Solution of Selected Exercises

4.1.1

3. Given $y = c_1 e^{4x} + c_2 x \ln x$, then
$y' = c_1 + c_2(1 + \ln x)$,
$y(1) = c_1 = 3$,
$y'(1) = c_1 + c_2 = -1$
From these two equations we get
$c_1 = 3, c_2 = -4$. Thus the solution is
$y = 3x - 4x \ln x$.

10. Since $a_0(x) = \tan x$ and $x_0 = 0$ the problem has a unique solution for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

12. Here $y = c_1 + c_2 x^2$. Therefore $y(0) = c_1 = 1, y'(1) = 2c_2 = 6$ which implies that $c_1 = 1$ and $c_2 = 3$. The solution is $y = 1 + 3x^2$.

13. (a) Since $y = c_1 e^x \cos x + c_2 e^x \sin x$ and so $y' = c_1 e^x(-\sin x + \cos x) + c_2 e^x(\cos x + \sin x)$. This implies that $y(0) = c_1 = 1, y'(0) = c_1 + c_2 = 0$ so that $c_1 = 1$ and $c_2 = -1$. Therefore the solution is $y = e^x \cos x - e^x \sin x$.

15. The functions are linearly dependent as $(-4)x + (3)x^2 + (1)(4x - 3x^2) = 0$.

23. $W(e^{-3x}, e^{4x}) = \begin{vmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{vmatrix} = 4e^x + 3e^x = 7e^x \neq 0.$
Hence e^{-3x} and e^{4x} are linearly independent solutions, so e^{-3x}, e^{4x} is a fundamental set of solutions. This gives us $y = c_1 e^{-3x} + c_2 e^{4x}$ as the general solution.

28. The functions satisfy the differential equation and their Wronskian

$$\begin{aligned} W(\cos(\ln x), \sin(\ln x)) &= \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix} \\ &= \frac{1}{x}\left[\cos^2 \ln x + \sin^2 \ln x\right] \\ &= \frac{1}{x} \qquad \text{as } \cos^2 \ln x + \sin^2 \ln x = 1 \\ &\neq 0 \qquad \text{for } 0 \leq x < \infty \end{aligned}$$

$\{\cos(\ln x), \sin(\ln x)\}$ is a fundamental set of solutions. The general solution is $y = c_1 cos(\ln x) + c_2 \sin(\ln x)$.

33. $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ form a fundamental set of solutions of the homogeneous equation $y'' - 4y' + 4y = 0$, and y_p is a particular solution of non-homogeneous equation $y'' - 4y' + 4y = 2e^{2x} + 4x - 12$.

4.2

3.

$$\begin{aligned} y &= u(x)\cos 4x, \quad \text{so} \\ y' &= -4u\sin 4x + u'\cos 4x \\ y'' &= u''\cos 4x - 8u' sin4x - 16u\cos 4x \end{aligned}$$

and

$$\begin{aligned} y'' + 16y &= (\cos 4x)u'' - 8(\sin 4x)u' = 0, \quad \text{or} \\ u'' - 8(\tan 4x)u' &= 0 \end{aligned}$$

If $w = u'$ we obtain the first-order equation $w' - 8(\tan 4x)w = 0$, which has the integrating factor $e^{-8}\int \tan 4x dx = \cos^2 4x$. Now, $\frac{d}{dx}\left[(\cos^2 4x)w\right] = 0$ gives $(\cos^2 4x)w = 0$. Therefore, $w = u' = c\sec^2 4x$ and $u = c_1 \tan 4x$. A second solution is $y_2 = \tan 4x \cos 4x = \sin 4x$.

14.

$$\begin{aligned} y'' - \frac{3x}{x^2}y' + \frac{5}{x} &= 0 \\ p(x) &= -\frac{3}{x}, \text{we have} \\ y_2 &= x^2\cos(\ln x)\int \frac{e^{-\int -3\frac{dx}{x}}}{x^4\cos^2(\ln x)}dx \\ &= x^2\cos(\ln x)\int \frac{x^3}{x^4\cos^2(\ln x)}dx \\ &= x^2\cos(\ln x)\tan(\ln x) \\ &= x^2\sin(\ln x) \end{aligned}$$

Therefore, a second solution is

$$y_2 = x^2\sin(\ln x)$$

19. Define $y = u(x)e^x$, so

$$y' = ue^x + u'e^x, y'' = u''e^x + 2u'e^x + ue^x$$

and

$$y'' - 3y' + 2y = e^x u'' - e^x u' = 0 \text{ or } u'' - u' = 0$$

If $w = u'$, we obtain the first order equation $w' - w = 0$, which has the integrating factor $e^{-\int dx} = e^{-x}$.
Now,

$$\frac{d}{dx}\left[e^{-x}w\right] \text{ gives } e^{-x}w = c$$

Therefore, $w = u' = ce^x$ and $u = ce^x$. A second solution is $y_2 = e^x e^x = e^{2x}$.
To find a particular solution we try $y_p = Ae^{3x}$. Then $y' = 3Ae^{3x}$, $y'' = 9Ae^{3x}$, and $9Ae^{3x} - 3(3Ae^{3x}) + 2Ae^{3x} = 5e^{3x}$. Thus $A = \frac{5}{2}$ and $y_p = \frac{5}{2}e^{3x}$.
The general solution is

$$y = c_1 e^x + c_2 e^{2x} + \frac{5}{2}e^{3x}$$

4.3

9. The auxiliary equation is

$$m^2 + 9 = 0 \Rightarrow m = 3i \text{ and } m = -3i$$

so that

$$y = c_1 \cos 3x + c_2 \sin 3x$$

10. The auxiliary equation is

$$2m^2 + 2m + 1 = 0 \Rightarrow m = -\frac{1}{2} \pm i^2$$

so that

$$y = e^{-\frac{x}{2}} \left(c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2} \right)$$

15. The auxiliary equation is

$$m^3 - 4m^2 - 5m = 0 \Rightarrow m = 0, m = 5 \text{ and } m = -1$$

so that

$$y = c_1 + c_2 e^{5x} + c_3 e^{-x}$$

34. The auxiliary equation is

$$m^2 - 2m + 1 = 0 \Rightarrow m = 1 \text{ and } m = -1$$

so that

$$y = c_1 e^x + c_2 x e^x$$

If $y(0) = 5$ and $y'(0) = 10$ then $c_1 = 5, c_1 + c_2 = 10$ so $c_1 = 5$, $y'(0) = 10$ then $y = 5e^x + 5xe^x$

40. The auxiliary equation is

$$m^2 - 2m + 2 = 0 \Rightarrow m = 1 \pm i$$

so that

$$y = e^x \left(c_1 \cos x + c_2 \sin x \right)$$

If $y(0) = 1$ and $y(\pi) = 1$ then $c_1 = 1$ and $y(\pi) = e^{\pi} \cos \pi = -e^{\pi}$. Since $-e^{\pi} \neq 1$, the boundary-value problem has no solution.

4.5

8.

$$\begin{aligned} y''' + 4y'' + 3y' &= x^2 \cos x - 3x \\ (D^3 + 4D^2 + 3D)y &= x^2 \cos x - 3x \\ Ly &= x^2 \cos x - 3x \\ \text{where,} \qquad L &= (D^3 + 4D^2 + 3D) \\ &= D(D^2 + 4D + 3) \\ &= D(D+1)(D+3) \end{aligned}$$

13.

$$\begin{aligned} &(D-2)(D+5)(e^{2x} + 3e^{-5x}) \\ &= (D-2)(2e^{2x} - 15e^{-5x} + 5e^{2x} + 15e^{-5x}) \\ &= (4e^{2x} + 75e^{-5x} + 10e^{2x} - 75e^{-5x}) - (4e^{2x} - 30e^{-5x} + 10e^{2x} + 30e^{-5x}) \\ &= 0 \end{aligned}$$

41.

$$y''' + y'' = 8x^2$$

Apply D^3 to the differential equation, we obtain

$$D^3(D^3 + D^2)y = D^5(D+1)y = 0.$$

Then

$$y = c_1 + c_2 x + c_3 e^{-x} + c_4 x^4 + c_5 x^3 + c_6 x^2$$

and

$$y_p = Ax^4 + Bx^3 + Cx^2$$

Substituting y_p into the differential equation yields

$12Ax^2 + (24A + 6B)x + (6B + 2C) = 8x^2$

Equating coefficients give

$$\begin{aligned} 12A &= 8 \\ 24A + 6B &= 0 \\ 6B + 2C &= 0 \end{aligned}$$

Then

$$\begin{aligned} A &= \frac{2}{3} \\ B &= -\frac{8}{3} \\ C &= 8 \end{aligned}$$

and the general solution is

$$y = c_1 + c_2x + c_3e^{-x} + \frac{2}{3}x^4 - \frac{8}{3}x^3 + 8x^2$$

48.

Applying $D(D^2 + 1)$ to the differential equation, we obtain

$$D(D^2 + 1)(D^2 + 4)y = 0$$

Then

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos x + c_4 \sin x + c_5$$

and

$$y_p = A \cos x + B \sin x + C$$

Substituting y_p into the differential equation yields

$$3A \cos x + 3B \sin x + 4C = 4 \cos x + 3 \sin x - 8$$

Equating coefficients gives

$$A = \frac{4}{3}$$
$$B = 1$$
$$C = -2$$

The general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{4}{3} \cos x + \sin x - 2$$

53. Applying $D^2 - 2D + 2$ to the differential equation, we obtain

$(D^2 - 2D + 2)(D^2 - 2D + 5)y = 0$

Then

$$y = e^x(c_1 \cos 2x + c_2 \sin 2x) + e^x(c_3 \cos x + c_4 \sin x)$$

and

$$y_p = Ae^x \cos x + Be^x \sin x.$$

Substituting y_p into the differential equation yields

$$3Ae^x \cos x + 3Be^x \sin x = e^x \sin x.$$

Equating coefficients give

$$A = 0$$
$$B = \frac{1}{3}$$

and the general solution is

$$y = e^x(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{3}e^x \sin x.$$

4.6

6. The auxiliary equation is $m^2 + 1 \;=\; 0$, so

$$\begin{aligned} y_c &= c_1 \cos x + c_2 \sin x, \quad \text{and} \\ W &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \end{aligned}$$

Identifying $f(x) = \sec^2 x$, we obtain

$$\begin{aligned} u_1' &= -\frac{\sin x}{\cos^2 x} \\ u_2' &= \sec x \end{aligned}$$

Then

$$\begin{aligned} u_1 &= -\frac{1}{\cos x} = -\sec x \\ u_2 &= \ln|\sec x + \tan x| \end{aligned}$$

and

$$\begin{aligned} y &= c_1 \cos x + c_2 \sin x - \cos x \sec x + \sin x \ln|\sec x + \tan x| \\ &= c_1 \cos x + c_2 \sin x - 1 + \sin x \ln|\sec x + \tan x| \end{aligned}$$

11. The auxiliary equation is $m^2 + 3m + 2 = (m + 1)(m + 2) = 0$, so

$$\begin{aligned} y_c &= c_1 e^{-x} + c_2 e^{-2x}, \quad \text{and} \\ W &= \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x} \end{aligned}$$

Identifying $f(x) = \frac{1}{(1+e^x)}$, we obtain

$$\begin{aligned} u_1' &= \frac{e^x}{1 + e^x} \\ u_2' &= -\frac{e^{2x}}{1 + e^x} = \frac{e^x}{1 + e^x} - e^x \end{aligned}$$

Then

$$\begin{aligned} u_1 &= \ln(1+e^x) \\ u_2 &= \ln(1+e^x) - e^x, \end{aligned}$$

and

$$\begin{aligned} y &= c_1e^{-x} + c_2e^{-2x} + e^{-x}\ln(1+e^x) + e^{-2x}\ln(1+e^x) - e^{-x} \\ &= c_3e^{-x} + c_2e^{-2x} + (1+e^{-x})e^{-x}\ln(1+e^x) \end{aligned}$$

12. The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$, so

$$\begin{aligned} y_c &= c_1e^x + c_2xe^x, \quad \text{and} \\ W &= \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x} \end{aligned}$$

Identifying $f(x) = \frac{e^x}{(1+x^2)}$, we obtain

$$\begin{aligned} u_1' &= \frac{xe^xe^x}{e^{2x}(1+x^2)} = -\frac{x}{1+x^2} \\ u_2' &= \frac{e^xe^x}{e^{2x}(1+x^2)} = \frac{1}{1+x^2} \end{aligned}$$

Then

$$\begin{aligned} u_1 &= -\frac{1}{2}\ln(1+x^2) \\ u_2 &= \tan^{-1}x, \end{aligned}$$

and

$$y = c_1e^x + c_2xe^x - \frac{1}{2}e^x\ln(1+x^2) + xe^x\tan^{-1}x$$

17. The auxiliary equation is $3m^2 - 6m + 6 = 0$, so

$$\begin{aligned} y_c &= e^x(c_1\cos x + c_2\sin x), \quad \text{and} \\ W &= \begin{vmatrix} e^x\cos x & e^x\sin x \\ e^x\cos x - e^x\sin x & e^x\cos x + e^x\sin x \end{vmatrix} = e^{2x} \end{aligned}$$

Identifying $f(x) = \frac{1}{1}e^x \sec x$, we obtain

$$u_1' = -\frac{(e^x \sin x)(e^x \sec x)/3}{e^{2x}} = -\frac{1}{3}\tan x$$
$$u_2' = \frac{(e^x \cos x)(e^x \sec x)/3}{e^{2x}} = \frac{1}{3}$$

Then

$$u_1 = -\frac{1}{3}\ln(\cos x)$$
$$u_2 = \frac{1}{3}x,$$

and

$$y = c_1 e^x \cos x + c_2 e^x \cos x + \frac{1}{3}\ln(\cos x)e^x \cos x + \frac{1}{3}xe^x \sin^x$$

24. Write the equation in the form

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = \frac{\sec(\ln x)}{x^2}$$

and identify $f(x) = \frac{\sec(\ln x)}{x^2}$. From

$$y_1 = \cos(\ln x). \quad \text{and}$$
$$y_2 = \sin(\ln x)$$

we compute

$$W = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix} = \frac{1}{x}$$

Now

$$u_1' = -\frac{\tan(\ln x)}{x}, \quad \text{so}$$
$$u_1 = \ln|\cos(\ln x)|$$

and

$$u_2' = \frac{1}{x}, \quad \text{so}$$
$$u_2 = \ln x$$

Thus, particular solution is

$$y_p = \cos(\ln x) \ln|\cos(\ln x)| + (\ln x)\sin(\ln x)$$

4.7

3. The auxiliary equation is $m^2 = 0$ so that $y = c_1 + c_2 \ln x$

4. The auxiliary equation is $m^2 - 4m = m(m-4) = 0$ so that $y = c_1 + c_2 x^4$

5. The auxiliary equation is $m^2 + 4 = 0$ so that
$y = c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)$

11. The auxiliary equation is $m^2 + 4m + 4 = (m+2)^2 = 0$ so that
$y = c_1 x^{-2} + c_2 x^{-2} \ln x$

14. The auxiliary equation is $m^2 - 8m + 41 = 0$ so that
$y = x^4[c_1 \cos(5 \ln x) + c_2 \sin(5 \ln x)]$

21. The auxiliary equation is $m^2 - 2m + 1 = 0$ or $(m-1)^2 = 0$, so that

$$y_c = c_1 x + c_2 x \ln x, \quad \text{and}$$
$$W(x, x\ln x) = \begin{vmatrix} x & x\ln x \\ 1 & 1 + \ln x \end{vmatrix} = x^2$$

Identifying $f(x) = \frac{2}{x}$, we obtain

$$u_1' = -2\frac{\ln x}{x}$$
$$u_2' = \frac{2}{x}$$

Then

$$u_1 = -(\ln x)^2$$
$$u_2 = 2\ln x$$

and

$$\begin{aligned} y &= c_1 x + c_2 x \ln x - x(\ln x)^2 + 2x(\ln x)^2 \\ &= c_1 x + c_2 x \ln x + x(\ln x)^2 \end{aligned}$$

22. The auxiliary equation is $m^2 - 3m + 2 = (m-1)(m-2) = 0$, so

$$y_c = c_1 x + c_2 x^2, \quad \text{and}$$
$$W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$$

Identifying $f(x) = x^2 e^x$, we obtain

$$u_1' = -x^2 e^x$$
$$u_2' = x e^x$$

Then

$$u_1 = -x^2 e^x + 2xe^x - 2e^x$$
$$u_2 = xe^x - e^x,$$

and

$$\begin{aligned} y &= c_1 x + c_2 x^2 - x^3 e^x + 2x^2 e^x - 2xe^x + x^3 e^x - x^2 e^x \\ &= c_1 x + c_2 x^2 + x^2 e^x - 2xe^x \end{aligned}$$

25. The auxiliary equation is $m^2 + 1 = 0$, so that

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x), \quad \text{and}$$
$$y' = -c_1 \frac{1}{x} \sin(\ln x) + c_2 \frac{1}{x} \cos(\ln x)$$

The initial conditions imply $c_1 = 1$ and $c_2 = 2$. Thus

$$y = \cos(\ln x) + 2\sin(\ln x)$$

26. The auxiliary equation is $m^2 - 4m + 4 = (m-2)^2 = 0$, so that

$$y = c_1 x^2 + c_2 x^2 \ln x, \quad \text{and}$$
$$y' = 2c_1 x + c_2(x + 2x \ln x)$$

The initial conditions imply $c_1 = 5$ and $c_2 + 10 = 3$. Thus

$$y = 5x^2 - 7x^2 \ln x$$

35. We have

$$4t^2\frac{d^2y}{dt^2}+y=0; y(t)\Big|_{t=1}=2,$$

$$y'(t)|_{t=1}=-4$$

auxiliary equation is $4m^2-4m+1=(2m-1)^2=0$, so that

$$y=c_1t^{\frac{1}{2}}+c_2t^{\frac{1}{2}}\ln t, \quad \text{and}$$

$$y'=\frac{1}{2}c_1t^{-\frac{1}{2}}+c_2(t^{-\frac{1}{2}}+\frac{1}{2}t^{-\frac{1}{2}}\ln t)$$

The initial conditions imply $c_1=2$ and $1+c_2=-4$. Thus

$$y=2t^{\frac{1}{2}}-5t^{\frac{1}{2}}\ln t=2(-x)^{\frac{1}{2}}-5(-x)^{\frac{1}{2}}\ln(-x), \quad x<0$$

36. The differential equation and initial conditions become

$$t^2\frac{d^2y}{dt^2}-4t\frac{dy}{dt}+6y=0; y(t)\Big|_{t=2}=8,$$

$$y'(t)|_{t=2}=0$$

The auxiliary equation is $m^2-5m+6=(m-2)(m-3)=0$, so that

$$y=c_1t^2+c_2t^3, \quad \text{and}$$

$$y'=2c_1t+3c_2t^2$$

The initial conditions imply

$$4c_1+8c_2=84c_1+12c_2=0$$

from which we find $c_1=6$ and $c_2=-2$. Thus

$$y=6t^2-2t^3=6x^2+2x^3, \quad x<0.$$

6.1

1. $\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}/(n+1)}{2^n x^n/n}\right| = \lim_{n\to\infty}\frac{2n}{n+1}|x| = 2|x|$
This series is absolutely convergent for $2|x| < 1$ or $|x| < 1/2$. At $x = -1/2$, the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by the alternating series test. At $x = 1/2$, the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series which diverges. Thus, the given series converges on $[-1/2, 1/2)$.

9.

$$\begin{aligned}\sum_{n=1}^{\infty}2nc_nx^{n-1}+\sum_{n=0}^{\infty}6c_nx^{n+1} &= 2\cdot 1\cdot c_1x^0+\underbrace{\sum_{n=2}^{\infty}2nc_nx^{n-1}}_{k=n-1}+\underbrace{\sum_{n=2}^{\infty}6c_nx^{n+1}}_{k=n+1}\\ &= 2c_1+\sum_{k=1}^{\infty}2(k+1)c_{k+1}x^k+\sum_{k=1}^{\infty}6c_{k-1}x^k\\ &= 2c_1+\sum_{k=1}^{\infty}[2(k+1)c_{k+1}+6c_{k-1}]x^k\end{aligned}$$

17. Substituting $y = \sum_{n=0}^{\infty}c_nx^n$ into the differential equation we have

$$\begin{aligned}y''+x^2y'+xy &= \underbrace{\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}}_{k=n-2}+\underbrace{\sum_{n=1}^{\infty}nc_nx^{n+1}}_{k=n+1}+\underbrace{\sum_{n=0}^{\infty}c_nx^{n+1}}_{k=n+1}\\ &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k+\sum_{k=2}^{\infty}(k-1)c_{k-1}x^k+\sum_{k=1}^{\infty}c_{k-1}x^k\\ &= 2c_2+(6c_3+c_0)x+\sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2}+kc_{k-1}]x^k\\ &= 0\end{aligned}$$

Thus

$$c_2 = 0\cdot 6c_3+c_0$$
$$(k+2)(k+1)c_{k+2}+kck-1 = 0$$

and

$$
\begin{aligned}
c_3 &= -\frac{1}{6}c_0 \\
c_{k+2} &= -\frac{k}{(k+2)(k+1)}c_{k-1}, \quad k = 2, 3, 4, \cdots
\end{aligned}
$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$
\begin{aligned}
c_3 &= -\frac{1}{6} \\
c_4 = c_5 &= 0 \\
c_6 &= \frac{1}{45}
\end{aligned}
$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$
\begin{aligned}
c_3 &= 0 \\
c_4 &= -\frac{1}{6} \\
c_5 = c_6 &= 0 \\
c_7 &= \frac{5}{252}
\end{aligned}
$$

and so on. Thus, two solutions are

$$
\begin{aligned}
y_1 &= 1 - \frac{1}{6}x^3 + \frac{1}{45}x^6 - \cdots \qquad \text{and} \\
y_2 &= x - \frac{1}{6}x^4 + \frac{5}{252}x^7 - \cdots
\end{aligned}
$$

18. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned}
y'' + 2x^2y' + 2y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + 2\underbrace{\sum_{n=1}^{\infty} nc_n x^{n+1}}_{k=n} + 2\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\
&= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k + 2\sum_{k=1}^{\infty} kc_k x^k + 2\sum_{k=0}^{\infty} c_k x^k \\
&= 2c_2 + 2c_0 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} + 2(k+1)c_k]x^k \\
&= 0
\end{aligned}$$

Thus

$$\begin{aligned}
2c_2 + 2c_0 &= 0 \\
(k+2)(k+1)c_{k+2} + 2(k+1)c_k &= 0
\end{aligned}$$

and

$$\begin{aligned}
c_2 &= -c_0 \\
c_{k+2} &= -\frac{2}{k+2}c_k, \quad k = 1, 2, 3, \cdots
\end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$\begin{aligned}
c_2 &= -1 \\
c_3 = c_5 = c_7 = \cdots &= 0 \\
c_4 &= \frac{1}{2} \\
c_6 &= -\frac{1}{6}
\end{aligned}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$
\begin{aligned}
c_2 = c_4 = c_6 = \cdots = 0 \\
c_3 &= -\frac{2}{3} \\
c_5 &= \frac{4}{15} \\
c_7 &= -\frac{8}{105}
\end{aligned}
$$

and so on. Thus, two solutions are

$$
\begin{aligned}
y_1 &= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots \qquad \text{and} \\
y_2 &= x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \cdots
\end{aligned}
$$

6.2

19. Substituting $\sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$3xy'' + (2-x)y' - y$$
$$= (3r^2 - r)c_0 x^{r-1} + \sum_{k=1}^{\infty}[3(k+r-1)(k+r)c_k + 2(k+r)c_k - (k+r)c_{k-1}]x^{k+r-1}$$
$$= 0,$$

which implies

$$3r^2 - r = r(3r-1) = 0$$

and

$$(k+r)(3k+3r-1)c_k - (k+r)c_{k-1} = 0$$

The indicial roots are $r = 0$ and $r = 1/3$. For $r = 0$ the recurrence relation is

$$c_k = \frac{c_{k-1}}{(3k-1)}, \quad k = 1, 2, 3, \cdots$$

and

$$c_1 = \frac{1}{2}c_0, \quad c_2 = \frac{1}{10}c_0, \quad c_3 = \frac{1}{80}c_0$$

For $r = 1/3$ the recurrence relation is

$$c_k = \frac{c_{k-1}}{3k}, \quad k = 1, 2, 3, \cdots$$

and

$$c_1 = \frac{1}{3}c_0, \quad c_2 = \frac{1}{18}c_0, \quad c_3 = \frac{1}{162}c_0$$

The general solution on $(0, \infty)$ is

$$y = C_1\left(1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{80}x^3 + \cdots\right) + C_2 x^{1/3}\left(1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \cdots\right)$$

20. Substituting $\sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned}
&x^2y'' - (x - \frac{2}{9})y \\
&= (r^2 - r + \frac{2}{9})c_0x^r + \sum_{k=1}^{\infty}[(k+r)(k+r-1)c_k + \frac{2}{9}c_k - c_{k-1}]x^{k+r} \\
&= 0,
\end{aligned}$$

which implies

$$r^2 - r + \frac{2}{9} = \left(r - \frac{2}{3}\right)\left(r - \frac{1}{3}\right) = 0$$

and

$$\left[(k+r)(k+r-1) + \frac{2}{9}\right] c_k - c_{k-1} = 0$$

8.1

4. Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$X' = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix} X$$

5. Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$X' = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} X + \begin{pmatrix} 0 \\ -3t^2 \\ t^2 \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ -t \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

8.

$$\frac{dx}{dt} = 7x+5y-9z-8e^{-2t}; \quad \frac{dy}{dt} = 4x+y+z+2e^{5t}; \quad \frac{dz}{dt} = -2y+3z+5e^{5t}-3e^{-2t}$$

13. Since

$$X' = \begin{pmatrix} 3/2 \\ -3 \end{pmatrix} e^{-3t/2} \quad \text{and} \quad \begin{pmatrix} -1 & 1/4 \\ 1 & -1 \end{pmatrix} X = \begin{pmatrix} 3/2 \\ -3 \end{pmatrix} e^{-3t/2}$$

we see that

$$X' = \begin{pmatrix} -1 & 1/4 \\ 1 & 1 \end{pmatrix} X$$

14. Since

$$X' = \begin{pmatrix} 5 \\ -1 \end{pmatrix} e^t + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^t \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} X = \begin{pmatrix} 5 \\ -1 \end{pmatrix} e^t + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^t$$

we see that

$$X' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} X$$

8.2

3. The system is

$$\mathbf{X}' = \begin{pmatrix} -4 & 2 \\ -5/2 & 2 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 1)(\lambda + 3) = 0$. For $\lambda_1 = 1$ we obtain

$$\left(\begin{array}{cc|c} -5 & 2 & 0 \\ -5/2 & 1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

For $\lambda_2 = -3$ we obtain

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ -5/2 & 5 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Then

$$X = c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t}$$

6. The system is

$$\mathbf{X}' = \begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda(\lambda + 5) = 0$. For $\lambda_1 = 0$ we obtain

$$\left(\begin{array}{cc|c} -6 & 2 & 0 \\ -3 & 1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

For $\lambda_2 = -5$ we obtain

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ -3 & 6 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Then

$$X = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-5t}$$

7. The system is

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A}-\lambda\mathbf{I}) = (\lambda-1)(2-\lambda)(\lambda+1) = 0$. For $\lambda_1 = 1, \lambda_2 = 2,$ and $\lambda_3 = -1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{K}_2 = \begin{pmatrix}2\\3\\1\end{pmatrix}, \text{ and } \mathbf{K}_3 = \begin{pmatrix}1\\0\\2\end{pmatrix},$$

so that

$$X = c_1\begin{pmatrix}1\\0\\0\end{pmatrix}e^{t} + c_2\begin{pmatrix}2\\3\\1\end{pmatrix}e^{2t} + c_3\begin{pmatrix}1\\0\\2\end{pmatrix}e^{-t}.$$

19. We have $\det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^2 = 0$. For $\lambda_1 = 0$ we obtain

$$\mathbf{K} = \begin{pmatrix}1\\3\end{pmatrix}$$

A solution of $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix}1\\2\end{pmatrix}$$

so that

$$X = c_1\begin{pmatrix}1\\3\end{pmatrix} + c_2\left[\begin{pmatrix}1\\3\end{pmatrix}t + \begin{pmatrix}1\\2\end{pmatrix}\right]$$

20. We have $\det(\mathbf{A}-\lambda\mathbf{I}) = (\lambda+1)^2 = 0$. For $\lambda_1 = -1$ we obtain

$$\mathbf{K} = \begin{pmatrix}1\\1\end{pmatrix}$$

A solution of $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix}0\\1/5\end{pmatrix}$$

so that

$$X = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t} + c_2\left[\begin{pmatrix}1\\1\end{pmatrix}te^{-t} + \begin{pmatrix}0\\1/5\end{pmatrix}e^{-t}\right]$$

21. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 2)^2 = 0$. For $\lambda_1 = 2$ we obtain

$$\mathbf{K} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}$$

so that

$$X = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} e^{2t} \right]$$

26. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (1 - \lambda)(\lambda - 2)^2 = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 2$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

A solution of $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

so that

$$X = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} e^{2t} \right]$$

27. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda - 1)^3 = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Solutions of $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{P}=\mathbf{K}$ and $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{Q}=\mathbf{P}$

$$\mathbf{P}=\begin{pmatrix}0\\1\\0\end{pmatrix} \quad \text{and} \quad \mathbf{Q}=\begin{pmatrix}1/2\\0\\0\end{pmatrix}$$

so that

$$X=c_1\begin{pmatrix}0\\1\\1\end{pmatrix}+c_2\left[\begin{pmatrix}0\\1\\1\end{pmatrix}te^t+\begin{pmatrix}0\\1\\0\end{pmatrix}e^t\right]+c_3\left[\begin{pmatrix}0\\1\\1\end{pmatrix}\frac{t^2}{2}e^t+\begin{pmatrix}0\\1\\0\end{pmatrix}te^t+\begin{pmatrix}1/2\\1\\0\end{pmatrix}e^t\right]$$

34. We have $\det(\mathbf{A}+\lambda\mathbf{I})=\lambda^2+1=0$. For $\lambda_1=i$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}-1-i\\2\end{pmatrix}$$

so that

$$\mathbf{X}_1=\begin{pmatrix}-1-i\\2\end{pmatrix}e^{it}=\begin{pmatrix}\sin t-\cos t\\2\cos t\end{pmatrix}+i\begin{pmatrix}\cos t-\sin t\\2\sin t\end{pmatrix}$$

Then

$$\mathbf{X}=c_1\begin{pmatrix}\sin t-\cos t\\2\cos t\end{pmatrix}+c_2\begin{pmatrix}\cos t-\sin t\\2\sin t\end{pmatrix}$$

35. We have $\det(\mathbf{A}-\lambda\mathbf{I})=\lambda^2-8\lambda+17=0$. For $\lambda_1=4+i$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}-1-i\\2\end{pmatrix}$$

so that

$$\mathbf{X}_1=\begin{pmatrix}-1-i\\2\end{pmatrix}e^{(4+i)t}=\begin{pmatrix}\sin t-\cos t\\2\cos t\end{pmatrix}e^{4t}+i\begin{pmatrix}-\sin t-\cos t\\2\sin t\end{pmatrix}e^{4t}$$

Then

$$\mathbf{X}=c_1\begin{pmatrix}\sin t-\cos t\\2\cos t\end{pmatrix}e^{4t}+c_2\begin{pmatrix}-\sin t-\cos t\\2\sin t\end{pmatrix}e^{4t}$$

36. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 10\lambda + 34 = 0$. For $\lambda_1 = 5 + 3i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 - 3i \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix} 1 - 3i \\ 2 \end{pmatrix} e^{(5+3i)t} = \begin{pmatrix} \cos 3t + 3 \sin t \\ 2 \cos 3t \end{pmatrix} e^{5t} + i \begin{pmatrix} \sin 3t - 3 \cos 3t \\ 2 \cos 3t \end{pmatrix} e^{5t}$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} \cos 3t + 3 \sin 3t \\ 2 \cos 3t \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} \sin 3t - 3 \cos 3t \\ 2 \cos 3t \end{pmatrix} e^{5t}$$

39. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda(\lambda^2 + 1) = 0$. For $\lambda_1 = 0$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_2 = i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} -i \\ i \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} -i \\ i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} \sin t \\ -\sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} -\cos t \\ \cos t \\ \sin t \end{pmatrix}$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\sin t \\ \cos t \end{pmatrix} + c_3 \begin{pmatrix} -\cos t \\ \cos t \\ \sin t \end{pmatrix}$$

41. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (1 - \lambda)(\lambda^2 - 2\lambda + 2) = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 + i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix} e^{(1+i)t} = \begin{pmatrix} \cos t \\ -\sin t \\ -\sin t \end{pmatrix} e^t + i \begin{pmatrix} \sin t \\ \cos t \\ \cos t \end{pmatrix}$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos t \\ -\sin t \\ -\sin t \end{pmatrix} e^t + c_3 \begin{pmatrix} \sin t \\ \cos t \\ \cos t \end{pmatrix} e^t$$

8.3

1. From

$$\mathbf{X}' = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi^{-1}} = \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi^{-1} F} dt = \int \begin{pmatrix} -11 \\ 5e^{-t} \end{pmatrix} dt = \begin{pmatrix} -11t \\ -5e^{-t} \end{pmatrix}$$

and

$$\mathbf{X_p} = \mathbf{\Phi U} = \begin{pmatrix} -11 \\ -11 \end{pmatrix} t + \begin{pmatrix} -15 \\ -10 \end{pmatrix}$$

2. From

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi^{-1}} = \begin{pmatrix} \frac{3}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ -\frac{1}{2}e^t & \frac{1}{2}e^t \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi^{-1} F} dt = \int \begin{pmatrix} -2te^{-t} \\ 2te^t \end{pmatrix} dt = \begin{pmatrix} 2te^{-t} + 2e^{-t} \\ 2te^t - 2e^t \end{pmatrix}$$

and

$$\mathbf{X_p} = \mathbf{\Phi U} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} t + \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

7. From

$$\mathbf{X}' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 12 \\ 12 \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t}$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi^{-1}} = \begin{pmatrix} \frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi^{-1}F} dt = \int \begin{pmatrix} 6te^{-3t} \\ 6te^{3t} \end{pmatrix} dt = \begin{pmatrix} -2te^{-3t} - \frac{2}{3}e^{-3t} \\ 2te^{3t} - \frac{2}{3}e^{3t} \end{pmatrix}$$

and

$$\mathbf{X_p} = \mathbf{\Phi U} = \begin{pmatrix} -12 \\ 0 \end{pmatrix} t + \begin{pmatrix} -4/3 \\ -4/3 \end{pmatrix}$$

8.4

1. For $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ we have

$$\begin{aligned}
\mathbf{A}^2 &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \\
\mathbf{A}^3 &= \mathbf{A}\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}, \\
\mathbf{A}^4 &= \mathbf{A}\mathbf{A}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix},
\end{aligned}$$

and so on. In general

$$\mathbf{A}^k = \begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix} \qquad \text{for } k = 1, 2, 3, \cdots$$

Thus

$$\begin{aligned}
e^{\mathbf{A}t} &= \mathbf{I} + \frac{\mathbf{A}}{1!}t + \frac{\mathbf{A^2}}{2!}t + \frac{\mathbf{A^3}}{3!}t + \cdots \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \frac{1}{1!}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} t + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} t^2 + \frac{1}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} t^3 + \cdots \\
&= \begin{pmatrix} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots & 0 \\ 0 & 1 + t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots \end{pmatrix} \\
&= \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}
\end{aligned}$$

and

$$e^{-\mathbf{A}t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

2. For $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we have

$$
\begin{aligned}
\mathbf{A}^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \\
\mathbf{A}^3 &= \mathbf{A}\mathbf{A}^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbf{A} \\
\mathbf{A}^4 &= (\mathbf{A}^2)^2 = \mathbf{I} \\
\mathbf{A}^5 &= \mathbf{A}\mathbf{A}^4 = \mathbf{A}\mathbf{I} = \mathbf{A}
\end{aligned}
$$

and so on. In general

$$
\mathbf{A}^k = \begin{cases} \mathbf{A}, & k = 1, 3, 5, \cdots \\ \mathbf{I}, & k = 2, 4, 6, \cdots \end{cases}
$$

Thus

$$
\begin{aligned}
e^{\mathbf{A}t} &= \mathbf{I} + \frac{\mathbf{A}}{1!}t + \frac{\mathbf{A^2}}{2!}t + \frac{\mathbf{A^3}}{3!}t + \cdots \\
&= \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{I}t^2 + \frac{1}{3!}\mathbf{A}t^3 + \cdots \\
&= \mathbf{I}\left(1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \cdots\right) + \mathbf{A}\left(t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots\right) \\
&= \mathbf{I}\cosh t + \mathbf{A}\sinh t \\
&= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
e^{-\mathbf{A}t} &= \begin{pmatrix} \cosh(-t) & \sinh(-t) \\ \sinh(-t) & \cosh(-t) \end{pmatrix} \\
&= \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}
\end{aligned}
$$

3. For

$$
\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix}
$$

we have

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, $\mathbf{A}^3 = \mathbf{A}^4 = \mathbf{A}^5 = \cdots = \mathbf{0}$ and

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} t & t & t \\ t & t & t \\ -2t & -2t & -2t \end{pmatrix} = \begin{pmatrix} t+1 & t & t \\ t & t+1 & t \\ -2t & -2t & -2t+1 \end{pmatrix}$$

9. To solve

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

we identify $t_0 = 0, \mathbf{F}(s) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, and use the results of the main equation to get

$$\begin{aligned}
\mathbf{X}(t) &= e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t} \int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)ds \\
&= \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} ds \\
&= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \int_0^t \begin{pmatrix} 3e^{-s} \\ -e^{-2s} \end{pmatrix} ds \\
&= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -3e^{-s} \\ \frac{1}{2}e^{-2s} \end{pmatrix} \Bigg|_0^t \\
&= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -3e^{-t} - 3 \\ \frac{1}{2}e^{-2t} - \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} -3 - 3e^t \\ \frac{1}{2} - \frac{1}{2}e^{2t} \end{pmatrix} \\
&= c_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}
\end{aligned}$$