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Topological Entropy of the Tangent Map*

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Abstract

In this work we consider the Artin-Mazur zeta function of piecewise monotone maps. We define a kneading determinant which has a clear relationship with the Artin-Mazur zeta function. With that definition we obtain a factorization of the zeta function. We apply our results to the family $f_\beta = -\beta \tanh(\beta \tan(x))$, related to the complex tangent family $\lambda \tan z$. This family depends on one real parameter $\frac{\pi}{2} \leq \beta \leq \frac{3\pi}{2}$ in the interval $[-\beta, \beta]$, with two discontinuities. We compute the topological entropy for this type of mappings.

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1 Introduction and statements

The Artin-Mazur zeta function is an important tool in one dimensional dynamics, it counts the fixed points of any iterate of a piecewise monotone map. In the first section of this paper we define the kneading determinant of a piecewise monotone map. This determinant is used to establish a factorization of the Artin-Mazur zeta function enlightening the role of the periodic orbits. In the second section we apply the result to a particular case: a discontinuous family of mappings arising in the study of the complex tangent map. Finally we use the relations obtained to compute easily the topological entropy for this map. Further results can be obtained for a large variety of families of mappings.

Definition 1 A *piecewise monotone* (shortly *PM*) map on $[a,b]$ is a map $f:\ [a,b]\setminus C_f \to [a,b]$, where $C_F = \{c_1, \ldots, c_k\} \subset [a,b]$ is a finite set containing $\{a,b\}$, such

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that f is continuous and strictly monotone on each connected component of $[a,b]\setminus C_f$.

Let $f:\ [a,b]\setminus C_f \to [a,b]$ be a PM map and n a positive integer. The $n-$th iterate of f is the map $f^n:\ [a,b]\setminus C_{f^n}\to[a,b]$ defined inductively by $f^n(x)=f(f^{n-1}(x))$ for all $x\in[a,b]\setminus C_{f^n}$, where

$$C_{f^n}=\left\{x\in[a,b]: f^k(x)\in C_f \text{ for some } k=0,\dots,n-1\right\}.$$

It can be easily seen that this map is PM as well. In what follows we will use the symbol $\mathrm{Fix}(f^n)$ to denote the set of all fixed points of f^n, that is,

$$\mathrm{Fix}(f^n)=\{x\in[a,b]\setminus C_{f^n}:\ f^n(x)=x\}.$$

If the set $\mathrm{Fix}(f^n)$ is finite, for all $n\geq 1$, one defines the Artin-Mazur zeta function of f, $\zeta_f(t)$, as the following formal power series in the indeterminate t:

$$\zeta_f(t)=\exp\sum_{n\geq 1}\frac{\#\mathrm{Fix}(f^n)}{n}t^n.$$

1.1 The kneading determinant of a PM map

Let $f:[a,b]\setminus C_f\to[a,b]$ be a PM map and $I=[x,y]$ (with $y>x$) be a subinterval of $[a,b]$. We say that f is monotone on I if $]x,y[\subseteq[a,b]\setminus C_f$. If f is monotone on $[x,y]$, we define the sign function $\epsilon([x,y])=\pm1$ according to whether f is increasing or decreasing on $]x,y[$. Moreover, for any $x\in[a,b]\setminus C_f$, put $\epsilon(x)=\pm1$ according to whether f is increasing or decreasing on some neighborhood of x and put $\epsilon(x)=0$ for every $x\in C_F$. By definition a *lap* of f is a maximal interval of monotonicity of f. That is to say, an interval $I=[c,d]\subseteq[a,b]$ (with $d>c$) is a lap of F if and only if $[c,d]\cap C_f=\{c,d\}$. In what follows we will use the symbol $\mathcal{L}_f$ to denote the set of all laps of f.

Next we define the kneading determinant of a PM map f on $[a,b]$. Let $a=c_1<c_2<\cdots<c_k=b$ be the elements of C_f, and $d_1,\dots,d_m$ the elements of $\{f(c_2-),\dots,f(c_k-)\}\cup\{f(c_1+),\dots,f(c_{k-1}+)\}$. For each $i=1,\dots,m$, put $C_i^+=\{j: f(c_j+)=d_i\}$ and $C_i^-=\{j: f(c_j-)=d_i\}$, and define the step function $\omega_i:[a,b]\to\mathbb{Q}$ by setting

$$\omega_i=\sum_{j\in C_i^-}\psi_j^-+\sum_{j\in C_i^+}\psi_j^+,$$

where the step functions $\psi_j^- : [a,b] \to \mathbb{Q}$ and $\psi_j^+ : [a,b] \to \mathbb{Q}$ are defined by:

$$\psi_j^-(x) = \begin{cases} \frac{\epsilon(c_j-)}{2} & \text{if} \quad x \geq c_j, \\ \frac{-\epsilon(c_j-)}{2} & \text{if} \quad x < c_j; \end{cases} \qquad \psi_j^+(x) = \begin{cases} \frac{-\epsilon(c_j+)}{2} & \text{if} \quad x > c_j, \\ \frac{\epsilon(c_j+)}{2} & \text{if} \quad x \leq c_j, \end{cases}$$

for $j = 1, \ldots, k$ (with $\epsilon(c_1-) = \epsilon(c_k+) = 0$). Finally, we define the kneading matrix of f, $N_f(t)$, as the $m \times m$-matrix, with entries in $\mathbb{Q}[[t]]$, and given by

$$N_f(t) = \begin{bmatrix} \sum_{n\geq 0} \epsilon_n(d_1)\,\omega_1(f^n(d_1))\,t^n & \cdots & \sum_{n\geq 0} \epsilon_n(d_m)\,\omega_1(f^n(d_m))\,t^n \\ \vdots & \ddots & \vdots \\ \sum_{n\geq 0} \epsilon_n(d_1)\,\omega_m(f^n(d_1))\,t^n & \cdots & \sum_{n\geq 0} \epsilon_n(d_m)\,\omega_m(f^n(d_m))\,t^n \end{bmatrix},$$

with $\epsilon_0(d_j) = 1$ and $\epsilon_n(d_j) = \prod_{i=0}^{n-1} \epsilon\left(f^i(d_j)\right)$. Denoting the $m \times m$-identity matrix by $\mathbf{I}$, we define the kneading determinant of f to be the formal power series given by

$$D_{N_f}(t) = \det(\mathbf{I} - t\ N_f(t)). \tag{1.1}$$

Definition 2 Let f be a PM map on $[a,b]$ and $\mathbb{I}$ the set whose elements are the pairs (x,X) such that $x \in [a,b]$ and X is a connected component of $[a,b]\setminus\{x\}$. The map f induces a map $F : \mathbb{I} \to \mathbb{I}$ defined by:

$$F(x,X) = (y,Y) \text{ if and only if } \lim_{z\in X, z\to x} f(z) = y$$

and there exists a neighborhood V of x such that $f(V \cap X) \subset Y$.

Notice that if $(x,X) \in \mathrm{Fix}(F^n)$ and $x \notin C_{f^n}$, then $x \in \mathrm{Fix}(f^n)$. Thus, if each iterate of f has finitely many fixed points, the zeta function of F is defined and

$$\zeta_F(t) = \exp \sum_{n\geq 1} \frac{\#\mathrm{Fix}(F^n)}{n} t^n = \prod_{\mathbf{o}\in\mathbf{O}} (1 - t^{\mathrm{per}(\mathbf{o})})^{-1},$$

where $\mathbf{O}$ denotes the set of all periodic orbits of F.

Definition 3 Let f be a PM map on $[a,b]$. If (x,X) is a fixed point of F and there exists a neighborhood V of x such that $f(y)$ lies between x and y, for all $y \in V \cap X$, then (x,C) will be called a *formally stable pair* of f. The number of all formally stable pairs of f is denoted by S_f. A periodic orbit $\mathbf{o}$ of F is formally stable if (x,X) is a formally stable fixed point of $F^{\mathrm{per}(\mathbf{o})}$, for all $(x,X) \in \mathbf{o}$.

So, if each iterate f^n has only finitely many fixed points, we can define the formal power series

$$\zeta_f^s(t) \overset{def}{=} \exp \sum_{n\geq 1} \frac{S_{f^n}}{n} t^n = \prod_{\mathbf{o}\in\mathbf{O}_s} (1 - t^{\mathrm{per}(\mathbf{o})})^{-1},$$

where $\mathbf{O}_s$ denotes the set of all formally stable periodic orbits of f. Finally define $\mathbb{I}_0 = \{(x, X) \in \mathbb{I} : x \in C_f\}$ and $\mathbf{O}_0 = \{\mathbf{o} \in \mathbf{O} : \mathbf{o} \cap \mathbb{I}_0 \neq \varnothing\}$. Notice that, F is invariant on the finite set $\cup_{\mathbf{o}\in\mathbf{O}_0}\mathbf{o}$, thus if we denote the restriction of F to $\cup_{\mathbf{o}\in\mathbf{O}_0}\mathbf{o}$ by F_0, we have

$$\zeta_{F_0}(t) = \exp \sum_{n\geq 1} \frac{\#\mathrm{Fix}(F_0^n)}{n} t^n = \prod_{\mathbf{o}\in\mathbf{O}_0} (1 - t^{\mathrm{per}(\mathbf{o})})^{-1}.$$

A proof of the following result can be seen in [1].

Theorem 1 *Let f be a PM map, then*

$$\zeta_f(t) = \frac{\zeta_f^s(t)}{D_{N_f}(t).\zeta_{F_0}(t)}.$$

2 The case of the tangent map

In this section we apply the previous theory to the case of the family of real mappings:

$$f_\beta(x) = -\beta \tanh(\beta \tan(x)),$$

where $x \in J$, with $J = [-\beta, \beta]$ and $\frac{\pi}{2} < \beta < \frac{3\pi}{2}$. In this section when we use expressions like "periodic orbit", "stable point", "iterate", we are always referring to f_β. We call F_β the induced map of f_β defined in Defintion 2.

This family arises in the study of the complex tangent map $\lambda \tan(z)$ when we use as initial condition $z_0 = x$, a real number, and the parameter $\lambda = i\beta$, a pure imaginary number. The family f_β represents the second iterate of $\lambda \tan(z)$ or the return mapping to the real axis. In Figure 1 we can see the graph of f_β. Any function in this family is an odd function in the variable x of the interval $J = [-\beta, \beta]$ on itself. There are two symmetric discontinuity points $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. We notice that f_β is a decreasing function in each one of the intervals $\left(-\beta, -\frac{\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \beta\right)$, which are the laps of f_β, so $C_{F_\beta} = \left\{-\beta, -\frac{\pi}{2}, \frac{\pi}{2}, \beta\right\}$. Moreover,

$$\lim_{x\to -\frac{\pi}{2}^\pm} f_\beta(x) = \lim_{x\to \frac{\pi}{2}^\pm} f_\beta(x) = \pm\beta$$

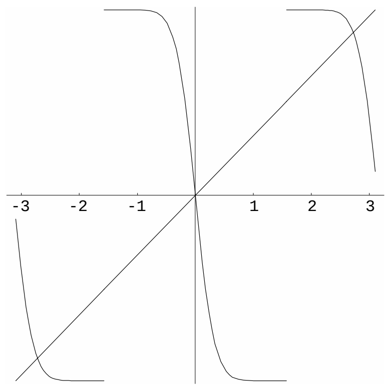

Figure 1: Graph of f_β when $\beta = 3.1$.

and

$$\lim_{x \to \pm\frac{\pi}{2}^\pm} \frac{d^n f_\beta(x)}{dx^n} = 0, \text{ for any integer } n \geq 1. \tag{2.1}$$

Every even (odd) iterate f_β^{2p} (f_β^{2p+1}) is increasing (decreasing) in every lap of f_β^{2p} with $p > 0$ integer.

We define a step function $\Phi : J \diagdown C_{F_\beta} \longmapsto \{-1, 0, 1\}$, such that

$$\Phi(x) = \begin{cases} -1 & \text{if} \quad -\beta < x < -\frac{\pi}{2}, \\ 0 & \text{if} \quad -\frac{\pi}{2} < x < -\frac{\pi}{2}, \\ 1 & \text{if} \quad \frac{\pi}{2} < x < \beta. \end{cases}$$

We define also a formal power series $u_{f_\beta}(t)$ in the indeterminate t:

$$u_{f_\beta}(t) = -t \sum_{n=0}^{+\infty} (-1)^n \Phi\left(f_\beta^n(\beta)\right) t^n.$$

The Milnor-Thurston kneading determinant $D_{MT}(t)$ [6], when the orbits of $-\beta$ and β do not visit any point of the set $C_{F_\beta} = \left\{-\beta, -\frac{\pi}{2}, \frac{\pi}{2}, \beta\right\}$, is

$$D_{MT}(t) = \frac{1 + 2u_{f_\beta}(t)}{1 + t},$$

see [7].

Theorem 2 *Let $f_\beta(x) = -\beta \tanh(\beta \tan(x))$, with x a real number and with a real parameter $\frac{\pi}{2} < \beta < \frac{3\pi}{2}$. Consider the case when the orbits of $-\beta$ and β do not visit* *any point of the set* $C_{F_\beta} = \left\{-\beta, -\frac{\pi}{2}, \frac{\pi}{2}, \beta\right\}$. *The kneading determinant* $D_{N_{f_\beta}}(t)$, *defined by* (1.1), *and the Milnor-Thurston kneading determinant* $D_{MT}(t)$ [6] , *are identical:*

$$D_{N_{f_\beta}}(t) = D_{MT}(t).$$

Proof. We use the definition (1.1) of kneading determinant to obtain the matrix

$$\mathbf{I} - tN_{f_\beta}(t) = \begin{bmatrix} 1-t & 1+\frac{u_{f_\beta}(t)}{t} & -1-\frac{u_{f_\beta}(t)}{t} & t \\ -\frac{t}{2} & 1-\frac{t}{2(1+t)} & -\frac{t}{2(1+t)} & \frac{t}{2} \\ \frac{t}{2} & -\frac{t}{2(1+t)} & 1-\frac{t}{2(1+t)} & -\frac{t}{2} \\ t & -1-\frac{u_{f_\beta}(t)}{t} & 1+\frac{u_{f_\beta}(t)}{t} & 1-t \end{bmatrix}.$$

Now we compute the determinant of this matrix and the result follows. □

A similar result holds when the orbits of points in C_{F_β} visit points in the same set, in that case the kneading determinant is simply a polynomial.

After this result we use the symbol $D_{f_\beta}(t)$ to denote the kneading determinant of f_β.

There are, at most, two stable orbits [8] of f_β.

Theorem 3 *Consider the case where the points* $\pm\beta$ *are pre-images of any point in the set* C_{F_β}. *The orbits of these points are stable and the Artin-Mazur zeta function for* f_β *is:*

$$\zeta_{f_\beta}(t) = \frac{1}{D_{f_\beta}(t)}.$$

Proof. In this case we have two symmetric orbits. In both cases these orbits are stable. We recall (2.1): the discontinuity points of f_β are flat points of F_β, so

$$\zeta^s_{f_\beta}(t) = \zeta_{F_{\beta 0}}(t)$$

and the result follows from Theorem 1. □

Consider the case where the orbit of the points $\pm\beta$ are not pre-images of points of C_{F_β}. The relation between the Artin-Mazur zeta function and the kneading determinant is

$$\zeta_{f_\beta}(t) = \frac{\zeta^s_{f_\beta}(t)}{D_{f_\beta}(t)}.$$

This expression is quite useful and simple to manage.

Example 1 Suppose that f_β has 2 symmetric stable periodic orbits $\{x_n\}_{n=0,1,\ldots}$ and $\{-x_n\}_{n=0,1,\ldots}$ attracting the points $\pm\beta$. We have two situations.

- The period p is even. In that case the function F_β has 4 stable orbits with period p. The zeta function $\zeta^s_{f_\beta}(t)$ for the stable orbits is

$$\zeta^s_{f_\beta}(t) = \frac{1}{(1-t^p)^4}.$$

The relation between the zeta function and the kneading determinant is straightforward to obtain

$$\zeta_{f_\beta}(t) = \frac{1}{D_{f_\beta}(t)\,(1-t^p)^4}.$$

- The period p is odd, now the relation is

$$\zeta_{f_\beta}(t) = \frac{1}{D_{f_\beta}(t)\,(1-t^{2p})^2},$$

because F_β has 2 stable periodic orbits with period $2p$.

Example 2 If f_β has 1 symmetric stable periodic orbit $\{x_n\}_{n=0,1,\ldots}$ attracting both points $\pm\beta$, the period p in this case is even. We have

$$\zeta_{f_\beta}(t) = \frac{1}{D_{f_\beta}(t)\,(1-t^p)^2}.$$

Now we define a formal power series

$$L_{f_\beta}(t) = \sum_{n\geq 0} \ell\left(f^n_\beta\right) t^n,$$

where $\ell\left(f^n_\beta\right)$ is the lap number of f^n_β and $\ell\left(f^0_\beta\right) = 1$. This lap number is related to the Artin-Mazur zeta function:

Proposition 1 *The series of the lap numbers and the Artin-Mazur zeta function are related by the expression*

$$\zeta_{f_\beta}(t) = \zeta^s_{f_\beta}(t)\left(1-t^2\right) L_{f_\beta}(t).$$

Proof. The proof is a consequence of the relation $L_{f_\beta}(t) = \frac{1}{(1-t^2)D_{f_\beta}(t)}$ obtained in [7]. □

With this proposition we see that the number of fixed points is directly related with the number of laps of the mapping of the family f_β.

Similarly we can define a formal power series

$$\Delta_{f_\beta}(t) = \sum_{n \geq 1} \delta\left(f_\beta^n\right) t^n,$$

where $\delta\left(f_\beta^n\right)$ is the number of points of discontinuity of f_β^n. This power series is related with $L_{f_\beta}(t) = \Delta_{f_\beta}(t) + \frac{1}{1-t}$. All the results of this section could be expressed in terms of the number of points of discontinuity of f_β instead of lap numbers.

The growth number of f_β is

$$s(\beta) = \lim_{n \to \infty} \left(\sqrt[n]{\ell\left(f_\beta^n\right)} \right)$$

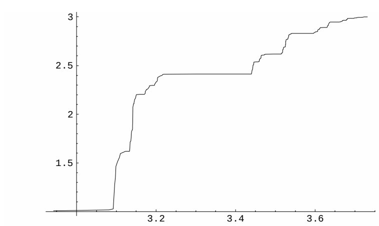

Figure 2: Growth number of f_β in function of β, for $\frac{\pi}{2} < \beta < \frac{3\pi}{2}$.

and can be computed using the least root in the unit interval of the kneading determinant, or the convergence radius of the zeta function or the power series $L_{f_\beta}(t)$ or the power series $\Delta_{f_\beta}(t)$. The topological entropy is $h_t(\beta) = \log s(\beta)$. In Figure 2 we see the graph of the growth number $s(\beta)$.

We hope to generalize this results on topological entropy to the case of the complex tangent map.

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