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ON VECTOR VARIATIONAL-LIKE INEQUALITIES

A. H. SIDDIQI, Q. H. ANSARI AND R. AHMAD

Department of Mathematics, Aligarh Muslim University, Aligarh 202 002

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In this paper, we introduce a more general form of variational inequalities and prove its existence in the setting of topological vector spaces with or without convexity assumptions.

Key Words : Vector Variational Inequality Problem; KKM-Maps; Topological Vector Spaces; Vector Variational-like Inequality Problem; H-spaces

1. INTRODUCTION

The vector variational inequality is a generalized form of variational inequality, which was introduced by Giannessi[1] in the finite dimensional Euclidean space with further applications. From that time on, in a general setting Chen and Cheng[2], Chen and Yang[3], Chen[4], Siddiqi *et al.*[5] and Yang[6,7] have studied vector variational inequalities and proved the existence of their solutions. They have also derived its equivalence with the vector extremum problem and the vector complementarity problem. Parida *et al.*[8] and Yang and Chen[9] studied the existence of solution of variational-like inequalities in R^n and showed a relationship between variational-like inequality problem and convex programming as well as with complementarity problem. Further, the existence of the solution of variational-like inequalities have been studied in reflexive Banach spaces and topological vector spaces with or without convexity assumptions by Siddiqi *et al.*[10] Inspired and motivated by the applications of the vector variational inequalities and variational-like inequalities, in this paper, we introduce the vector variational-like inequalities and prove the existence of their solutions in the setting of topological vector spaces with or without convexity assumptions.

Let X be a topological vector space and Y be an ordered topological vector space. Let K be a nonempty convex subset of X, and $T : K \to L(X, Y)$ and $\eta : K \times K \to X$ be continuous mappings, where $L(X, Y)$ is the space of all linear continuous operators from X into Y. Let $\{C(u) : u \in K\}$ be a family of closed pointed convex cones in Y with int $C(u) \neq \phi$ for every $u \in K$, where int $C(u)$ is the interior of the set $C(u)$.

We consider the problem of finding $u_0 \in K$ such that

$$\langle T(u_0), \eta(u, u_0) \rangle \notin -\operatorname{int} C(u_0), \text{ for all } u \in K. \quad \dots (1.1)$$

We shall call it the vector variational-like inequality problem, where $\langle T(u), v \rangle$ denotes the evaluation of the linear operator $T(u)$ at v. Hence $\langle T(u), v \rangle \in Y$.

Special Cases

(i) If $\eta(u, u_0) = u - g(u_0)$, where $g : K \to K$, then the problem (1.1) is equivalent to find $u_0 \in K$ such that

$$\langle T(u_0), u - g(u_0) \rangle \notin -\operatorname{int} C(u_0), \text{ for all } u \in K, \quad \dots (1.2)$$

which is known as general vector variational inequality problem, studied by Siddiqi *et al.*[5]

(ii) If $\eta(u, u_0) = u - u_0$, then the problem (1.1) becomes the problem of finding $u_0 \in K$ such that

$$\langle T(u_0), u - u_0 \rangle \notin -\operatorname{int} C(u_0), \text{ for all } u \in K. \quad \dots (1.3)$$

Such type of problem is known as vector variational inequality problem, considered and studied by Chen[4] and Yang[11].

(iii) If $Y = R$, $L(X, Y) = X^*$, $C(u) = R_+$, for all $u \in K$, then the problem (1.1) reduces to the problem of finding $u_0 \in K$ such that

$$\langle T(u_0), \eta(u, u_0) \rangle \geq 0, \text{ for all } u \in K, \quad \dots (1.4)$$

is called variational-like inequality problem[7, 8, 10].

Lemma 1.1 (Chen[4]) — Let (Y, P) be an ordered topological vector space equipped with a closed, pointed and convex cone P such that $\operatorname{int} P \neq \phi$. Then for all $v, z \in Y$, we have

(i) $v - z \in \operatorname{int} P$ and $v \notin \operatorname{int} P \Rightarrow z \notin \operatorname{int} P$;

(ii) $v - z \in P$ and $v \notin \operatorname{int} P \Rightarrow z \notin \operatorname{int} P$;

(iii) $v - z \in -\operatorname{int} P$ and $v \notin -\operatorname{int} P \Rightarrow z \notin -\operatorname{int} P$;

(iv) $v - z \in -P$ and $v \notin -\operatorname{int} P \Rightarrow z \notin -\operatorname{int} P$.

2. Existence Theorems in Topological Vector Spaces

We will use the following concept and results :

Definition 2.1 (Fan[11]) — A mapping $F : X \to 2^X$ is called a KKM - map, if for every finite subset $\{u_1, u_2, \dots, u_n\}$ of X, $\operatorname{conv}(\{u_1, u_2, \dots, u_n\}) \subset \bigcup_{i=1}^{n} F(u_i)$, where $\operatorname{conv}(\{u_1, u_2, \dots, u_n\})$ is a convex hull of the finite set $\{u_1, u_2, \dots, u_n\}$, and 2^X is a set of all nonempty subsets of X.

Lemma 2.1 (Fan[11]) — Let A be an arbitrary nonempty set in a topological vector space X and $F: A \to 2^X$ be a KKM-map. If $F(u)$ is closed for all $u \in A$ and is compact for at least one $u \in A$ then

$$\bigcap_{u \in A} F(u) \neq \phi.$$

Theorem 2.1 (Fan[11]) — *Let E be a nonempty compact convex set in a topological vector space X and A be a subset of $E \times E$ with the following properties :*

(1) *For each $u \in E$, $(u, u) \in A$.*

(2) *For any fixed $u \in E$, the set $A_u = \{v \in E : (u, v) \in A\}$ is closed in E.*

(3) *For each fixed $v \in E$, the set $A_v = \{u \in E : (u, v) \notin A\}$ is convex.*

Then there exists a point $v_0 \in E$ such that $E \times \{v_0\} \subset A$.

Let K be a nonempty compact convex subset of X. The bilinear form $\langle \cdot, \cdot \rangle$ is supposed to be continuous.

Theorem 2.2 — *Let $T: K \to L(X, Y)$ and $\eta: K \times K \to X$ be two continuous maps and let $f(u) \mapsto \langle T(v), \eta(u, v) \rangle$ be affine, for each fixed $v \in K$. Let the multivalued mapping $W(u) = Y \setminus \{- \text{int } C(u)\}$ is upper semicontinuous on K and $\langle T(u), \eta(u, u) \rangle \notin - \text{int } C(u)$, for every $u \in K$. Then there exists $u_0 \in K$ such that*

$$\langle T(u_0), \eta(u, u_0) \rangle \notin - \text{int } C(u_0), \textit{ for all } u \in K.$$

PROOF : Let

$$A = \{(u, v) \in K \times K : \langle T(v), \eta(u, v) \rangle \notin - \text{int } C(v)\}.$$

Our theorem will be proved if we show that the assumptions (1), (2) and (3) of Theorem 2.1 are satisfied.

For every $u \in K$, $(u, u) \in A$, if and only if $\langle T(u), \eta(u, u) \rangle \notin - \text{int } C(u)$, by the assumption and definition of A. Now, let $A_u = \{v \in K : (u, v) \in A\}$, for each fixed $u \in K$, then we show that A_u is closed.

Let $\{v_n\}$ be a net in A_u such that $v_n \to v$. Then $v \in K$, because K is compact. Since $v_n \in A_u$, we have

$$\langle T(v_n), \eta(u, v_n) \rangle \notin - \text{int } C(v_n).$$

Hence, $\langle T(v_n), \eta(u, v_n) \rangle \in W(v_n) = Y \setminus \{- \text{int } C(v_n)\}$. Since T, η and $\langle \cdot, \cdot \rangle$ are continuous, we have

$$\langle T(v_n), \eta(u, v_n) \rangle \to \langle T(v), \eta(u, v) \rangle.$$

The upper semicontinuity of multivalued map W implies that

$$\langle T(v), \eta(u, v) \rangle \in W(v)$$

i.e. $\langle T(v), \eta(u, v) \rangle \notin - \text{int } C(v)$. Thus $v \in A_u$ and hence A_u is closed.

To finish the proof we show that for each fixed $v \in K$, $A_v = \{u \in K : (u, v) \notin A\}$ is convex. Indeed, if $u_1, u_2 \in A_v$ and $\alpha, \beta \in R_+$ such that $\alpha + \beta = 1$, and since $C(v)$ is a cone, we have

$$\alpha \langle T(v), \eta(u_1, v) \rangle \in -\text{int } C(v) \qquad \text{... (2.1)}$$

and

$$\beta \langle T(v), \eta(u_2, v) \rangle \in -\text{int } C(v). \qquad \text{... (2.2)}$$

Adding (2.1) and (2.2), and by using the affiness of $f(\cdot)$, we have

$$\langle T(v), \eta(\alpha u_1 + \beta u_2, v) \rangle \in -\text{ int } C(v)$$

and hence $\alpha u_1 + \beta u_2 \in A_v$, showing that A_v is convex.

Now, from Theorem 2.1, there exists $u_0 \in K$ such that $K \times \{u_0\} \subset A$, which implies that

$$u_0 \in K : \langle T(u_0), \eta(u, u_0) \rangle \notin -\text{ int } C(u_0), \text{ for all } u \in K.$$

Remark 2.1 : (i) If $\eta(u, u_0) = u - g(u_0)$, where $g : K \to K$, then Theorem 2.2 reduces to Theorem 2.1[5].

(ii) If $Y = R$, $L(X, Y) = X^*$ and $C(u) = R_+$, for all $u \in K$ then Theorem 2.2 becomes Theorem 3.2[10].

(iii) If $Y = R$, $L(X, Y) = X^*$, $C(u) = R_+$ and $\eta(u, u_0) = u - g(u_0)$, where $g : K \to K$, for all $u \in K$, then Theorem 2.2 reduces to the Proposition 2[12].

In the case where K is not necessarily compact, we have the following result :

Theorem 2.3 — *Assume that*

(1) *K is a nonempty closed convex subset of X;*

(2) *the mappings $T : K \to L(X, Y)$ and $\eta : K \times K \to X$ are continuous;*

(3) *$C : K \to 2^Y$ is a multivalued map such that for every $u \in K$, $C(u)$* is a closed, pointed convex cone with int $C(u) \neq \phi$;

(4) *$W : K \to 2^Y$ is an upper semicontinuous multivalued map defined as*

$$W(u) := Y \setminus \{-\text{ int } C(u)\}, \text{ for all } u \in K;$$

(5) *there exists a function $h : K \times K \to Y$ such that*

(i) $h(u, v) - \langle T(v), \eta(u, v) \rangle \in -$ int $C(v)$, *for every* $(u, v) \in K \times K$;

(ii) *the set* $\{u \in K : h(u, v) \in -\text{ int } C(v)\}$ *is convex, for every* $v \in K$;

(iii) $h(u, u) \notin -$ int $C(u)$, *for all* $u \in K$;

(iv) *there exists a nonempty compact convex subset D of K such that for every $v \in K \setminus D$, there exists $u \in D$ with*

$$\langle T(v), \eta(u, v) \rangle \in -\text{ int } C(v).$$

Then there exists $u_0 \in D \subset K$ such that

$$\langle T(u_0), \eta(u, u_0) \rangle \notin - \text{int } C(u_0), \textit{ for all } u \in K.$$

PROOF : For each element $u \in K$, we define

$$D(u) = \{v \in D : \langle T(v), \eta(u, v) \rangle \notin - \text{int } C(v)\}$$

and from assumptions (2) and (4), we have that $D(u)$ is closed in D. Since every element $u_0 \in \bigcap_{u \in K} D(u)$ is a solution of vector variational-like inequality problem (1.1), we have to prove that $\bigcap_{u \in K} D(u) \neq \phi$. Since D is compact it is sufficient to show that the family $\{D(u)\}_{u \in K}$ has finite intersection property. Indeed, let $u_1, u_2, ..., u_m \in K$ be given. We put $A = Conv\ (D \cup \{u_1, u_2, ..., u_m\})$ and we have that A is a compact convex subset of K.

We consider the following multivalued maps :

$$F_1(u) = \{v \in A : \langle T(v), \eta(u, v) \rangle \notin - \text{int } C(v)\}$$

and

$$F_2(u) = \{v \in A : h(u, v) \notin - \text{int } C(v)\}$$

for every $u \in K$. Since the bilinear form $\langle \cdot, \cdot \rangle$ is continuous and from assumptions (2) and (4), we have that F_1 is closed subset of a compact convex set A. Hence, $F_1(u)$ is compact.

From assumptions (5*i*) and (5*iii*), we have

$$h(u, u) - \langle T(u), \eta(u, u) \rangle \in - \text{int } C(u)$$

and

$$h(u, u) \notin - \text{int } C(u).$$

Then by Lemma 1.1(iii), we have

$$\langle T(u), \eta(u, u) \rangle \notin - \text{int } C(u).$$

Hence, $F_1(u)$ is nonempty.

Now we prove that F_2 is a KKM-map. Indeed, if we suppose that there exist $x_1, x_2, ..., x_n \in A$ and $\alpha_i \geq 0$, $i = 1, 2, ..., n$, with $\sum_{i=1}^{n} \alpha_i = 1$, such that

$$\sum_{i=1}^{n} \alpha_i x_i \notin \bigcup_{i=1}^{n} F_2(x_i)$$

then we have

$$h\left(x_i, \sum_{i=1}^{n} \alpha_i x_i \right) \in - int\, C\left(\sum_{i=1}^{n} \alpha_i x_i \right).$$

By assumption (5*ii*), we have

$$h\left(\sum_{i=1}^{n} \alpha_i x_i,\ \sum_{i=1}^{n} \alpha_i x_i\right) \in -\,int\, C\left(\sum_{i=1}^{n} \alpha_i x_i\right)$$

which is a contradiction to assumption (5iii). Therefore, F_2 is a KKM-map.

From assumption (5*i*) and Lemma 1.1(iii), we have $F_2(u) \subset F_1(u)$, for every $u \in K$. Then we obtain that F_1 is also a KKM-map. Applying Lemma 2.1 to F_1, we get $\bigcap_{u \in A} F_1(u) \neq \phi$, that is, there exists $v_0 \in A$ such that

$$\langle T(v_0), \eta(u, v_0) \rangle \notin -\ \text{int}\ C(v_0),\ \textit{for all}\ u \in A.$$

By assumption (5*iv*), we have that $v_0 \in D$ and moreover $v_0 \in D(u_i)$, for every $1 \leq i \leq m$. Hence $\{D(u)\}_{u \in K}$ has the finite intersection property and the proof is finished.

Remark 2.1 : If $Y = R$, $L(X, Y) = X^*$, $C(u) = R_+$, for all $u \in K$ and $\eta(u, u_0) = u - g(u_0)$, where $g : K \to K$, then Theorem 2.3 reduces to Theorem 8 of Isac[12].

3. An Existence Theorem Without Convexity

In this section, we prove an existence theorem for a special case of vector variational-like inequality problem 1.1 replacing convexity assumptions with merely topological properties. We use the technique of Chen[4] to prove the main result of this section.

The following definitions can be found in Bardaro and Ceppitelli[13].

Definition 3.1 — Let X be a topological space and $\{\Gamma_A\}$ a given family of nonempty contractible subsets of X, indexed by finite subsets of X.

A pair $(X, \{\Gamma_A\})$ is said to be a H-space, if $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

A subset $D \subset X$ is called H-convex, if for every finite subset A of D, it follows that $\Gamma_A \subset D$.

A subset $D \subset X$ is called weakly H-convex, if $\Gamma_A \bigcap D$ is nonempty and contractible for every finite subset $A \subset D$. This is equivalent to saying that the pair $(D, \{\Gamma_A \bigcap D\})$ is a H-space.

A subset $K \subset X$ is called H-compact, if there exists a compact and weakly H-convex set $D \subset X$, such that $K \bigcup A \subset D$ for every finite subset A of X.

Let $(X, \{\Gamma_A\})$ be an H-space. A multivalued map $F : X \to 2^X$ is called H-KKM, if $\Gamma_A \subset \bigcup_{x \in A} F(x)$, for every finite subset $A \subset X$.

Theorem 3.1 (Bardaro and Ceppitelli[13]) — *Let $(X, \{\Gamma_A\})$ be an H-space and $F : X \to 2^X$ be an H-KKM multivalued map such that :*

(*i*) *for each* $x \in X$, $F(x)$ *is compactly closed, that is,* $B \cap F(x)$ *is closed in* B *for every compact set* $B \subset X$;

(*ii*) *there exists a compact set* $L \subset X$ *and an H-compact set* $K \subset X$ *such that, for each weakly H-convex set* D *with* $K \subset D \subset X$, *we have* $\bigcap_{x \in D} (F(x) \cap D) \subset L$.

Then $\bigcap_{x \in X} F(x) \neq \phi$. We now consider a special case of (1.1), but in a more general context.

(VVLIP)′ : Find $u_0 \in X$ such that

$$\langle T(u_0), \eta(u, u_0) \rangle \notin -\operatorname{int} P, \text{ for all } u \in X, \qquad \dots (3.1)$$

where $(X, \{\Gamma_A\})$ is an H-space, (Y, P) is an ordered topological vector space with a closed pointed convex cone P such that int $P \neq \phi$ and $T : X \to L(X, Y)$, $\eta : X \times X \to X$ are given maps.

Theorem 3.2 — *Let* $(X, \{\Gamma_A\})$ *be an H-space, and let* (Y, P) *be an ordered topological vector space with a closed pointed convex cone* P *such that* int $P \neq \phi$. *Let* $T : X \to L(X, Y)$ *and* $\eta : X \times X \to X$ *be two continuous maps. Assume that*

(1) $\langle T(v), \eta(v, v) \rangle \notin -\operatorname{int} P$ *for all* $v \in X$;

(2) *for each* $v \in X$, $B_v = \{u \in X : \langle T(v), \eta(u, v) \rangle \in -\operatorname{int} P\}$ *is H-convex or empty*;

(3) *there exists a compact set* $L \subset X$ *and an H-compact set* $K \subset X$ *such that for every weakly H-compact set* D *with* $K \subset D \subset X$, $\{v \in D : \langle T(v), \eta(u, v) \rangle \notin -\operatorname{int} P\} \subset L$, $\forall\, u \in D$.

Then the (VVLIP)′ *is solvable.*

PROOF : Let

$$F(u) = \{v \in X : \langle T(v), \eta(u, v) \rangle \notin -\operatorname{int} P\}, \text{ for all } u \in X.$$

If we prove that $\bigcap_{u \in X} F(u) \neq \phi$, then our theorem is proved, since every element $u_0 \in \bigcap_{u \in X} F(u)$ is a solution of (VVLIP)′. It can be followed from Theorem 3.1, if we prove that F is an H-KKM map and the conditions (i) and (ii) of Theorem 3.1 hold.

Suppose that F is not an H-KKM map. Then there exists a finite subset $A \subset X$ such that $\Gamma_A \not\subset \bigcup_{u \in A} F(u)$. Thus there exists $z \in \Gamma_A$ such that

$$z \notin F(u), \text{ for all } u \in A,$$

i.e. $\langle T(z), \eta(u, z) \rangle \in -\operatorname{int} P$, for all $u \in A$. By assumption (2) and since B_z is H-convex, we have $\Gamma_A \subset B_z$, for $A \subset B_z$. Therefore, $z \in B_z$ and hence $\langle T(z), \eta(z, z) \rangle \in -\operatorname{int} P$, which is a contradiction to assumption (1). Thus

$\Gamma_A \subset \bigcup_{u \in A} F(u)$, for every finite subset $A \subset X$, so that F is an H-KKM mapping.

Next, we prove that for every $u \in X$, $F(u)$ is closed. Indeed, suppose that $\{v_n\}$ be a net in $F(u)$ such that $v_n \to v$. As T, η and $\langle \cdot, \cdot \rangle$ are continuous, we have

$$\langle T(v_n), \eta(u, v_n) \rangle \to \langle T(v), \eta(u, v) \rangle .$$

Since $\langle T(v_n), \eta(u, v_n) \rangle \notin -\text{ int } P$, for all n, that is $\langle T(v_n), \eta(u, v_n) \rangle \in W = Y \setminus \{-\text{ int } P\}$. But $W = Y \setminus \{-\text{ int } P\}$ is closed, we have $\langle T(v), \eta(u, v) \rangle \in W$ that is,

$$\langle T(v), \eta(u, v) \rangle \notin -\text{ int } P.$$

Hence, $v \in F(u)$ and therefore $F(u)$ is closed for every $u \in X$, that is, the condition (i) of the Theorem 3.1 holds. It is easy to see that the assumption (3) of this theorem is a condition (ii) of Theorem 3.1. Thus by Theorem 3.1,

$$\bigcap_{u \in X} F(u) \neq \phi,$$

that is, there exist $u_0 \in X$ such that

$$\langle T(u_0), \eta(u, u_0) \rangle \notin -\text{ int } P, \text{ for all } u \in X.$$

Remark 3.1 : If $\eta(u, u_0) = u - g(u_0)$, where $g : X \to X$, then Theorem 3.2 reduces to Theorem 3.1 of Siddiqi *et al.*[5]

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