VARIATIONAL-LIKE INEQUALITIES FOR MULTIVALUED MAPS

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In this paper, we introduce a more general form of variational-like inequalities for multivalued maps and prove the existence of its solution in the setting of reflexive Banach spaces.

Key Words: Inequalities — Variational like; Multivalued Maps; Banach spaces; Nonempty subsets; Convex mathematical programming

1. INTRODUCTION

Let X be a reflexive Banach space with its dual X^* and K and K and K be nonempty subsets of X and K^* , respectively. Given two maps $M: K \times C \to X^*$ and $K \times K \to X$, and a multivalued map $K: K \to X^*$, then we consider the following problem:

Problem 1 — Find $x_0 \in K$ such that for each $y \in K$, $\exists u_0 \in T(x_0)$ such that

$$\langle M(x_0, u_0), \eta(y, x_0) \rangle + b(x_0, y) - b(x_0, x_0) \ge 0,$$
 ... (1.1)

where $b: K \times K \to \mathbb{R}$ is not necessarily differentiable and satisfies some proper conditions, and $\langle \cdot, \cdot \rangle$ is the pairing between X^* and X.

If $b \equiv 0$, then Problem 1 reduces to the problem of finding $x_0 \in K$ such that for each $y \in K$, $\exists u_0 \in T(x_0)$ such that

$$\langle M(x_0, u_0), \eta(y, x_0) \rangle \ge 0.$$
 ... (1.2)

This poblem is the weak formulation of generalized variational-like inequality problem (GVLIP), introduced by Parida and Sen³ in finite dimensional spaces. They have also shown the relationship betwen (GVLIP) and convex mathematical programming. It has been further studied by Yao^{5,6} with applications in complementarity problems.

If we take M(x, u) = u and $\eta(x, y) = g(y) - g(x)$, $\forall x, y \in K$, where $g: K \to K$ then Problem 1 is equivalent to find $x_0 \in K$ such that for each $y \in K$, $\exists u_0 \in T(x_0)$ such that

$$\langle u_0, g(y) - g(x_0) \rangle + b(x_0, y) - b(x_0, x_0) \ge 0.$$
 ... (1.3)

Such problem was introduced and studied by Ding and Tarafdar¹ in the setting of locally convex Hausdorff topological vector spaces.

If M(x, u) = u and b(x, y) = h(y), $\forall x \in K$ then Problem 1 becomes to the problem of finding $x_0 \in K$ such that for each $y \in K$, $\exists u_0 \in T(x_0)$ such that

$$\langle u_0, \eta(y, x_0) \rangle + h(y) - h(x_0) \ge 0.$$
 ... (1.4)

It has been introduced and studied by Siddiqi, Ansari and Ahmad⁴.

In this paper, we prove the existence of solution of Problem 1, which is more general and unifying one. We also derive the existence theorem for a special case of Problem 1.

We need the following concept and result for the proof of our main result. We denote conv (A), $\forall A \subset X$, the convex hull of A.

Definition 1.1 — A map $T: X \to 2^X$ is called KKM-map, if for every finite subset $\{x_1, ..., x_n\}$ of X, conv $(\{x_1, ..., x_n\}) \subset \bigcup_{i=1}^n T(x_i)$.

Lemma 1.1 (KKM-FAN²) — Let A be an arbitrary nonempty set in a topological vector space E and $T: A \to 2^X$ be a KKM-map. If T(x) is closed for all $x \in A$ and is compact for at least one $x \in A$ then $\bigcap_{x \in A} T(x) \neq 0$.

2. EXISTENCE RESULTS

First, we give some definitions which are necessary for the proof of existence theorem for Problem 1.

Definition 2.1 — Let X be a normed space with its dual X^* , C be a nonempty subset of X^* and K be a nonempty convex subset of X. Given two maps $M: K \times C \to X^*$ and $\eta: K \times K \to X$, then a multivalued map $T: K \to 2^C$ is called:

- (i) η -monotone with respect to M if for every pair of points $x \in K$, $y \in K$ and for all $u \in T(x)$, $v \in T(y)$ such that $\langle M(x, u) M(y, v), \eta(x, y) \rangle \ge 0$; and
- (ii) V-hemicontinuous with respect to M if $\forall x, y \in K$, $\alpha \ge 0$ and $u_{\alpha} \in T(x + \alpha y)$, there exists $u_0 \in T(x)$ such that for any $z \in K$, $\langle M(x, u_{\alpha}), z \rangle \rightarrow \langle M(x, u_0), z \rangle$ as $\alpha \rightarrow 0^+$.

Remark 2.1: If M(x, u) = u and $\eta(y, x) = y - x$, $\forall x, y \in K$ then above definitions (i) and (ii) reduce to the definitions of monotonicity and V-hemicontinuity of T, respectively.

Now we prove the main result of this paper.

Theorem 2.1 --- Assume that

- 1° K is a nonempty. closed bounded convex subset of a reflexive Banach space X;
- 2° C is a nonempty subset of X^{*} ;
- 3° $M: K \times C \rightarrow X^*$ is continuous and affine in the first argument;
- $\eta: K \times K \to X$ is continuous and affine in both the argument such that $\eta(x, x) = 0$, $\forall x \in K$
- 5° $T: K \to 2^C$ is η -monotone and V-hemicontinuous with respect to M such that T(x) is compact, $\forall x \in K$;

 6° b: $K \times K \rightarrow \mathbb{R}$ is continuous and convex in the second argument;

7° the set $\{x \in K : \exists v \in T(y) \text{ such that } \langle M(y, v), \eta(y, x) \rangle + b(x, y) - b(x, x) \ge 0, \forall y \in K \}$, is convex.

Then there exists a solution of Problem 1.

PROOF: For each $y \in K$, we define

$$F_1(y) = \{x \in K : \exists u \in T(x) \text{ such that } \langle M(x, u), \eta(y, x) \rangle + b(x, y) - b(x, x) \ge 0\}.$$

Then F_1 is a KKM-map. Indeed, let $\{x_1, ..., x_n\} \subset K$, $\alpha_i \ge 0 \quad \forall i = 1, 2, ..., n$ with

$$\sum_{i=1}^{n} \alpha_{i} = 1 \text{ and } \overline{x} = \sum_{i=1}^{n} \alpha_{i} x_{i} \notin \bigcup_{i=1}^{n} F_{1}(x_{i}). \text{ Then for any } \overline{u} \in T(\overline{x}), \text{ we have }$$

$$\langle M(\overline{x}, \overline{u}), \eta(x_i, \overline{x}) \rangle + b(\overline{x}, x_i) - b(\overline{x}, \overline{x}) < 0, \forall i = 1, 2, ..., n.$$

or
$$\sum_{i=1}^{n} \alpha_{i} \langle M(\overline{x}, \overline{u}), \eta(x_{i}, \overline{x}) \rangle_{+} + \sum_{i=1}^{n} \alpha_{i} b(\overline{x}, x_{i}) - b(\overline{x}, \overline{x}) < 0.$$

Since $\eta(\cdot, \cdot)$ is affine and $b(\cdot, \cdot)$ is convex in the second argument, we have

$$\left\langle M(\overline{x}, \overline{u}), \eta \left(\sum_{i=1}^{n} \alpha_{i} x_{i}, \overline{x} \right) \right\rangle + b \left(\overline{x}, \sum_{i=1}^{n} \alpha_{i} x_{i} \right) - b(\overline{x}, \overline{x})$$

$$\leq \sum_{i=1}^{n} \alpha_{i} \langle M(\overline{x}, \overline{u}), \eta(x_{i}, \overline{x}) + \sum_{i=1}^{n} \alpha_{i} b(\overline{x}, x_{i})(\overline{x}, \overline{x}) < 0.$$

This implies that $\langle M(\overline{x}, \overline{u}), \eta(\overline{x}, \overline{x}') + b(\overline{x}, \overline{x}) - b(\overline{x}, \overline{x}) < 0$. But, since $\eta(x, x) = 0$, $\forall x \in K$, we have

$$\langle M(\overline{x}, \overline{u}), \eta(\overline{x}, \overline{x}) \rangle = 0.$$

Therefore, we reach to a contradiction. Hence, F_1 is a KKM-map.

Define a multivalued map $F_2: K \to 2^K$ as, for each $y \in K$,

$$F_2(y) = \{x \in K : \exists v \in T(y) \text{ such that } \langle M(y,v), \eta(y,x) \rangle + b(x,y) - b(x,x) \ge 0\}.$$

Then $F_1(y) \subset F_2(y)$, $\forall y \in K$:

Let $x \in F_1(y)$ then $\exists u \in T(x)$ such that

$$\langle M(x, u), \eta(y, x) \rangle + b(x, y) - b(x, x) \ge 0.$$

For all $v \in T(y)$, we have

$$\left\langle \ M(y,\,v)-M(x,\,u),\,\eta(y,\,x)\ \right\rangle \leq \left\langle \ M(y,\,v),\,\eta(y,\,x)\ \right\rangle +b(x,\,y)-b(x,\,x).$$

Since T is η -monotone with respect to M, we have

$$\langle M(y, v), \eta(y, x) \rangle + b(x, y) - b(x, x) \ge 0.$$

So, $x \in F_2(y)$. Therefore, $F_1(y) \subset F_2(y)$, $\forall y \in K$ and hence $F_2(y)$ is also a KKM-map.

 $F_2(y)$, $\forall y \in K$ is closed: Let $\{x_n\}$ be sequence in $F_2(y)$ such that $x_n \to x_0$. Then $x_0 \in K$. Since $x_n \in F_2(y)$ $\forall n$, $\exists v_n \in T(y)$ such that

$$\langle M(y, v_n), \eta(y, x_n) \rangle + b(x_n, y) - b(x_n, x_n) \ge 0.$$

Since T(y) is compact, without loss of generality, we assume that there exists $v_0 \in T(y)$ such that $v_n \to v_0$. Since $M(\cdot, \cdot)$, $\eta(\cdot, \cdot)$, $b(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ are continuous, we have

$$\langle M(y, v_n), \eta(y, x_n) \rangle + b(x_n, y) - b(x_n, x_n) \rightarrow \langle M(y, v_0), \eta(y, x_0) + b(x_0, y) - b(x_0, x_0).$$

Therefore, $\langle M(y, v_0), \eta(y, x_0) \rangle + b(x_0, y) - b(x_0, x_0) \ge 0$. So, $x_0 \in F_2(y)$ and hence $F_2(y)$ is closed.

By assumption 7° , $F_2(y)$ is convex. Now we equip X with weak topology. Then K, as a closed convex subset in the reflexive Banach space X, is weakly compact. Since $F_2(y)$ is a closed convex subset of a reflexive Banach space then $F_2(y)$ is weakly closed. $F_2(y) \subset K$ and weak closedness of $F_2(y)$, we have $F_2(y)$ is weakly compact. Then by Lemma 1.1, we have $\bigcap_{y \in K} F_2(y) \neq 0$.

Let $x \in \bigcap_{y \in K} F_2(y)$. Then for any $y \in K$, $\exists v_y \in T(y)$ such that $\langle M(y, v_y), \eta(y, x) \rangle + b(x, y) - b(x, x) \ge 0$.

By convexity of K, for any $\alpha \in (0, 1)$ there exists $v_{\alpha} \in T(\alpha y + (1 - \alpha)x)$ such that

$$\langle M(\alpha y + (1-\alpha)x, \nu_{\nu}), \eta(\alpha y + (1-\alpha)x, x) \rangle + b(x, \alpha y + (1-\alpha)x) - b(x, x) \ge 0.$$

Since $M(\cdot, \cdot)$ and $\eta(\cdot, \cdot)$ are affine in the first argument and $b(\cdot, \cdot)$ is convex in the second argument, we have

$$\alpha^{2} \langle M(y, v_{\alpha}), \eta(y, x) \rangle + \alpha(1 - \alpha) \langle M(y, \alpha), \eta(x, x) \rangle + \alpha(1 - \alpha) \langle M(x, v_{\alpha}), \eta(y, x) \rangle$$

$$+ (1 - \alpha)^{2} \langle M(x, v_{\alpha}), \eta(x, x) \rangle + \alpha b(x, y) + (1 - \alpha) b(x, x) - b(x, x) \ge 0.$$

Since $\eta(x, x) = 0$, $\forall x \in K$, we have

$$\alpha^2 \left< M(y, v_{\alpha}), \, \eta(y, x) \right> + \alpha(1 - \alpha) \left< M(x, v_{\alpha}), \, \eta(y, x) \right> + \alpha b(x, y) - \alpha b(x, x) \ge 0.$$

Dividing by α , we get

$$\alpha \left< M(y, v_{\alpha}), \, \eta(y, x) \right> + (1 - \alpha) \left< M(x, v_{\alpha}), \, \eta(y, x) \right> + b(x, y) - b(x, x) \ge 0.$$

Taking $\alpha \to 0^+$ and by V-hernicontinuity of Twith respect to M, there exists $v_0 \in T(x)$ such that

$$\langle M(x_0, u_0), \eta(y, x_0) \rangle + b(x_0, y) - b(x_0, x_0) \ge 0.$$

Theorem 2.2 — Assume that

- 1° K is a nonempty closed bounded convex subset of a reflexive Banach space X;
- 2° C is a nonempty subset of X^{*} ;
- 3° $M: K \times C \rightarrow X^*$ is continuous and affine in the first argument;
- 4° $\eta: K \times K \to X$ is continuous and affine in both the argument such that $\eta(x, x) = 0$, $\forall x \in K$;
- 5° $T: K \to 2^C$ is η -monotone and V-hemicontinuous with respect to M such that T(x) is compact, $\forall x \in K$;
- 6° h: $K \to IR$ is convex and lower semicontinuous proper functional.

Then there exist $x_0 \in K$ such that for each $y \in K$, $\exists u_0 \in T(x_0)$ such that

$$\langle M(x_0, u_0), \eta(y, x_0) \rangle + h(y) - h(x_0) \ge 0.$$

PROOF: Take b(x, y) = h(y), $\forall x \in K$ in Theorem 2.1, then the proof follows by the proof of Theorem 2.1, if we prove that the set

$$A = \{x \in K : \exists v \in T(y) \text{ such that } \langle M(y, v), \eta(y, x) \rangle + h(y) - h(x) \ge 0, \ \forall y \in K\}$$

is convex.

Indeed, let $x_1, x_2 \in A$, $\alpha, \beta \ge 0$ such that $\alpha + \beta = 1$. Then for all $y \in K$, $\exists v \in T(y)$ such that

$$\langle M(y, v), \eta(y, x_1) \rangle + h(y) - h(x_1) \ge 0$$
 ... (2.1)

and

$$\langle M(y, v), \eta(y, x_2) \rangle + h(y) - h(x_2) \ge 0.$$
 ... (2.2)

Multiplying (2.1) and (2.2) by α and β , respectively and then adding, we get

$$\alpha \left\langle M(y, v), \eta(y, x_1) \right\rangle + \beta \left\langle M(y, v), \eta(y, x_2) \right\rangle + \alpha h(y) + \beta h(y) - \alpha h(x_1) - h(x_2) \ge 0.$$

Since $\eta(\cdot, \cdot)$ is affine and h is convex, we have

$$\langle M(y, v), \eta(y, \alpha x_1 + \beta x_2) \rangle + h(y) - h(\alpha x_1 - \beta x_2) \ge 0.$$

This implies that $\alpha x_1 + \beta x_2 \in A$ and hence A is convex.

Corollary 2.1 - Assume that

- 1° K is a nonempty closed bounded convex subset of a reflexive Banach space X;
- 2° C is a nonempty subset of X^* ;

- 3° $M: K \times C \rightarrow X^*$ is continuous and affine in the first argument;
- 4° $\eta: K \times K \to X$ is continuous and affine in both the argument such that $\eta(x, x) = 0$, $\forall x \in K$;
- 5° $T: K \to 2^C$ is η -monotone and V-hemicontinuous with respect to M such that T(x) is compact, $\forall x \in K$.

Then there exist $x_0 \in K$ such that for each $y \in K$, $\exists u_0 \in T(x_0)$ such that

$$\langle M(x_0, u_0), \eta(y, x_2) \rangle + \geq 0.$$

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$$|p'(z_0) + mn \beta z_0^{n-1}| \le \frac{n}{1+k^{\mu}} [M(p, 1) + m |\beta|],$$
 ... (3.2)

where

$$|p'(z_0)| = M(p', 1).$$

Now choosing the argument of β suitably in (3.2) and finally letting $|\beta| \to \frac{1}{k^n}$. Theorem 2 follows.

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