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WIENER-HOPF EQUATIONS AND GENERAL MILDLY NONLINEAR VARIATIONAL INEQUALITIES

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The general mildly nonlinear variational inequality problem is equivalent to the problem of solving the Wiener-Hopf equations. This equivalence is used to suggest and analyze a number of iterative algorithms for solving general mildly nonlinear variational inequalities including many known algorithms as special cases for solving general variational inequalities. The convergence criteria for these algorithms are discussed and be present results reflect the extension and improvement of Noor's results.[8]

Key Words : Wiener-Hopf Equations; General Mildly Nonlinear Variational Inequalities, Iterative Algorithms, Convergence Criteria

1. INTRODUCTION

Variational inequality theory introduced by Stampacchia[1] is a powerful tool of the current mathematical technology and has been extended and generalized to study a wide class of problems arising in mechanics, optimization and control problems, operations research and engineering sciences, etc. An important and useful generalization is the general variational inequality introduced by Noor[2, 3] in the study of odd-order obstacle problems. Now there has existed many known iterative methods for solving variational inequalities[4-7]. Among the most effective methods is the projection technique and its variant forms. But there is no such method for general variational inequalities, except those of Noor[2, 3]. Recently, Noor[8] proved that general variational inequality problem is equivalent to solving the Wiener-Hopf equations (Speck[9]). Using this equivalence, he suggested and analyzed a number of iterative algorithms for solving general variational inequalities. And also he discussed the convergence criteria for these algorithms. On other hand, in 1991, Noor[10] introduced and studied a class of variational inequalities which is known as the general mildly nonlinear variational inequality problem and is the extended form of variational inequalities. This class of variational inequalities enables us to study differential equations of both odd and even order. Inspired and motivated by the recent research work of Noor[8, 10], we show that the general mildly nonlinear variational inequality problem is equivalent to the problem of solving Wiener-Hopf equations. This equivalence is useful from the numerical and approximation point of views and enables us to suggest and analyze a number of new iterative algorithms for computing approximate solutions of general mildly nonlinear variational inequalities. We also study the conditions under which the approximate solution obtained from the iterative algorithms converges to the exact solution of the general mildly nonlinear variational inequalities.

In section 2, we formulate the general mildly nonlinear variational inequality related to the third order two point boundary value problems and review some basic results. In section 3, we prove the equivalence between mildly nonlinear variational inequality problem and the problem of solving the Wiener-Hopf equations. This equivalence is used to suggest some new iterative algorithms. Convergence criteria are also discussed.

2. Preliminaries

Let H be a real Hilbert space with norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let C be a nonempty closed convex subset of H and T and g be nonlinear continuous operators from H into itself. Then we consider the problem of finding $u\in H$ such that $g(u)\in C$, and

$$\langle Tu, g(v)-g(u)\rangle \geq \langle A(u), g(v)-g(u)\rangle, \text{ for all } g(v)\in C \quad \dots (1)$$

where A is a nonlinear continuous mapping from H into itself. The inequality (1) is known as the general mildly nonlinear variational inequality.

Remark 2.1 : A large number of differential equation problems of odd and even order can be characterized by a class of vairational inequalities of the type (1). For simplicity, we consider the third order two-point boundary value problem

$$\left.\begin{aligned} T(u) &\geq f(x, u(x)) && \text{in } D\\ u(x) &\geq \psi(x) && \text{in } D\\ [Au-f(x,u(x))]\,[u(x)-\psi(x)] &= 0 && \text{in } D\\ u=0 \text{ and } u' &= 0 && \text{on } D\end{aligned}\right\} \quad \dots (2)$$

where D is a domain in $\mathbb{R}^2$ with boundary $S=[0,1]$, $T=-d^3/dx^3$ is the differential operator of third order, f is a given nonlinear function of x, and $\psi(x)$ is the given obstacle function. To study the problem (2) in the variational inequality framework, we define

$$C = \{u\in H_0^2(\Omega);\ u(x)\geq \psi(x) \text{ on } D\},$$

which is a closed convex set in $H_0^2(\Omega)$. Now using the technique of K-positive definite operators, as developed by Noor[11], we can show that the problem (2) is equivalent to finding $u\in H_0^2(\Omega)$ such that $Ku\in C$ and

$$\langle Tu, Kv-Ku\rangle \geq \langle Au, Kv-Ku\rangle, \text{ for all } Kv\in C, \quad \dots (3)$$

where

$$\langle Tu, Kv \rangle = -\int_0^1 D^3u\, Dv\, dx = \int_0^1 D^2u\, Dv\, dx \qquad \text{... (4)}$$

and

$$\langle Au, Kv \rangle = \int_0^1 j_x(x, u(x))\, Dv\, dx \qquad \text{... (5)}$$

with $K = d/dx = D$.

It is clear that with $g = K$, we have the variational inequality problem (1).

Special Cases

(i) If $g(x) = x$ and $A(x) = 0$ for all $x \in H$, then problem (1) is equivalent to finding $u \in C$ such that

$$\langle Tu, v-u \rangle \geq 0, \text{ for all } v \in C, \qquad \text{... (6)}$$

which is known as the variational inequality problem, introduced and studied by Stampacchia[1] and Hartman and Stampacchia[12]; see also Noor[2].

(ii) If $A(x) = 0$ for all $x \in H$, then problem (1) is equivalent to finding $u \in C$ sucht hat $g(u) \in C$ and

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \text{ for all } g(v) \in C, \qquad \text{... (7)}$$

which is known as the general variational inequality problem introduced by Noor[2] in the study of odd-order obstacle problems; see also Noor[8].

(iii) If $g(x) = x \in C$, then problem (1) is equivalent to finding $u \in C$ such that

$$\langle Tu, v-u \rangle \geq \langle A(u), v-u \rangle, \text{ for all } v \in C. \qquad \text{... (8)}$$

Inequalities (8) are known as mildly (strongly) nonlinear variational inequalities, which were introduced and considered by Noor[13, 14] in the theory of mildly constrained but strongly-nonlinear differential equations. For the finite element error estimates of these variational inequalities, see Noor[11].

(iv) If $C^* = \{u \in H : \langle u, v \rangle \geq 0, \text{ for all } v \in C\}$ is a polar cone of the convex cone C in H, $C \subset g(C)$, then problem (1) is equivalent to finding $u \in H$ such that

$$g(u) \in C,\ (Tu - A(u)) \in C^*,\ \langle Tu - A(u), g(u) \rangle = 0 \qquad \text{... (9)}$$

which is known as the general mildly nonlinear complementarity problem, introduced and studied by Noor[10]. Problem (9) includes many previously known complementarity problems as special cases (see References[1-4, 6, 8, 9, 15-18]).

(v) If $C = H$ and $A(x) = 0$ for all $x \in H$, then problem (1) is equivalent to finding $u \in H$ such that

$$\langle Tu, g(v) \rangle = 0, \text{ for all } g(v) \in H, \quad \ldots (10)$$

which is known as the weak formulation of the odd-order boundary-valued problems.

We need the following concepts :

Definition 2.1 — A mapping $T : H \to H$ is said to be

(a) Strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v \rangle \geq \alpha \| u - v \|^2, \text{ for all } u, v \in H.$$

(b) Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\| Tu - Tv \| \leq \beta \| u - v \|, \text{ for all } u, v \in H.$$

In particular it follows that $\alpha \leq \beta$.

We also need the following well-known results.

Lemma 2.1 (Noor[14]) — If $C \subset H$ is a closed convex set and $z \in H$ is a given point, then $u \in C$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0 \text{ for all } v \in C$$

if and only if

$$u = P_C z, \quad \ldots (11)$$

where P_C is the projection of H onto C.

Lemma 2.2 (Noor[14]) — The mapping P_C defined by (11) is nonexpansive, that is

$$\| P_C u - P_C V \| \leq \| u - v \| \text{ for all } u, v \in H.$$

Let P_C be the projection of H onto C, and let $Q_C = I - P_C$, where I is the identity operator. If g^{-1} exists, then we consider the problem of finding $z \in H$ such that

$$(T - A)\, g^{-1} P_C z + \rho^{-1} Q_C z = 0, \quad \ldots (12)$$

where $\rho > .0$ is a constant. Equations of the type (12) are called more general Wiener-Hopf equations.

Special Cases

(i) If $A(x) = 0$ for all $x \in H$, then problem (12) is equivalent to finding $z \in H$ such that

$$T g^{-1} P_C z + \rho^{-1} Q_C z = 0, \quad \ldots (13)$$

where $\rho > 0$, is a constant. Equations of the type (13) are called general Wiener-Hopf equations, introduced and studied by Noor[8].

(ii) If $g(x) = x$ and $A(x) = 0$ for all $x \in H$, then problem (12) is equivalent to finding $z \in H$ such that

$$TP_C z + \rho^{-1} Q_C z = 0, \quad \text{... (14)}$$

where $\rho > 0$, is a constant. Equations of the type (14) are called the Wiener-Hopf equations.

Remark 2.2 : For the application and general treatment of Wiener-Hopf equations, see Speck[9]. Using essentially the projection technique, Shi[19] has shown that the variational inequality problem (6) is equivalent the Wiener-Hopf eq. (14). This equivalence has been used by Shi[20] and Noor[21] to suggest and analyze a number of new iterative algorithms for solving variational inequalities. For computation results, see Pitonyak *et al.*[22].

3. Main Results

First of all, using the technique of Shi[19] as extended by Noor[8,21], we prove the following result :

Theorem 3.1 — *The general mildly nonlinear variational inequality (1) has a solution $u \in H$ such that $g(u) \in C$, if and only if the more general Wiener-Hopf equation (12) has a solution $z \in H$, where*

$$z = g(u) - \rho(Tu - A(u)) \quad \text{... (15)}$$

and

$$g(u) = P_C z, \quad \text{... (16)}$$

where P_C is the projection of H onto C and $\rho > 0$ is a constant.

Proof : Let $u \in H$ be such that $g(u) \in C$ is a solution of (1). Then by Lemma 2.1, it follows that

$$g(u) = P_C[g(u) - \rho(Tu - A(u))]. \quad \text{... (17)}$$

Using $Q_C = I - P_C$ and applying (17) respectively, we obtain

$$\begin{aligned} Q_C\,[g(u) - \rho(Tu - A(u))] &= g(u) - \rho(Tu - A(u)) \; - \; P_C\,[g(u) - \rho(Tu - A(u))] \\ &= -\,\rho(Tu - A(u)) \\ &= -\,\rho(T - A)\,g^{-1}\,P_C\,[g(u) - \rho(Tu - A(u))], \end{aligned}$$

from which it follows that

$$(T - A)\,g^{-1}\,P_C\,z + \rho^{-1}\,Q_C z = 0,$$

where

$$z = g(u) - \rho(Tu - A(u)),$$

and g^{-1} is the inverse of the operator g.

Conversely, let $z \in H$ be a solution of (12), then, we have

$$\rho(T-A)\,g^{-1}\,P_C\,z = -\,Q_C\,z = P_C\,z - z \qquad \dots (18)$$

Now from (18) and Lemma (2.1) for all $g(v) \in C$, we obtain

$$0 \le \langle P_C\,z - z, g(v) - P_C z \rangle = \langle \rho\,(T-A)\,g^{-1}\,P_C\,z,\; g(v) - P_C z \rangle,$$

from which it follows that

$$\langle (T-A)\,g^{-1}\,P_C z,\;\; g(v) - P_C z \rangle \ge 0, \text{ for all } g(v) \in C.$$

Thus, $g(u) = P_C(z)$ is a solution of (1), and from (18) we have

$$z = g(u) - \rho(Tu - A(u)).$$

Remark 3.1 : It is obvious that general mildly nonlinear variational inequalities and more general Wiener-Hopf equations are equivalent. This equivalence is very useful from the numerical point of view. Using this equivalence and by an appropriate rearrangement we suggest a number of new iterative algorithms for solving general mildly nonlinear variational inequalities (1).

(i) The more general Wiener-Hopf eqs. (12) can be written as

$$Q_C z = -\,\rho(T-A)\,g^{-1}\,P_C z,$$

which implies that, using (16),

$$z = P_C z - \rho(T-A)\,g^{-1}\,P_C z$$

$$= g(u) - \rho(Tu - A(u)). \qquad \dots (19)$$

This formulation enables us to suggest the following iterative algorithm for solving the general mildly nonlinear variational inequalities (1).

Algorithm 3.1 — For a given $z_0 \in H$, compute z_{n+1} by the iterative scheme

$$g(u_n) = P_C z_n \qquad \dots (20)$$

$$z_{n+1} = g(u_n) - \rho(Tu_n - A(u_n)) \qquad \dots (21)$$

(ii) By an appropriate rearrangement, the more general Wiener-Hopf eqs. (12) can be written in the following form,

$$z = P_C z - \rho(T+A)\,g^{-1}\,P_C z + (I - \rho^{-1})\,Q_C z$$

$$= u - \rho(Tu - A(u)) + (I - \rho^{-1})\,Q_C z$$

by using (16). Using this formulation, we suggest and propose the following iterative algorithm.

Algorithm 3.2 — For a given $z_0 \in H$, compute z_{n+1} by the iterative schemes

$$g(u_n) = P_C z_n$$

$$z_{n+1} = u_n - \rho(Tu_n - A(u_n)) + (I - \rho^{-1}) Q_C u_n, \quad n = 0, 1, 2, \ldots$$

(iii) If $A(x) = 0$ for all $x \in H$, the operator T is linear and T^{-1} exists, then the more general Wiener-Hopf equations can be written as

$$z = (I - \rho^{-1} g T^{-1}) Q_C z.$$

This formulation enables us to suggest the following iterative schemes for solving general variational inequalities

Algorithm 3.3 (Noor[8]) — For a given $z_0 \in H$, compute z_{n+1} by the iterative scheme

$$g(u_n) = P_C(z_n)$$

$$z_{n+1} = (I - \rho^{-1} g T^{-1}) Q_C z_n, \quad n = 0, 1, 2, \ldots$$

We now study those conditions under which the approximate solution z_{n+1} obtained from Algorithm 3.1 converges to the exact solution z of the more general Wiener-Hopf eqs. (12).

Theorem 3.2 — *Let T, $g : H \to H$ be both strongly monotone and Lipschitz continuous operators, and $A : H \to H$ be a Lipschitz continuous operator. If z_{n+1} is obtained from the iterative scheme (20)-(21), and if $z \in H$ is the exact solution of the more general Wiener-Hopf equations (12), then*

$$z_{n+1} \to z, \quad \text{strongly in } H,$$

for

$$\left| \rho - \frac{\alpha + \nu(k-1)}{\beta^2 - \nu^2} \right| < \frac{\sqrt{(\alpha + \nu(k-1))^2 - (\beta^2 - \nu^2) k(2-k)}}{\beta^2 - \nu^2}, \quad k < 1,$$

$$\alpha > \nu(1-k) + \sqrt{(\beta^2 - \nu^2) k(2-k)}, \quad \nu(1-k) < \alpha,$$

and

$$k = 2\sqrt{1 - 2\delta + \sigma^2},$$

where σ, β and ν are Lipschitz continuity constants of g, T and A, respectively and δ and α are strong monotonicity constant of g and T, respectively.

PROOF : Let $z \in H$ satisfy the more general Wiener-Hopf eq. (12). Note that equations (12) can be written as (16) and (19). Hence, from (19) and (21), we have

$$\| z_{n+1} - z \| = \| g(u_n) - g(u) - \rho(Tu_n - Tu) + \rho(A(u_n) - A(u)) \|$$

$$\leq \| u_n - u - (g(u_n) - g(u)) \| + \| u_n - u - \rho(Tu_n - Tu) \|$$

$$+ \| \rho(A(u_n) - A(u)) \| . \qquad \ldots (22)$$

Since T, g are both strongly monotone and Lipschitz continuous, by using the technique of Noor[13], we have

$$\| u_n - u - (g(u_n) - g(u)) \|^2 \le (1 - 2\delta + \sigma^2) \| u_n - u \|^2, \qquad \text{... (23)}$$

and

$$\| u_n - u - \rho(Tu_n - Tu) \|^2 \le (1 - 2\rho\alpha + \rho^2\beta^2) \| u_n - u \|^2, \qquad \text{... (24)}$$

where σ and β are the strong monotonicity constants of g and T, respectively, and δ and α are the Lipschitz continuity constant of g and T, respectively.

From (22), (23), (24) and by using the Lipschitz continuity of A, we obtain

$$\| z_{n+1} - z \| \le \left\{ \sqrt{1 - 2\delta + \sigma^2} + \rho\nu + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} \right\} \| u_n - u \|$$

$$= \left\{ \frac{1}{2} k + \rho\nu + t(\rho) \right\} \| u_n - u \|, \qquad \text{... (25)}$$

where

$$k = 2\sqrt{1 - 2\delta + \sigma^2} < 1,$$

$$t(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2},$$

and ν is the Lipschitz continuity constant of A.

From (20), we have

$$\| u_n - u \| = \| u_n - u - (g(u_n) - g(u)) + P_C z_n - P_C z \|$$

$$\le \sqrt{1 - 2\delta + \sigma^2} \; \| u_n - u \| + \| z_n - z \|$$

$$\le \frac{1}{2} k \| u_n - u \| + \| z_n - z \|$$

Using (24), from which it follows that

$$\| u_n - u \| \le (1/(1 - k/2)) \| z_n - z \| \qquad \text{... (26)}$$

From (25) and (26), we obtain

$$\| z_{n+1} - z \| \le \left\{ (k/2) + \rho\nu)/(1 - k/2) \right\} \| z_n - z \| = \theta \| z_n - z \|,$$

where

$$\theta = \{(k/2 + \rho\nu + t(\rho))/(1 - k/2)\} < 1,$$

which is equivalent to

$$k + \rho\nu + t(\rho) < 1.$$

Now $t(\rho)$ assumes its minimum value for $\bar{\rho} = \alpha/\beta^2$ with $t(\rho) = \sqrt{(1 - \alpha^2/\beta^2)}$. We have to show that $k + \rho\nu + t(\rho) < 1$. For $\rho = \bar{\rho}$, $k + \rho\nu + t(\bar{\rho}) < 1$ implies that $k < 1$ and $\alpha > \nu(1 - k) + \sqrt{(\beta^2 - \nu^2)\, k(2 - k)}$. Thus it follows that $k + \rho\nu + t(\rho) < 1$ for all ρ with

$$\left| \rho - \frac{\alpha + \nu(k-1)}{\beta^2 - \nu^2} \right| < \frac{\sqrt{(\alpha + \nu(k-1))^2 - (\beta^2 - \nu^2)\, k(2-k)}}{\beta^2 - \nu^2}$$

$$\alpha > \nu(1-k) + \sqrt{(\beta^2 - \nu^2)\, k(2-k)}, \ \ \nu(1-k) < \alpha, \text{ and } k < 1.$$

Since $\theta < 1$, so the mapping defined by (19) has a fixed point z, which is the solution of (12). Furthermore, it also follows that $z_{n+1} \to z$ strongly in H, is the required result.

Remark 3.2 :

(i) If $A(x) = 0$ for each $x \in H$, then Theorem (3.1—3.2) reduces to Theorem (3.1—3.2) of Noor[8]. Therefore, our results improve and extend Noor's results[8];

(ii) The iterative methods suggested and analyzed in this paper are very convenient and are reasonably easy to use for the computation. Pitonyak *et al.*[22] have presented some numerical examples of solutions to obstacle problems for the membrane and the elastic string using a special case of Algorithm 3.3. In order to develop efficient and implementable algorithms, further research efforts are needed.

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