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# On Functionally Equivalent Impulsive Delay Differential Equations

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**Abstract**

By means of certain functional relations, the equivalence of impulsive delay differential equations and impulsive differential equations is established. Based on some well known results for impulsive differential equations and for delay differential equations, nontrivial consequences on existence and nonexistence of periodic solutions of impulsive delay differential equations are obtained.

**Key words:** Impulse, Delay, Periodic solutions, Functional equivalence.

## 1 Introduction

Impulsive delay differential equations may express the evolution of some real world simulation processes which depend on their prehistory and are subject to short time impulses. The past dependence causes the presence of the delays in the differential equation as well as in the impulsive conditions and this often turns out to be the cause of phenomena substantially affecting the motions. Such processes occur in the theory of optimal control, theoretical physics, population dynamics, biotechnologies, economics, etc. Most of the equations considered in the literature have involved the delays only through the state variable and impulsive conditions of the form $\Delta x(\theta_k) = I_k(x(\theta_k))$ have been considered, where the solution value in the impulses is determined by its limit from the left and independent of prehistory, we name here [1, 4, 7]. The present paper presents a new class of impulsive delay differential equation of the form

$$\begin{cases} x'(t) = f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_m)), & t \neq \theta_k, \\ \Delta x(\theta_k) = I_k(x(\theta_k), x(\theta_{k-q_1}), \ldots, x(\theta_{k-q_m})), & k \in \mathbb{Z}, \end{cases} \tag{1}$$

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in which the delays appear through the state variable and through the jumps. These impulsive conditions are natural for the equation considered since at any discontinuity point the solution value is also defined by its prehistory. By means of certain functional relations, the equivalence of impulsive delay differential equations and impulsive differential equations is established. Based on some well known results for impulsive differential equations [5, 6] and for delay differential equations [2, 3], nontrivial consequences on existence and nonexistence of periodic solutions of impulsive delay differential equations are obtained. We present, moreover, some general remarks.

## 2 Preliminaries

Let $\mathbb{R}^n$ be the Euclidean $n$-space, $\mathbb{R}$ the set of real numbers and $\mathbb{R}^+$ denotes the positive real numbers, $\mathbb{Z}$ is the set of integer numbers and $\mathbb{N}$ denotes the set of natural numbers. Let a sequence $\{\theta_k\},\ k \in \mathbb{Z}$, be fixed in $\mathbb{R}$ such that $\theta_{k+1} > \theta_k$ with $|\theta_k| \to \infty$ as $|k| \to \infty$. We mean by $C(\mathbb{R}^n)$ the space of continuous functions over the Euclidean $n$-space. Assume, moreover, that the equation (1) satisfies the conditions

(A1) $f \in C(\mathbb{R}^{1+n(m+1)}; \mathbb{R}^n)$ and $f$ is $\omega$-periodic in $t$;

(A2) $I_k \in C(\mathbb{R}^{n(m+1)}; \mathbb{R}^n),\ k \in \mathbb{Z}$, and $I_k$ are $p$-periodic in $k$.

The constant parameters $\tau_i,\ i = 1, \ldots, m$, represent the delays in the differential equation and $q_r,\ r = 1, \ldots, m$, represent the delays in the jumps. We assume that the following conditions are valid throughout the rest of the paper.

(H1) There exists $k_i \in \mathbb{N},\ i = 1, \ldots, m$, such that $\tau_i = k_i\omega$ for $\omega > 0$;

(H2) There exists $j_r \in \mathbb{N},\ r = 1, \ldots, m$, such that $q_r = j_r p$ for $p \in \mathbb{N}$;

(H3) There exists $p \in \mathbb{N}$ such that $\theta_{k+p} = \theta_k + \omega,\ k \in \mathbb{Z}$.

The novelty of this paper is based on the consideration of the delays in the jumps. Thus, we assume that every interval of length $\tau$ contains more than $l$ points $\theta_k,\ k \in \mathbb{Z}$, where $l = \max\{q_r :\ r = 1, \ldots, m\}$ and $\tau = \max\{\tau_i : i = 1, \ldots, m\}$. By $\Delta x(t)$ we mean as usual the difference $x(t^+) - x(t^-)$, where $x(t^+) = \lim_{h \to 0^+} x(t+h)$ and $x(t^-) = \lim_{h \to 0^-} x(t+h)$. We assume that the solutions are left continuous and hence write $x(t^-) = x(t)$. An absolutely continuous function $x(t)$ in every interval $[\theta_i, \theta_{i+1})$ is said to be a *solution* of (1) if it satisfies the differential equation in (1) for almost every $t$ and the impulsive conditions for $t = \theta_k$. Associated with equation (1) we consider the impulsive differential equation of the form

$$\begin{cases} x'(t) = f(t, x(t), x(t), \ldots, x(t)), & t \neq \theta_k, \\ \Delta x(\theta_k) = I_k(x(\theta_k), x(\theta_k), \ldots, x(\theta_k)), & k \in \mathbb{Z}. \end{cases} \tag{2}$$

**Definition 1** Equations (1) and (2) are said to be *functionally equivalent* relative to the functional relation

$$x(t+\omega) \equiv x(t) \tag{3}$$

if the $\omega$-periodic solutions of equation (1) are also $\omega$-periodic solutions of equation (2) and the $\omega$-periodic solutions of equation (2) satisfy (1).

## 3 Existence of periodic solutions

In this section, we shall use the definition of functional equivalence in the considered sense of systems (1) and (2) to establish existence and nonexistence of periodic solutions of impulsive delay differential equations.

**Theorem 1** *Let conditions* $(A1)$, $(A2)$ *and* $(H1)$–$(H3)$ *be fulfilled. Then, equations* (1) *and* (2) *are functionally equivalent relative to relation* (3).

**Proof.** It suffices to prove that any $\omega$-periodic solution $\phi(t)$ of equation (1) is an $\omega$-periodic solution of (2) and vice versa. Let $\phi(t)$ be an $\omega$-periodic solution of equation (1). Then

$$\begin{cases} \phi'(t) = & f(t, \phi(t), \phi(t-\tau_1), \ldots, \phi(t-\tau_m)), \quad t \neq \theta_k, \\ \Delta\phi(\theta_k) = & I_k(\phi(\theta_k), \phi(\theta_{k-q_1}), \ldots, \phi(\theta_{k-q_m})), \quad k \in \mathbb{Z}. \end{cases}$$

From conditions (H1)–(H3) it follows that

$$\begin{cases} \phi'(t) = & f(t, \phi(t), \phi(t-k_1\omega), \ldots, \phi(t-k_m\omega)), \quad t \neq \theta_k, \\ \Delta\phi(\theta_k) = & I_k(\phi(\theta_k), \phi(\theta_k - j_1\omega), \ldots, \phi(\theta_k - j_m\omega)), \quad k \in \mathbb{Z}. \end{cases}$$

The periodicity of $\phi$ implies that

$$\begin{cases} \phi'(t) = & f(t, \phi(t), \phi(t), \ldots, \phi(t)), \quad t \neq \theta_k, \\ \Delta\phi(\theta_k) = & I_k(\phi(\theta_k), \phi(\theta_k), \ldots, \phi(\theta_k)), \quad k \in \mathbb{Z}, \end{cases}$$

which is equation (2). The converse is proved by observing that any periodic solution $\phi(t)$ can be written as $\phi(t - k_m w)$ which is (by condition (H1)) equal to $\phi(t-\tau_m)$.

**Theorem 2** *Let conditions* $(A1)$, $(A2)$ *and* $(H1)$–$(H3)$ *be fulfilled. Then, equation* (1) *does not have more than an $n$-parameter family of $\omega$-periodic solutions.*

**Proof.** Let $\phi$ be an $\omega$-periodic solution of equation (1). Then,

$$\begin{cases} \phi'(t) &= f(t, \phi(t), \phi(t-\tau_1), \ldots, \phi(t-\tau_m)), \quad t \neq \theta_k, \\ \Delta\phi(\theta_k) &= I_k(\phi(\theta_k), \phi(\theta_{k-q_1}), \ldots, \phi(\theta_{k-q_m})), \quad k \in \mathbb{Z}. \end{cases}$$

Similar to the arguments of the previous proof, it follows that

$$\begin{cases} \phi'(t) &= f(t, \phi(t), \phi(t), \ldots, \phi(t)), \quad t \neq \theta_k, \\ \Delta\phi(\theta_k) &= I_k(\phi(\theta_k), \phi(\theta_k), \ldots, \phi(\theta_k)), \quad k \in \mathbb{Z}. \end{cases} \tag{4}$$

Equation (4) is a system of $n$ first order impulsive differential equations whose general solution contains $n$ arbitrary constants. Thus, since (4) has not more than an $n$-parametric family of $\omega$-periodic solutions, (1) cannot have more than an $n$-parametric family of $\omega$-periodic solutions.

Consider the linear equation

$$\begin{cases} x'(t) + A_0(t)x(t) + \sum_{i=1}^{m} A_i(t)x(t-\tau_i) &= h(t), \ t \neq \theta_k, \\ \Delta x(\theta_k) + B_k^0 x(\theta_k) + \sum_{r=1}^{m} B_k^r x(\theta_{k-q_r}) &= J_k, \quad k \in \mathbb{Z}, \end{cases} \tag{5}$$

together with the following conditions:

(B1) $h \in C(\mathbb{R}^n)$ and $h$ is $\omega$-periodic in $t$;

(B2) $J_k$ are $p$-periodic in $k$;

(B3) $A_i \in C(\mathbb{R}; \mathbb{R}^{n\times n}),\ i = 0, \ldots, m$, and $\sum_{i=0}^{m} A_i(t)$ is $\omega$-periodic in $t$;

(B4) $\sum_{r=0}^{m} B_k^r$ are $p$-periodic in $k$.

**Theorem 3** *Let conditions (B1)–(B4) and (H1)–(H3) be fulfilled. Then, equation (5) has a unique $\omega$-periodic solution if the equation*

$$\begin{cases} x'(t) + \sum_{i=0}^{m} A_i(t)x(t) &= 0, \quad t \neq \theta_k, \\ \Delta x(\theta_k) + \sum_{r=0}^{m} B_k^r x(\theta_k) &= 0, \quad k \in \mathbb{Z}, \end{cases} \tag{6}$$

*has no periodic solutions of period $\omega$, except the trivial one.*

**Proof.** Consider the equation

$$\begin{cases} x'(t) + \sum_{i=0}^{m} A_i(t)x(t) &= h(t), \quad t \neq \theta_k, \\ \Delta x(\theta_k) + \sum_{r=0}^{m} B_k^r x(\theta_k) &= J_k, \quad k \in \mathbb{Z}. \end{cases} \tag{7}$$

Then, by Theorem 1, equation (7) is functionally equivalent to (5). However, by [6, Theorem 53], equation (7) has a unique $\omega$-periodic solution. Thus, equation (5) has a unique $\omega$-periodic solution.

In the case $n = 1$, we may establish a more specific condition for equation (5) not to have an $\omega$-periodic solution. Rewrite (5) as follows:

$$\begin{cases} x'(t) + a_0(t)x(t) + \sum_{i=1}^{m} a_i(t)x(t-\tau_i) = 0, & t \neq \theta_k, \\ \Delta x(\theta_k) + b_{0k}x(\theta_k) + \sum_{r=1}^{m} b_{rk}x(\theta_{k-q_r}) = 0, & k \in \mathbb{Z}, \end{cases} \tag{8}$$

where the following conditions are satisfied:

(C1) $a_i \in C(\mathbb{R})$ and $\sum_{i=0}^{m} a_i(t)$ is $\omega$-periodic in $t$;

(C2) $\sum_{r=0}^{m} b_{rk}$ are $p$-periodic in $k$;

(C3) $\sum_{r=0}^{m} b_{rk} > -1$.

**Theorem 4** *Let conditions (C1)–(C3) and (H1)–(H3) be satisfied. If*

$$\int_0^{\omega} \sum_{i=0}^{m} a_i(r)\,dr - \sum_{0 \le \theta_k < \omega} \ln\Big(1 + \sum_{r=0}^{m} b_{rk}\Big) \neq 0,$$

*then, equation (8) has no nontrivial $\omega$-periodic solutions.*

**Proof.** Suppose, on the contrary, that equation (8) has an $\omega$-periodic solution $\phi(t)$ such that $\phi(0) \neq 0$, otherwise, by the uniqueness theorem we will have $\phi(t) = 0$, see [5] for more details. Clearly

$$\begin{cases} \phi'(t) + \sum_{i=0}^{m} a_i(t)\phi(t) = 0, & t \neq \theta_k, \\ \Delta\phi(\theta_k) + \sum_{r=0}^{m} b_{rk}\phi(\theta_k) = 0, & k \in \mathbb{Z}. \end{cases}$$

It follows that

$$\phi(t) = x_0 \exp\left(-\int_0^t \sum_{i=0}^{m} a_i(r)\,dr\right) \prod_{0 \le \theta_k < t} \Big(1 + \sum_{r=0}^{m} b_{rk}\Big),$$

where $\phi(0) = x_0 \neq 0$. From the periodicity of $\phi$, we have

$$\exp\left(-\int_0^{\omega}\sum_{i=0}^{m} a_i(r)\,dr\right)\prod_{0\le\theta_k<\omega}\Big(1+\sum_{r=0}^{m} b_{rk}\Big) = 1.$$

It readily follows that

$$\int_0^{\omega}\sum_{i=0}^{m} a_i(r)\,dr - \sum_{0\le\theta_k<\omega}\ln\Big(1+\sum_{r=0}^{m} b_{rk}\Big) = 0,$$

which is a contradiction. This completes the proof.

**Corollary 1** *Let conditions* $(C1)$*–*$(C3)$ *and* $(H1)$*–*$(H3)$ *be satisfied. If equation* (8) *has nontrivial* $\omega$*-periodic solutions, then*

$$\int_0^{\omega}\sum_{i=0}^{m} a_i(r)\,dr - \sum_{0\le\theta_k<\omega}\ln\Big(1+\sum_{r=0}^{m} b_{rk}\Big) = 0.$$

Consider the scalar equation

$$\begin{cases} x'(t) &= a(t)x(t)+b(t)x(u(t)), \quad t\neq\theta_k, \\ \Delta x(\theta_k) &= c_k x(\theta_k)+d_k x(\theta_{k-j}), \quad k\in\mathbb{Z}, \end{cases} \tag{9}$$

where $a(t), b(t) \in C(\mathbb{R})$ and $u(t) \in C(\mathbb{R}^+)$ such that $u(t) \le t$. Let there exist a positive number $q$ and a monotonic differentiable function $s = \psi(t)$ for $t \neq \theta_k$ such that $\psi(t) - \psi(u(t)) = q$ and $\psi(\theta_k) = \xi_k$. Assume that $x(t) = z(\psi(t)) = z(s)$, $t \neq \theta_k$, then

$$x'(t) = z'(s)\psi'(\psi^{-1}(s)), \quad s\neq\xi_k,$$

and

$$x(u(t)) = z(s-q), \quad s\neq\xi_k.$$

Therefore

$$z'(s) = a_1(s)z(s)+b_1(s)z(s-q), \quad s\neq\xi_k,$$

where

$$a_1(s) = \frac{a(\psi^{-1}(s))}{\psi'(\psi^{-1}(s))} \quad\text{and}\quad b_1(s) = \frac{b(\psi^{-1}(s))}{\psi'(\psi^{-1}(s))}.$$

Moreover, since

$$\Delta x(\theta_k) = \Delta z(\psi(\theta_k)) = \Delta z(\xi_k) \quad\text{and}\quad x(\theta_{k-j}) = z(\psi(\theta_{k-j})) = z(\xi_{k-j}),$$

we have

$$\Delta z(\xi_k) = c_k z(\xi_k)+d_k z(\xi_{k-j}),$$

where

$$c_k = c_k(\psi^{-1}(\xi_k)) \quad \text{and} \quad d_k = d_k(\psi^{-1}(\xi_k)).$$

Hence, (9) is reduced to

$$\left\{\begin{array}{rcl} z'(s) &=& a_1(s)z(s) + b_1(s)z(s-q), \quad s \neq \xi_k, \\ \Delta z(\xi_k) &=& c_k z(\xi_k) + d_k z(\xi_{k-j}), \quad k \in \mathbb{Z}. \end{array}\right. \tag{10}$$

The equation

$$\left\{\begin{array}{rcl} z'(s) &=& (a_1(s) + b_1(s))z(s), \quad s \neq \xi_k, \\ \Delta z(\xi_k) &=& (c_k + d_k)z(\xi_k), \quad k \in \mathbb{Z}, \end{array}\right. \tag{11}$$

is equivalent to (10) relative to $z(s) = z(s-q)$. Thus, if the function $d(s) = a_1(s) + b_1(s)$ is $q$-periodic and $e_k = c_k + d_k > -1$ is $j$-periodic (the delays are the periods) such that $\xi_{k-j} = \xi_k - q$ and satisfy the condition

$$\int_0^{\omega} d(s)\, ds + \sum_{0 \le \xi_k < \omega} \ln(1 + e_k) = 0,$$

then, equation (11) has the $q$-periodic solution

$$z(s) = c \exp\left(\int_0^s d(\xi)\, d\xi\right) \prod_{0 \le \xi_k < s} (1 + e_k).$$

Consequently, equation (9) has a solution of the form

$$x(t) = c \exp\left(\int_0^{\psi(t)} d(s)\, ds\right) \prod_{0 \le \theta_k < \psi(t)} (1 + e_k),$$

possessing the property $x(t) = x(u(t))$.

**Remark 1** The presentations considered above can be extended to impulsive differential equations with deviating arguments of the form

$$\left\{\begin{array}{rcl} x^{(n)}(t) &=& f(t, x(t), x(t-p_i), x(t-q_j), \ldots, \\ && \qquad x^{(n)}(t-p_i), x^{(n)}(t-q_j)), \qquad t \neq \theta_k, \\ \Delta x^{(n-1)}(\theta_k) &=& I_k(x(\theta_k), x(\theta_{k-u_r}), x(\theta_{k-v_q}), \ldots, \\ && \qquad x^{(n-1)}(\theta_{k-u_r}), x^{(n-1)}(\theta_{k-v_q})), \quad k \in \mathbb{Z}. \end{array}\right. \tag{12}$$

Suppose that there exist $k_i \in \mathbb{N},\ i = 1, \ldots, m$, such that $p_i = k_i\omega$ for $\omega > 0$ and the constants $q_j > 0,\ j = 1, 2, \ldots, s$, and there exist $j_r \in \mathbb{N},\ r = 1, \ldots, m$, such that

$u_r = j_r p$ for $p > 0$, the constants $v_q > 0$, $q = 1, 2, \ldots, s$, and $\theta_{k+p} = \theta_k + \omega$. In this case, we may consider a functionally equivalent equation as

$$\begin{cases} x^{(n)}(t) &= f(x(t), x(t), x(t-q_j), \ldots, \\ & \qquad x^{(n)}(t), x^{(n)}(t-q_j)), \qquad t \neq \theta_k, \\ \Delta x^{(n-1)}(\theta_k) &= I_k(x(\theta_k), x(\theta_k), x(\theta_{k-v_q}), \ldots, \\ & \qquad x^{(n-1)}(\theta_k), x^{(n-1)}(\theta_{k-v_q})), \quad k \in \mathbb{Z}, \end{cases} \tag{13}$$

where functional equivalence is understood in a similar manner.

**Remark 2** The definition of functional equivalence, generally speaking, is not precise. Indeed, if equation (12) is of neutral type, then its periodic solutions may be $n$ times piecewise differentiable functions. Then, as for equation (13), there is a possibility of the case when its order turns out to be less than $n$, consequently, some of its solutions may remain piecewise differentiable a smaller number of times and may even be only piecewise continuous. There is also the possibility of the case, for instance if equation (13) has order $n$ and $s = 0$, for which all solutions of equation (13) are $n$ times piecewise differentiable. Consequently, it is impossible to assert that equations (12) and (13) have in common all periodic solutions of period $\omega$.

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