 .

$$\det(J_D - \lambda I) = \begin{vmatrix} \bar{u}_1 F'_{1u_1} - d_1 \rho_1 - \lambda & \bar{u}_1 F'_{1u_2} - d_1 \rho'_1 \bar{u}_1 & d_1 \rho_1 & d_1 \rho'_1 \bar{u}_1 \\ \bar{u}_2 F'_{2u_1} - d_2 \rho'_2 \bar{u}_2 & \bar{u}_2 F'_{2u_2} - d_2 \rho_2 - \lambda & d_2 \rho'_2 \bar{u}_2 & d_2 \rho_2 \\ d_1 \rho_1 & d_1 \rho'_1 \bar{u}_1 & \bar{u}_1 F'_{1u_1} - d_1 \rho_1 - \lambda & \bar{u}_1 F'_{1u_2} - d_1 \rho'_1 \bar{u}_1 \\ d_2 \rho'_2 \bar{u}_2 & d_2 \rho_2 & \bar{u}_2 F'_{2u_1} - d_2 \rho'_2 \bar{u}_2 & \bar{u}_2 F'_{2u_2} - d_2 \rho_2 - \lambda \end{vmatrix}. \quad (8)$$

Using the properties of determinant we get

$$\begin{vmatrix} \bar{u}_1 F'_{1u_1} - \lambda & \bar{u}_1 F'_{1u_2} & d_1 \rho_1 & d_1 \rho'_1 \bar{u}_1 \\ \bar{u}_2 F'_{2u_1} & \bar{u}_2 F'_{2u_2} - \lambda & d_2 \rho'_2 \bar{u}_2 & d_2 \rho_2 \\ 0 & 0 & \bar{u}_1 F'_{1u_1} - 2d_1 \rho_1 - \lambda & \bar{u}_1 F'_{1u_2} - 2d_1 \rho'_1 \bar{u}_1 \\ 0 & 0 & \bar{u}_2 F'_{2u_1} - 2d_2 \rho'_2 \bar{u}_2 & \bar{u}_2 F'_{2u_2} - 2d_2 \rho_2 - \lambda \end{vmatrix} \quad (9)$$

$$= D_2(\lambda)(\lambda^2 - \lambda(\bar{u}_1 F'_{1u_1} + \bar{u}_2 F'_{2u_2} - 2(d_1 \rho_1 + d_2 \rho_2)) + (\bar{u}_1 F'_{1u_1} - 2d_1 \rho_1)(\bar{u}_2 F'_{2u_2} - 2d_2 \rho_2) - (\bar{u}_2 F'_{2u_1} - 2d_2 \rho'_2 \bar{u}_2)(\bar{u}_1 F'_{1u_2} - 2d_1 \rho'_1 \bar{u}_1).$$

We know that $D_2(\lambda)$ has two roots with negative real parts. By (6), clearly, $(\bar{u}_1 F'_{1u_1} + \bar{u}_2 F'_{2u_2} - 2(d_1 \rho_1 + d_2 \rho_2)) < 0$. The other polynomial will have a negative and a positive root if the constant term is negative. There are three important special cases of interactions.

3.1 Predator-prey interaction

For predator-prey interaction, $\rho_1 \in C^1$ is a positive increasing function of u_2 , the density of the predator, $\rho'_1 > 0$, and $\rho_2 \in C^1$ is a positive decreasing function of u_1 , the density of the prey, $\rho'_2 < 0$. The idea is that the dependence of the diffusion coefficient on the density of the other species reflects the inclination of a prey (or an activator) to leave from a certain patch because of the danger (or the inhibition) and the tendency of a predator (or the inhibition) to stay at a certain patch because of the abundance of prey (or an activator) (see [1, 3]). The functions ρ_i model the cross-diffusion effect. We say that the cross diffusion is strong if $|\rho'_{iu_k}|$ ($i \neq k$) is large. If by varying a parameter $|\rho'_{iu_k}|$ ($i \neq k$) is increasing, then we say that the cross diffusion effect is increasing. Suppose that

$$F'_{1u_1} > 0, F'_{1u_2} < 0, F'_{2u_1} > 0, F'_{2u_2} < 0, \rho'_1 > 0 \text{ and } \rho'_2 < 0. \quad (10)$$

Now (all functions are taken at $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$),

$$\begin{aligned} & \det(J_D - \lambda I) \\ = & D_2(\lambda)(\lambda^2 - \lambda(\bar{u}_1 F'_{1u_1} + \bar{u}_2 F'_{2u_2} - 2(d_1\rho_1 + d_2\rho_2)) \\ & + (\bar{u}_1 F'_{1u_1} - 2d_1\rho_1)(\bar{u}_2 F'_{2u_2} - 2d_2\rho_2) - (\bar{u}_2 F'_{2u_1} - 2d_2\rho'_2\bar{u}_2)(\bar{u}_1 F'_{1u_2} - 2d_1\rho'_1\bar{u}_1) \end{aligned} \quad (11)$$

$$\begin{aligned} = & D_2(\lambda)[\lambda^2 - \lambda(\bar{u}_1 F'_{1u_1} + \bar{u}_2 F'_{2u_2} - 2(d_1\rho_1 + d_2\rho_2)) \\ & + \bar{u}_1\bar{u}_2(F'_{1u_1}F'_{2u_2} - F'_{2u_1}F'_{1u_2}) - 2d_1\bar{u}_2F'_{2u_2}\rho_1 + 2d_2\rho'_2F'_{1u_2}\bar{u}_1\bar{u}_2 \\ & + 2d_1\rho'_1\bar{u}_1(\bar{u}_2F'_{2u_1} - 2d_2\rho'_2\bar{u}_2) + 2d_2\rho_2(2d_1\rho_1 - \bar{u}_1F'_{1u_1})]. \end{aligned} \quad (12)$$

Suppose that the parameters have been chosen so that

$$2d_1\rho_1 - \bar{u}_1F'_{1u_1} < 0. \quad (13)$$

By the properties of the model and conditions (6) the other terms in the constant of the polynomial square brackets are positive. If we have achieved this, we may increase ρ_2 and the constant term becomes negative. In other words, the equilibrium $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ will be unstable (see Example 1).

3.2 Competitive interaction

For competitive interaction, $\rho_1 \in C^1$ is a positive increasing function of u_2 , the density of the competitor, with analogous conditions on ρ_2 . The idea is that high density of the competitor increases the diffusion rate of the species, then

$$F'_{1u_2} < 0, \quad F'_{2u_1} < 0, \quad \rho'_1 > 0 \quad \text{and} \quad \rho'_2 > 0, \quad (14)$$

and

$$\begin{aligned} & \det(J_D - \lambda I) \\ = & D_2(\lambda)(\lambda^2 - \lambda(\bar{u}_1 F'_{1u_1} + \bar{u}_2 F'_{2u_2} - 2(d_1\rho_1 + d_2\rho_2)) \\ & + (\bar{u}_1 F'_{1u_1} - 2d_1\rho_1)(\bar{u}_2 F'_{2u_2} - 2d_2\rho_2) - (\bar{u}_2 F'_{2u_1} - 2d_2\rho'_2\bar{u}_2)(\bar{u}_1 F'_{1u_2} - 2d_1\rho'_1\bar{u}_1) \\ = & D_2(\lambda)[\lambda^2 - \lambda(\bar{u}_1 F'_{1u_1} + \bar{u}_2 F'_{2u_2} - 2(d_1\rho_1 + d_2\rho_2)) \\ & + \bar{u}_1\bar{u}_2(F'_{1u_1}F'_{2u_2} - F'_{2u_1}F'_{1u_2}) + 2d_1\bar{u}_2(\rho'_1 F'_{2u_1}\bar{u}_1 - F'_{2u_2}\rho_1) \\ & + 2d_2\bar{u}_1(\rho'_2 F'_{1u_2}\bar{u}_2 - F'_{1u_1}\rho_2) + 4d_1d_2\rho_1\rho_2(1 - \frac{\rho'_1\rho'_2}{\rho_1\rho_2}\bar{u}_1\bar{u}_2)]. \end{aligned} \quad (16)$$

We know that $D_2(\lambda)$ has two roots with negative real parts. The other polynomial will have a negative and a positive root if the constant term is negative. Clearly, $(\rho'_1 F'_{2u_1} \bar{u}_1 - F'_{2u_2} \rho_1) = \rho_1 \left(\frac{\rho'_1}{\rho_1} F'_{2u_1} \bar{u}_1 - F'_{2u_2} \right) < 0$ if $\frac{\rho'_1}{\rho_1}$ is big enough, $(\rho'_2 F'_{1u_2} \bar{u}_2 - F'_{1u_1} \rho_2) = \rho_2 \left(\frac{\rho'_2}{\rho_2} F'_{1u_2} \bar{u}_2 - F'_{1u_1} \right) < 0$ if $\frac{\rho'_2}{\rho_2}$ is big enough and $\rho_1 \rho_2 \left(1 - \frac{\rho'_1 \rho'_2}{\rho_1 \rho_2} \bar{u}_1 \bar{u}_2 \right) < 0$ if $\frac{\rho'_1 \rho'_2}{\rho_1 \rho_2}$ is big enough. If we have achieved this, we may increase d_1 and/or d_2 and the constant term becomes negative. In other words, the equilibrium $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ will be unstable (see Example 2).

3.3 Mutualistic interaction

For mutualistic interaction, $\rho_1 \in C^1$ is a positive decreasing function of u_2 , with analogous conditions on ρ_2 . The idea is that these migration functions describe the inclination of individuals of one species to stay at a certain patch due to the attraction by the other species in the patch, then

$$F'_{1u_2} > 0, \quad F'_{2u_1} > 0, \quad \rho'_1 < 0 \quad \text{and} \quad \rho'_2 < 0, \quad (17)$$

and

$$\begin{aligned} & \det(J_D - \lambda I) \\ = & D_2(\lambda)(\lambda^2 - \lambda(\bar{u}_1 F'_{1u_1} + \bar{u}_2 F'_{2u_2} - 2(d_1 \rho_1 + d_2 \rho_2)) \\ & + (\bar{u}_1 F'_{1u_1} - 2d_1 \rho_1)(\bar{u}_2 F'_{2u_2} - 2d_2 \rho_2) - (\bar{u}_2 F'_{2u_1} - 2d_2 \rho'_2 \bar{u}_2)(\bar{u}_1 F'_{1u_2} - 2d_1 \rho'_1 \bar{u}_1)) \\ = & D_2(\lambda)[\lambda^2 - \lambda(\bar{u}_1 F'_{1u_1} + \bar{u}_2 F'_{2u_2} - 2(d_1 \rho_1 + d_2 \rho_2)) \\ & + \bar{u}_1 \bar{u}_2 (F'_{1u_1} F'_{2u_2} - F'_{2u_1} F'_{1u_2}) + 2d_1 \bar{u}_2 (\rho'_1 F'_{2u_1} \bar{u}_1 - F'_{2u_2} \rho_1) \\ & + 2d_2 \bar{u}_1 (\rho'_2 F'_{1u_2} \bar{u}_2 - F'_{1u_1} \rho_2) + 4d_1 d_2 \rho_1 \rho_2 (1 - \frac{\rho'_1 \rho'_2}{\rho_1 \rho_2} \bar{u}_1 \bar{u}_2)]. \end{aligned} \quad (18)$$

We know that $D_2(\lambda)$ has two roots with negative real parts. The other polynomial will have a negative and a positive root if its constant term is negative. Clearly,

$$\rho_1 \rho_2 \left(1 - \frac{\rho'_1 \rho'_2}{\rho_1 \rho_2} \bar{u}_1 \bar{u}_2 \right) < 0 \quad (20)$$

if $\frac{\rho'_1 \rho'_2}{\rho_1 \rho_2}$ is big enough. If we have achieved this, we may increase d_1 and/or d_2 and the constant term becomes negative. In other words, the equilibrium $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ will be unstable (see Example 3).

4 Numerical investigations

In this section we apply our analytical approach of Section 3 to the following examples of migration function and we are looking for conditions which imply Turing instability (diffusion driven instability).

Example 1 (predator-prey) We choose

$$\begin{aligned} F_1 &= r_1 - a_{11}u_1 - \frac{u_2}{a + u_1}, & F_2 &= -r_2 - a_{22}u_2 + \frac{bu_1}{a + u_1}, \\ \rho_1(u_2) &= \frac{m_1u_2}{1 + u_2}, & \rho_2(u_1) &= m_2 \exp\left(\frac{-u_1}{m_2}\right), \quad m_1, m_2 > 0. \end{aligned} \quad (21)$$

If $r_1 = 10.5$, $r_2 = 1$, $a_{11} = 0.5$, $a_{22} = 5$, $m_1 = 1$, $d_2 = 1$, $d_1 = 1$, $a = 2.3$, $b = 547.8$, then $\bar{u}_1 = 1$, $\bar{u}_2 = 33$.

We consider m_2 as a bifurcation parameter. In this case, at $m_{2crit} \cong 923.0945$ we have four eigenvalues λ_i ($i = 1, 2, 3, 4$) such that $\text{Re } \lambda_i < 0$ ($i = 1, 2, 3$) and $\lambda_4 = 0$.

If $m_2 < m_{2crit} \Rightarrow \text{Re } \lambda_i < 0$ ($i = 1, 2, 3, 4$), then $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ is asymptotically stable.

In this example $\left| \rho'_{2u_1}(u_1, u_2) \right| = \exp\left(-\frac{u_1}{m_2}\right)$. As we see, if m_2 is increased for fixed u_1 , this derivative is increasing, *i.e.*, the cross diffusion effect is increasing.

If $m_2 > m_{2crit} \Rightarrow \text{Re } \lambda_i < 0$ ($i = 1, 2, 3$) and $\lambda_4 > 0$, then $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ is unstable.

Thus, as m_2 is increased through $m_2 = m_{2crit}$, then the cross migration response is strong and the spatially homogeneous equilibrium loses its stability. Numerical calculations show that two new spatially nonconstant equilibria emerge (see Figure 1), and these equilibria are asymptotically stable.

Example 2 (competition) We choose

$$\begin{aligned} F_1 &= r_1 - a_{11}u_1 - a_{12}u_2, & F_2 &= r_2 - a_{21}u_1 - a_{22}u_2, \\ \rho_1(u_2) &= \exp(m_1u_2), & \rho_2(u_1) &= \exp(m_2u_1), \quad m_1, m_2 > 0. \end{aligned} \quad (22)$$

If $r_1 = 11$, $r_2 = 10$, $a_{11} = 5$, $a_{22} = 4$, $a_{12} = 4$, $a_{21} = 3$, $m_1 = 2$, $m_2 = 1$, $d_1 = 1$, then $\det A = a_{11}a_{22} - a_{21}a_{12} = 8$, $\bar{u}_1 = \frac{r_1a_{22} - r_2a_{12}}{\det A} = 0.5$, $\bar{u}_2 = \frac{r_2a_{11} - r_1a_{21}}{\det A} = \frac{17}{8}$.

We consider d_2 as a bifurcation parameter. In this case, at $d_{2crit} \cong 0.582712$ we have four eigenvalues λ_i ($i = 1, 2, 3, 4$) such that $\lambda_i < 0$ ($i = 1, 2, 3$) and $\lambda_4 = 0$.

If $d_2 < d_{2crit} \Rightarrow \lambda_i < 0$ ($i = 1, 2, 3, 4$), then $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ is asymptotically stable.

If $d_2 > d_{2crit} \Rightarrow \lambda_i < 0$ ($i = 1, 2, 3$) and $\lambda_4 > 0$, then $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ is unstable.

Thus, as d_2 is increased through $d_2 = d_{2crit}$, then the spatially homogeneous equilibrium loses its stability. Numerical calculations show that two new spatially nonconstant equilibria emerge (see Figure 2), and these equilibria are asymptotically stable.

Example 3 (cooperation) We choose

$$\begin{aligned} F_1 &= r_1 - a_{11}u_1 - a_{12}u_2, & F_2 &= r_2 - a_{21}u_1 - a_{22}u_2, \\ \rho_1(u_2) &= m_1 \exp(-u_2/m_1), & \rho_2(u_1) &= m_2 \exp(-u_1/m_2), \end{aligned} \quad (23)$$

If $r_1 = 2$, $r_2 = 1$, $a_{11} = 5$, $a_{22} = 4$, $a_{12} = -4$, $a_{21} = -3$, $m_1 = 1$, $m_2 = 1$, $d_1 = 1$, then $\det A = a_{11}a_{22} - a_{21}a_{12} = 8$, $\bar{u}_1 = \frac{r_1 a_{22} - r_2 a_{12}}{\det A} = \frac{3}{2}$, $\bar{u}_2 = \frac{r_2 a_{11} - r_1 a_{21}}{\det A} = \frac{11}{8}$.

We consider d_2 as a bifurcation parameter. In this case, at $d_{2crit} \cong 28.11725408$ we have four eigenvalues λ_i ($i = 1, 2, 3, 4$) such that $\lambda_i < 0$ ($i = 1, 2, 3$) and $\lambda_4 = 0$.

If $d_2 < d_{2crit} \Rightarrow \lambda_i < 0$ ($i = 1, 2, 3, 4$), then $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ is asymptotically stable .

If $d_2 > d_{2crit} \Rightarrow \lambda_i < 0$ ($i = 1, 2, 3$) and $\lambda_4 > 0$, then $(\bar{u}_1, \bar{u}_2, \bar{u}_1, \bar{u}_2)$ is unstable.

Thus, as d_2 is increased through $d_2 = d_{2crit}$, then the spatially homogeneous equilibrium loses its stability. Numerical calculations show that two new spatially nonconstant equilibria emerge (see Figure 3), and these equilibria are asymptotically stable.

5 Conclusions

In the present article our interest is to study the dynamics of two species living in a habitat of two identical patches linked by migration in which the per capita migration rate of each species is influenced not only by its own but also by the other one's density, *i.e.*, there is cross diffusion present. We show that at a critical value of the bifurcation parameter the system undergoes a Turing bifurcation and the cross migration response is an important factor that should not be ignored when pattern emerges, also as the bifurcation parameter is increased through a critical value the spatially homogeneous equilibrium loses its stability and two new stable equilibria emerge. The result of predator-prey interaction does not contradict that of [3] where a situation is treated in which the spatially homogeneous equilibrium is stable for *all* values of the “self-diffusion” coefficients (without cross diffusion). Here this is not the case.

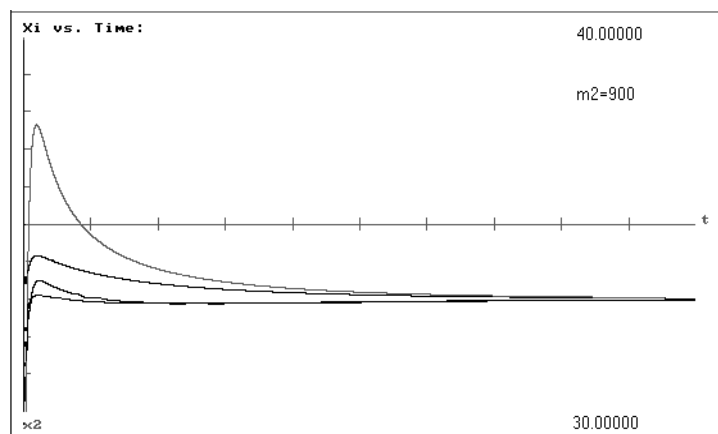
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Figure 1. Graphs of the coordinate $u_2(t, 1)$ of four solutions of Example 1 corresponding to the respective initial conditions $(1.31, 34.20, 0.80, 33.30)$, $(0.77, 33.10, 1.30, 34.00)$, $(1.30, 37.00, 0.77, 33.10)$, $(0.784, 33.20, 1.32, 34.20)$; (a) $m_2 = 900$, all solutions tend to 33.0, (b) $m_2 = 1000$; the first two tend to 34.1178, the last two tend to 33.1758 (Figure produced by applying PHASER).



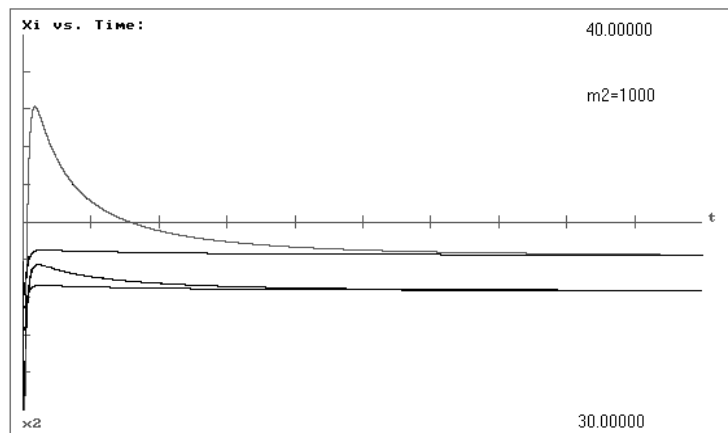


Figure 2. Graphs of the coordinate $u_2(t, 1)$ of five solutions of Example 2 corresponding to the respective initial conditions $(0.55, 1.50, 0.20, 0.60)$, $(0.30, 1.50, 0.40, 1.50)$, $(0.40, 2.40, 0.50, 1.00)$, $(1.00, 2.20, 0.40, 2.50)$, $(0.70, 2.00, 0.10, 2.00)$; (a) $d_2 = 0.5$, all solutions tend to 2.125, (b) $d_2 = 0.7$; the first three tend to 2.2579, the last two tend to 2.0005 (Figure produced by applying MATLAB).

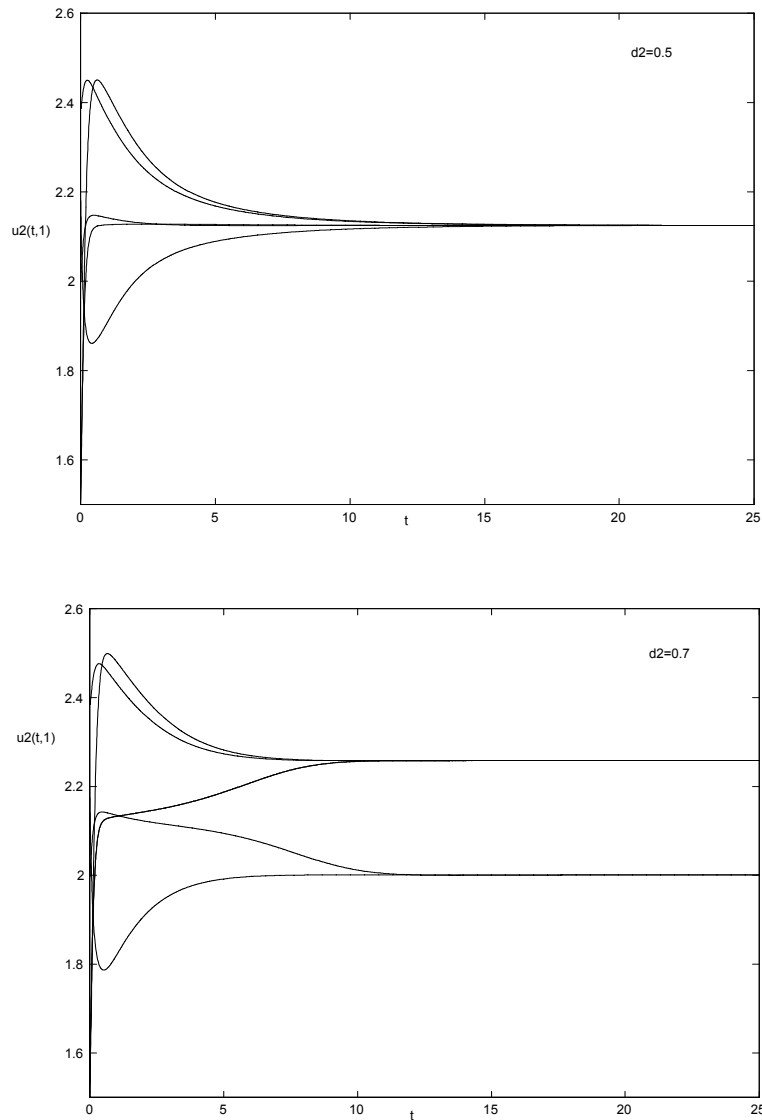


Figure 3. Graphs of the coordinate $u_1(t, 1)$ of five solutions of Example 3 corresponding to the respective initial conditions $(1.80, 1.60, 1.50, 1.25)$, $(1.20, 1.10, 1.59, 1.47)$, $(1.58, 1.45, 1.36, 1.22)$, $(1.00, 1.10, 1.585, 1.47)$, $(1.65, 1.100, 1.320, 1.500)$; (a) $d_2 = 28$, all solutions tend to 1.5; (b) $d_2 = 30$; three solutions tend to 1.6022576 and two solutions tend to 1.376189 (Figure produced by applying MATLAB).

