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Asymptotic Behavior of the Solution, when $t \to +\infty$, of a Class of Nonlinear Equations

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Abstract

The objective of this work is the study of the asymptotic behavior of the solution, when $t \to +\infty$, of a class of parabolic equations. We show that if the initial condition is not null, the solution is exactly exponential when $t \to +\infty$ and the decrease rate is characterized by an element of the operator spectrum.

1 Introduction

It is well known that the solutions of the nonlinear equations of the type

$$\begin{cases} u_t + Au + f(u) = 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

(A is an unbounded operator of the domain $D(A)$, and f is a nonlinear operator) in $\Omega \times \mathbb{R}$ (Ω open bounded) associated with conditions at the edges are usually regular enough and convergent towards their state of equilibrium when $t \to +\infty$.

The objective of this work is to show that this convergence towards $u \equiv 0$ is exactly of the exponential type and the rate of this decrease is characterized by an eigenvalue of the operator A.

More precisely, we study the limit of the quotient $\frac{\int \left|A^{1/2}u\right|^2 dx}{\int |u|^2 dx}$, denoted $\frac{\|u\|^2}{|u|^2}$, and we show that it is an eigenvalue of the operator A, we deduce that there exists an eigensubspace such that $\frac{u(x,t)}{|u(x,t)|}$ is found concentrated under this subspace, and the solution $u(t)$ behaves exactly like the function $t \mapsto e^{-\Lambda t}$ when $t \to +\infty$ ($\Lambda \in \sigma(A)$).

2 Notations and recalls of certain results

Let V and H be two separable Hilbert spaces such that:

$$V \hookrightarrow H \ \text{ with compact injection,} \tag{2.1}$$

$$V \text{ is dense in } H. \tag{2.2}$$

We denote by $\|\cdot\|$ and $|\cdot|$ the corresponding norms.

Consider the unbounded operator A with a range in H:

$$D(A) = \{u \in V,\ Au \in H\}. \tag{2.3}$$

Supplying $D(A)$ with the graph norm, A is then an isomorphism of $D(A)$ in H, so there exists a sequence of eigenvalues of A

$$0 < \lambda_1 < \lambda_2 < \cdots, \tag{2.4}$$

each with a finite multiplicity.

On the other hand, if R_j denotes the orthogonal projection onto the associated eigenspaces at j, then

$$R_j R_k = 0 \text{ if } i \neq j,\ R_1 \oplus R_2 \oplus \cdots = I. \tag{2.5}$$

We denote by

$$0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_j < \cdots \tag{2.6}$$

the sequence of eigenvalues of multiplicity m_k, and by $\{S(t)\}_{t\geq 0}$ the nonlinear semigroup defined by

$$S(t):\ V \to V, \quad u_0 \mapsto S(t)u_0. \tag{2.7}$$

Consider the problem given by

$$(P) \quad \begin{cases} u_t + Au + f(u) = 0, \\ u(0) = u_0, \end{cases} \tag{2.8}$$

A is an unbounded positive self-adjoint operator of the domain $D(A)$, $f(u)$ is a nonlinear operator.

From [3], we have the global existence results of uniform estimations in time as well as the asymptotic behavior when $t \to +\infty$.

Indeed, if $u_0 \in V$ and if f satisfies

(i) f continuous,

(ii) $(f(u) - f(v), u - v) + \lambda |u - v|^2 \geq 0 \ \forall u, v \in V, \lambda \in \mathbb{R}$,

(iii) $\exists \theta \in [0, 2[$, $\forall B$ bounded in H,

$$|f(u)| \leq C_B \|u\|^{2-\theta} |Au|^{\theta/2} \ \forall u, v \in D(A), \tag{2.9}$$

the problem given by the system of equations (2.8) possesses a unique solution u that satisfies

$$u \in C_b(\mathbb{R}_+, V) \cap L^2(0, \infty; D(A)), \tag{2.10}$$

$$u_t \in L^2(0,\infty;H) \tag{2.11}$$

$(C_b = C^0 \cap L^\infty)$.

On the other hand, when $t \to +\infty$, the solution u tends to its equilibrium state exponentially, and we have:

$$\text{there exists a constant } c_0 : \ \|u\| \le C_0 e^{-\lambda(t-t_0)} \ \ \forall t \ge t_0, \ t_0 \ge 0. \tag{2.12}$$

3 The behavior of the quotient $\frac{\|S(t)u_0\|^2}{|S(t)u_0|^2}$ when $t \to +\infty$

For $u_0 \in V, \ u_0 \neq 0$, the quotient

$$\lambda(t) = \frac{\|S(t)u_0\|^2}{|S(t)u_0|^2} \tag{3.1}$$

is defined for $t \ge 0$. The behavior of $\lambda(t)$ is given by the following theorem.

Theorem 3.1

$$\lim_{t\to+\infty} \lambda(t) = \Lambda(u_0), \tag{3.2}$$

where $\Lambda(u_0)$ *is the eigenvalue of the operator* A.

Proof. If $\lambda(t)$ is differentiated with respect to time, writing

$$\frac{1}{2}\frac{d}{dt}(\lambda(t)) = \left(Au, \frac{du}{dt}\right)\frac{|u|^2}{|u|^4} - \left(\frac{du}{dt}, \lambda u\right)\frac{|u|^2}{|u|^4}, \tag{3.3}$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}(\lambda(t)) = \frac{1}{|u|^2}\left(Au - \lambda u, \frac{du}{dt}\right), \tag{3.4}$$

or $\frac{du}{dt} = -(Au - f(u))$, and (3.4) will be

$$\frac{1}{2}\frac{d}{dt}(\lambda(t)) = \frac{1}{|u|^2}(Au - \lambda u, Au + f(u)). \tag{3.5}$$

However, we have $(Au - \lambda u, \lambda u) = (Au - \frac{\|u\|^2}{|u|^2}\cdot u, \ \frac{\|u\|^2}{|u|^2}\cdot u) = 0$.

So we can write using (3.5):

$$\frac{1}{2}\frac{d}{dt}(\lambda(t)) = -\frac{1}{|u|^2}(Au - \lambda u, \ Au - \lambda u) + (Au - \lambda u, \ f(u)). \tag{3.6}$$

Then by putting $v(t) = \frac{u(t)}{|u(t)|}$,

$$\frac{1}{2}\frac{d}{dt}(\lambda(t)) + |(A-\lambda)v|^2 = -((A-\lambda)v,\ \frac{f(u)}{|u|}). \tag{3.7}$$

Majorizing the term on the right-hand side and applying the inequality of Young, we obtain

$$\frac{d}{dt}(\lambda(t)) + |(A-\lambda)v|^2 \leq \frac{|f(u)|^2}{|u|^2}. \tag{3.8}$$

Taking into account $|f(u)| \leq \eta(u)\,\|u\|$ with $\eta(u) = \bigcirc(e^{-\alpha t}),\ \alpha > 0$, we have :

$$\frac{d}{dt}(\lambda(t)) + |(A-\lambda)v|^2 \leq c_1 e^{-\alpha t}\lambda(t),\ \alpha > 0. \tag{3.9}$$

Omitting the term $|(A-\lambda)v|^2$ in (3.9), we get

$$\frac{d}{dt}(\lambda(t)) \leq c_1 e^{-\alpha t}\lambda(t), \tag{3.10}$$

which will be integrated:

$$\Lambda_1 \leq \lambda(t) \leq \lambda(t_0)e^{c_1\int_{t_0}^{t} e^{-\alpha s}\,ds},\ t > t_0. \tag{3.11}$$

Taking the upper limit as $t \to +\infty$, it will be

$$\Lambda_1 \leq \limsup_{t\to+\infty}\lambda(t) = \lambda(t_0)e^{c_1\int_{t_0}^{+\infty} e^{-\alpha s}\,ds} < +\infty, \tag{3.12}$$

then the lower limit as $t \to +\infty$,

$$\Lambda_1 \leq \limsup_{t\to+\infty}\lambda(t) \leq \liminf_{t\to+\infty}\lambda(t) < +\infty. \tag{3.13}$$

We deduce then that $\lambda(t)$ converges toward a limit $\Lambda(u_0)$. Moreover, we have:

$$\lambda(t) \geq \Lambda(u_0)e^{-c_1\int_{t_0}^{+\infty} e^{-\alpha s}\,ds}. \tag{3.14}$$

To show that $\Lambda(u_0) \in \sigma(A) = \{\Lambda_1, \Lambda_2, \ldots\}$, we take again and integrate the inequality (3.9), it becomes then

$$\lambda(t) - \lambda(t_0) + \int_{t_0}^{t} |(A-\lambda)v|^2(s)\,ds \leq \sup_{t\geq t_0}\lambda(t)\int_{t_0}^{t} c_1 e^{-\alpha s}\,ds,\ \alpha > 0. \tag{3.15}$$

It results from it that

$$|(A-\lambda)v| \in L^2(t_0,\infty). \tag{3.16}$$

We then deduce that there exists a sequence $t_j \to +\infty$ such that

$$|(A-\lambda)(t_j)v(t_j)| \to 0 \text{ when } t_j \to +\infty. \tag{3.17}$$

Thus,

$$|(A-\Lambda(u_0))v(t_j)| \to 0 \text{ when } t_j \to +\infty.$$

The sequence $|(Av(t_j))|$ is bounded, by an A^{-1} capacity, we are sure of the existence of a subsequence, denoted by (t_j), such that

$$v(t_j) \to \overline{v} \text{ in } H \text{ strongly and in } V \text{ weakly.} \tag{3.18}$$

It results from it that

$$Av(t_j) = (A-\lambda(t_j))(v(t_j)) + (\lambda(t_j)-\Lambda(u_0))(v(t_j)) + \Lambda(u_0)(v(t_j)) \to \Lambda(u_0)\overline{v}, \tag{3.19}$$

with $\overline{v} \in D(A)$ (A is maximum accretive, so closed).

Moreover, $|v(t_j)| = 1$. Thus $|\overline{v}| = 1$, it results from (3.17) and (3.20):

$$A\overline{v} = \Lambda\overline{v} \tag{3.20}$$

(Thus λ is an eigenvalue of A).

Corollary 3.1

$$\lim_{t\to+\infty} \frac{\log\|u(t)\|}{t} = \lim_{t\to+\infty} \frac{|\log u(t)|}{t} = -\Lambda(u_0). \tag{3.21}$$

Let us use the equation

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \|u\|^2 + (f(u),u) = 0. \tag{3.22}$$

According to Theorem 3.1, we deduce that there exists t_ε ($\varepsilon > 0$) such that $\forall t > t_\varepsilon$ we have

$$-\varepsilon + \Lambda(u_0) \le \frac{\|u(t)\|^2}{|u(t)|^2} \le \Lambda(u_0) + \varepsilon. \tag{3.23}$$

From (3.23) and (3.22), taking into account $|f(u)| \le \eta(t)\|u\|$, $\eta(t) = \bigcirc(e^{-\alpha t})$, $\alpha > 0$, it follows that

$$\left\{-c_1e^{-\alpha t} - 2(\varepsilon + \Lambda(u_0))\right\}|u|^2 \le \frac{d}{dt}|u|^2 \le \left\{c_1e^{-\alpha t} - 2(\Lambda(u_0)+\varepsilon)\right\}|u|^2, \tag{3.24}$$

and from it, that

$$-(\varepsilon_1 + \Lambda(u_0)) \leq \frac{\log |u(t)|}{t} \leq (\varepsilon_1 - \Lambda(u_0)), \ \ \forall t > t_\varepsilon, \tag{3.25}$$

where

$$\lim_{t\to+\infty} \frac{\log |u(t)|}{t} = -\Lambda(u_0), \tag{3.26}$$

and since $\frac{\|u(t)\|}{|u(t)|}$ is bounded, thus we have

$$\lim_{t\to+\infty} \frac{\log \|u(t)\|}{t} = -\Lambda(u_0). \tag{3.27}$$

4 The behavior of $S(t)u_0$ when $t \to +\infty$

We recall that for the eigenvalue λ of the operator A, we denote by R_λ the orthogonal projection onto the eigenspace associated with λ:

$$R_\lambda \omega = \sum_{\lambda_j = \lambda} (\omega, \omega_j)\omega_j. \tag{4.1}$$

As well as if $\lambda_m < \lambda < \lambda_{m+1}$ and $\Lambda_M < \lambda < \Lambda_{M+1}$, then

$$R_\lambda = P_M - P_m \tag{4.2}$$

(Where P_m is an orthogonal projection).

The following propositions show that the solution u behaves asymptotically like $R_\Lambda u$.

Proposition 4.1

$$\lim_{t\to\infty} \left\| (I - R_{\Lambda(u_0)}) \frac{S(t)u_0}{|S(t)u_0|} \right\| = \lim_{t\to\infty} \left| (I - R_{\Lambda(u_0)}) \frac{S(t)u_0}{|S(t)u_0|} \right| = 0. \tag{4.3}$$

We start by establishing a lemma that specifies Corollary 3.1.

Lemma 4.1 *There exist two positive constants c_1 and c_2 such that*

$$|u(t)| = c_1 e^{-\Lambda(u_0)(t-t_0)} \ \ \forall t \geq t_0, \tag{4.4}$$

$$\|u(t)\| = c_2 e^{-\Lambda(u_0)(t-t_0)} \ \ \forall t \geq t_0. \tag{4.5}$$

Proof. Since $\frac{\|u(t)\|}{|u(t)|}$ is bounded, it is sufficient to prove (4.4).
Suppose the inequality

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \|u\|^2 \leq \eta(t)\lambda^{1/2}|u|^2. \tag{4.6}$$

By virtue of $\|u\|^2 \geq \lambda(t)|u|^2$ and (4.6) we obtain

$$\frac{d}{dt}\log(|u|^2 e^{2\Lambda t}) \leq 2\eta\lambda^{1/2} + 2(\lambda - \Lambda). \tag{4.7}$$

From (3.2) and (4.7) there follows (4.4).

Proof of Proposition 4.1. We start by establishing an evolution equation for $v(t) = \frac{S(t)u_0}{|S(t)u_0|} = \frac{u(t)}{|u(t)|}$,

$$\frac{dv}{dt} = \frac{1}{|u(t)|}\frac{du}{dt} - \frac{1}{|u(t)|^2}\frac{d}{dt}|u(t)| \cdot u, \tag{4.8}$$

but $2(\frac{du}{dt}, u) = 2|u|\frac{d}{dt}|u| = \frac{d}{dt}|u|^2$; $\frac{du}{dt} = -(Au + f(u))$ and (4.8) takes the form

$$\frac{dv}{dt} + (A - \lambda)v = \frac{1}{|u(t)|^2}(f(u), u) \cdot v - \frac{f(u)}{|u(t)|}. \tag{4.9}$$

We put

$$\rho = \left| \frac{1}{|u(t)|^2}(f(u), u) \cdot v - \frac{f(u)}{|u(t)|} \right|. \tag{4.10}$$

We deduce from the estimation of $f(u)$ and of the bounded $\lambda(t)$:

$$\rho \leq \frac{|f(u)|}{|u|^2}\|u\|\,|v| + \frac{f(u)}{|u|} \leq \sup \lambda(t) c_1 e^{-\alpha t}, \quad \alpha > 0, \tag{4.11}$$

i.e.,

$$\rho \ \leq c_5 e^{-\alpha t}, \quad \alpha > 0.$$

We denote $q = (I - P_\Lambda)v$, thus if we apply $(I - P_\Lambda)$ to (4.9) and we take the scalar product in H of the results by Aq, we have, using (4.10),

$$\frac{1}{2}\frac{d}{dt}\|q\|^2 + |Aq|^2 - \lambda(t)\|q\|^2 \leq \rho|Aq|. \tag{4.12}$$

Applying the inequality $2\rho|Aq| \leq 2\varepsilon|Aq|^2 + \frac{\rho^2}{2\varepsilon}$ $(\varepsilon > 0)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|q\|^2 + 2(1-\varepsilon)|Aq|^2 - 2\lambda(t)\|q\|^2 \leq c_6 e^{-\alpha t}, \quad \alpha > 0. \tag{4.13}$$

Denoting by $\Lambda' > \Lambda$ the first eigenvalue that is strictly greater than Λ, $\delta = \Lambda' - \Lambda > 0$, and since $|Aq|^2 \geq \Lambda' \|q\|^2$, we deduce from (4.13)

$$\frac{d}{dt}\|q\|^2 + 2\left[(1-\varepsilon)\Lambda' - \Lambda\right]\|q\|^2 \leq c_6 e^{-\alpha t}, \quad \alpha > 0. \tag{4.14}$$

Choosing $\varepsilon = \frac{\delta}{2\Lambda'}$, (4.14) will become

$$\frac{d}{dt}\|q\|^2 + \delta\|q\|^2 \leq c_6 e^{-\alpha t}, \quad \alpha > 0, \tag{4.15}$$

that will be integrated:

$$\|q(t)\|^2 \leq \|q(t_0)\|^2 e^{-\delta(t-t_0)} + e^{-\delta}\int_{t_0}^{t} e^{(\alpha-\delta)s}\,ds. \tag{4.16}$$

We deduce then

$$\|q(t)\| = \|(I - P_\Lambda)v\| \to 0 \qquad \text{as} \qquad t \to +\infty, \tag{4.17}$$

$$\int_0^{+\infty} \|q(t)\|^2\,dt < +\infty. \tag{4.18}$$

On the other hand, denoting $\Psi(t) = P_{\Lambda''}$, where $\Lambda'' < \Lambda$ is the first eigenvalue of A strictly smaller than Λ, $\delta' = \Lambda - \Lambda'' > 0$, the scalar product of $P_{\Lambda''}$ applied to (4.9) by $\Psi(t)$ will be written as

$$\frac{1}{2}\frac{d}{dt}|\Psi|^2 + \|\Psi\|^2 - \lambda(t)|\Psi|^2 \geq -\rho|\Psi|^2. \tag{4.19}$$

Applying the inequality of Young to the right-hand side, (4.19) becomes, using also $\|\Psi\|^2 \geq \Lambda''|\Psi|^2$:

$$\frac{d}{dt}|\Psi|^2 \geq (2\lambda(t) - 2(1+\varepsilon')\Lambda'')|\Psi|^2 - c_7 e^{-\alpha t}, \tag{4.20}$$

or, according to $\lambda(t) \to \Lambda(u_0)$ as $t \to +\infty$, we can write for $t \gg 1$

$$\frac{d}{dt}|\Psi|^2 \geq 2(\Lambda - (1+\varepsilon')\Lambda'')|\Psi|^2 - c_7 e^{-\alpha t}. \tag{4.21}$$

Choosing $\varepsilon' = \frac{\delta'}{2\Lambda''}$, (4.21) will be written as

$$\frac{d}{dt}|\Psi|^2 \geq \delta'|\Psi|^2 - c_7 e^{-\alpha t}, \tag{4.22}$$

that will be integrated:

$$|\Psi|^2 + \frac{e^{\delta' t}}{(\delta' + \alpha)}(e^{-(\alpha-\delta')t_0} - e^{-(\alpha-\delta')t}) \geq |\Psi(t_0)|^2 \, e^{\delta'(t-t_0)}. \tag{4.23}$$

Then, multiplying (4.23) by $e^{-\delta'(t-t_0)}$, for $t \to +\infty$ we obtain

$$|\Psi(t_0)| \leq c_8 e^{-\delta' t_0}, \quad t_0 \geq 0. \tag{4.24}$$

We deduce from (4.24):

$$|\Psi(t)| = |P_{\Lambda''} v(t)| \to 0 \quad \text{ as } \quad t \to +\infty, \tag{4.25}$$

$$\int_0^{+\infty} |P_{\Lambda''} v|^2 \, (s) \, ds < +\infty. \tag{4.26}$$

Since $(I - R_\Lambda)v = P_{\Lambda''}v + (I - P_\Lambda)v$, from (4.17) and (4.25) there results (4.3).

Corollary 4.1

$$\int_0^{+\infty} \|(I - R_\Lambda)v\| \, (s) \, ds < +\infty. \tag{4.27}$$

Proof. (4.27) results from (4.18) and (4.26).

The following corollary makes precise the convergence of $\|v(t)\|^2$ towards $\Lambda(u_0)$.

Corollary 4.2

$$\int_0^{+\infty} |\lambda(t) - \Lambda(u_0)| \, dt < +\infty. \tag{4.28}$$

Proof. We note that $|\lambda - \Lambda(u_0)| \leq (1 + \frac{1}{\Lambda_1}) \, \|(I - R_\Lambda)v\|^2$, thus from (4.27) we deduce (4.28).

Theorem 4.1 $\lim_{t\to\infty} e^{\Lambda(u_0)t} \, |S(t)u_0|$ *exists, is finite and not null.*

More precisely: $e^{\Lambda(u_0)t}S(t)u_0$ converges in H and V towards the eigenvector $\Lambda(u_0)$ of A associated with $\Lambda(u_0)$.

Proof of Theorem 4.1. Apllying R_Λ to the equation $\frac{du}{dt} + Au + f(u) = 0$, it becomes

$$\frac{d}{dt}(e^{\Lambda t} R_\Lambda u(t)) e^{\Lambda t} R_\Lambda f(u) = 0, \tag{4.29}$$

that will be integrated:

$$e^{\Lambda t}R_\Lambda u(t) - e^{\Lambda s}R_\Lambda u(s) = -\int_s^t e^{\Lambda\sigma}R_\Lambda f(u(\sigma))\,d\sigma, \tag{4.30}$$

but $|f(u)| \le c_9 e^{-(\Lambda+\alpha)t}$ according to (4.5) and then

$$\left|\int_s^t e^{\Lambda\sigma}R_\Lambda f(u(\sigma))\,d\sigma\right| \le c_1 \int_s^t e^{-\alpha\sigma}\,d\sigma \le c_1 \int_s^{+\infty} e^{-\alpha\sigma}\,d\sigma < +\infty. \tag{4.31}$$

Consequently, the integral on the right-hand side of (4.30) is convergent and $e^{\Lambda t}R_\Lambda u(t)$ converges.

If $\lim_{t\to+\infty} e^{\Lambda t}R_\Lambda u(t) = U_\Lambda = 0$, then the equation (4.30) will be:

$$e^{\Lambda t}R_\Lambda u(t) = \int_t^{+\infty} e^{\Lambda\sigma}R_\Lambda f(u(\sigma))\,d\sigma, \tag{4.32}$$

and since $|f(u)| \le c_9 e^{-(\Lambda+\alpha)t}$, $\alpha > 0$, we deduce that

$$|R_\Lambda u(t)| \le c_{10} e^{-(\Lambda+\alpha)t}, \tag{4.33}$$

but (4.33) contradicts Corollary 3.1, since according to Proposition 4.1 we have $|R_\Lambda u(t)|$ is equivalent to $c\,|u(t)|$ when $t \to \infty$; consequently, the limit $U_\Lambda(u_0) \ne 0$, and we have

$$\lim_{t\to+\infty} e^{\Lambda t}R_\Lambda u(t) = U_\Lambda \ne 0. \tag{4.34}$$

We simply verify that U_Λ is an eigenvalue of the operator A.

Because of Proposition 4.1, we deduce that $R_\Lambda u(t) = u(t) + \varepsilon(t)u(t)$, where $\varepsilon(t) \to 0$ $(t \to +\infty)$ and we obtain

$$\lim_{t\to+\infty} e^{\Lambda t}u(t) = U_\Lambda \ne 0, \tag{4.35}$$

which leads to the proof of Theorem 4.1.

Remark 4.1 Theorem 4.1 proves that for $t \to +\infty$, $|u(t)| = \bigcirc(e^{-\mu t})$ $\forall \mu > 0$, so $u \equiv 0$.

References

[1] AISSAOUI M. Z., *Comportement asymptotique et variétés spectrales associées à une classe d'équations d'évolution à non linéarité polynômiale*. (to appear)

[2] AISSAOUI M. Z., *Forme normale pour une classe d'équations d'évolution à non linéarité polynômiale*. (to appear).

[3] AISSAOUI M. Z., *Comportement asymptotique et forme normale pour une classe d'équations paraboliques abstraites*, Thèse de Doctorat, Université de Paris IX, Centre d'Orsay, 1987.

[4] FOIAS C. AND SAUT J.-C., *Asymptotic behaviour, as $t \to +\infty$, of solutions of Navier-Stokes equations and nonlinear spectral manifolds*, Indiana Univ. Math. J., **33** (1984), No. 3, 459–477.

[5] FOIAS C. AND SAUT J.-C., *On the smoothness of the nonlinear spectral manifolds to the Navier-Stokes equations*, Indiana Univ. Math. J., **33** (1984), No. 6, 911–926.

[6] FOIAS C. AND SAUT J.-C., *Linearization and normal form of the Navier-Stokes equation with polynomial forces*, Ann. Inst. Henri-Poincaré Anal. Non Linéaire, **4** (1987), No. 1, 1–47.