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Pointwise Convergence of Wavelet Expansions Associated with Dilation Matrix

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The wavelet expansion associated with dilation matrix of a function is studied. This expansion is shown to converge uniformly on compact subsets.

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1. INTRODUCTION

There has been rapid growth in the theory and applications of wavelets [2, 6] in recent years. An interesting introduction of the wavelets is given by Daubechies [3]. Kelly et al [4, 5] and Walter [7] have studied the convergence of wavelet expansions. A comprehensive discussion on two-dimensional wavelet expansions incorporating rotation can be found in Antoine et al. [1]. A detailed account of multidimensional multiresolution analysis is presented by Wojtaszczyk [8]. In this paper we study the convergence of multiwavelet expansion associated with the multiresolution analysis with dilation matrix. Our theorem is a generalization of Walter's results. However analogous results to Theorems 3.4 and 3.7 [4] require further investigation and will be presented separately. The main theorem of Kelly et al. is analogous to the famous Carleson result of Fourier series; Fourier series was shown to converge almost everywhere, but not for all Lebesgue points. In their theorem, pointwise convergence results state that, with few conditions on the wavelets, a wavelet expansion for a function $f \in L_p(R^d)$, $d \geq 1$, converges on the Lebesgue set of f. This result is a general form of Walter's result, (Theorem 1 and Corollary 1 [7]).

2. MULTIDIMENSIONAL MULTIRESOLUTION ANALYSIS

Let A be any real expansive $n \times n$ matrix (equivalently, all eigenvalues of A are required to have absolute value > 1). A wavelet set associated with the matrix A, called dilation matrix, is a finite set of functions $\psi^r(x) \in L_2(R^d)$, $r = 1, 2, 3, \ldots, s$ such that the system

$$\left\{|\det A|^{j/2}\psi^r(A^j x - \gamma)\right\} \tag{2.1}$$

with $r = 1, 2, \ldots, s$, $j \in Z$ and $\gamma \in Z^d$ (Z denotes a set of positive integers) is an orthonormal basis in $L_2(R^d)$. It is a generalization of the notion of wavelet. By analogy with the one-dimensional case, we may use the notation: for a function $F(\varphi, \psi$, etc.) on R^d, by $F_{j\gamma}$, we mean

$$F_{j\gamma}(x) = |\det A|^{j/2} F(A^j x - \gamma), \quad \text{where } j \in Z, \text{ and } \gamma \in Z^d.$$

We shall omit r as there is no ambiguity.

A multiresolution associated with dilation matrix A is a sequence of closed subspaces $\{V_j\}_{j \in Z}$ of $L_2(R^d)$ satisfying

(i) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$,

(ii) $\bigcup_{j \in Z} V_j$ is dense in $L_2(R^d)$,

(iii) $\bigcap_{j \in Z} V_j = \{0\}$.

(iv) $f \in V_j$ if and only if $f(Ax) \in V_{j+1}$, that is, $V_j = U_A^j V_0$,

(v) $f \in V_0$ if and only if $f(x - \gamma) \in V_0$ for all $\gamma \in Z^d$, and

(vi) there exists a function $\varphi \in V_0$ called a scaling function such that the system $\{\varphi(t-\gamma)\}_{\gamma \in Z^d}$ is an orthonormal basis in V_0.

The following results are relevant to our discussion:
THEOREM A [8, p. 116]. *For every multiresolution on R^d associated with a dilation matrix A, there exists an associated wavelet set (consisting of $q-1$ functions, where $q = |det\ A|$).*

A function F on R^d is called r-regular if F is of class C^r, $r = -1, 0, 1, \dots$ and

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} F(x)\right| \le \frac{C_k}{(1+|x|)^k} \tag{2.2}$$

for each $k = 0, 1, 2, \dots$ and each multi-index α with $|\alpha| \le \max(r, 0)$ and some constant C_k. As usual, C^{-1} means a measurable function and class C^0 means a function.

A multiresolution analysis on R^d is called r-regular if it has an r-regular scaling function.

THEOREM B [8, p. 118]. *For every r-regular multiresolution analysis on R^d associated with a dilation matrix A, $|det\ A| = q$, such that $2q - 1 > d$, there exists an associated wavelet set consisting of $q-1$ r-regular functions.*

COROLLARY A [8, p. 120]. *Assume that we have a multiresolution on R^d associated with a dilation matrix A, $|det\ A| = q$. Assume further that this multiresolution analysis has an r-regular scaling function $\varphi(x)$ such that its Fourier transform $\hat{\varphi}(s)$ is real. Then there exists a wavelet set associated with this multiresolution analysis consisting of $q-1$ r-regular functions.*

THEOREM C [8, p. 136]. *Suppose A is a dilation matrix such that for some set of digits $S = \{k_1, k_2, \dots, k_q\}$ a subset of R^d, Q defined by*

$$Q = \{x \in R^d : x = \sum_{j=1}^{\infty} A^{-j} s_j \quad where \quad s_j \in S\} \tag{2.3}$$

has measure 1, that is, the characteristic function χ_Q of Q is a scaling function of a multiresolution analysis. Then for each natural number $r = 1, 2, 3, \dots$ there exists an r-regular wavelet set (consisting of $|det\ A| - 1$ functions) associated with the dilation matrix A.

Examples [8, pp. 127–129].

(i) Let us take $d = 2$ and the simplest dilation $A = 2Id$. Taking the set of digits S as $\{(0,0),(0,1),(1,0),(1,1)\}$, we get $Q = [0,1]^2$. This clearly gives scaling function of a multiresolution analysis. Choosing the set S as

$$\Big\{(0,0),(1,1),(0,1),(1,2)\Big\},$$

we obtain as the set Q the parallelogram with vertices from the set S. This also gives a scaling function of a multiresolution analysis. We take $S = \{(0,0),(1,0),(0,1),(-1,-1)\}$, then Q will be like the Sierpinski triangle (see Figure 5.3 in [8, p. 129]).

(ii) Let us take $d = 2$ and the dilation given by the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Geometrically speaking, this dilation is a rotation by 45° and expansion by the factor $\sqrt{2}$. Since det $A = 2$, we obtain one wavelet generating an orthonormal basis in $L_2(R^2)$ provided there is a scaling function. Taking $S = \{(0,0),(1,0)\}$, we get Q as the fractal set known as the 'twin dragon' [8, pp. 129–130].

3. WAVELET EXPANSION ASSOCIATED WITH DILATION MATRIX

Associated with the V_j spaces in the definition of multiresolution analysis, there is the orthogonal complement of V_j in V_{j+1}, denoted by W_j such that $V_{j+1} = V_j \oplus W_j$. Thus $L_2(R^d) = \overline{\sum \oplus W_j}$. We define P_j and $Q_j = P_{j+1} - P_j$, respectively, to be the orthogonal projections onto the spaces V_j and W_j, with kernels $P_j(x,y)$ and $Q_j(x,y)$. By Theorem A there exists an associated wavelet set consisting of $q-1$ functions, where $q = |\det A|$. Corollary A guarantees the existence of r-regular wavelet sets. The sequence of projections $\{P_j f(x)\}$ is called the *multiresolution expansion* of f. The *scaling expansion* of f is defined as

$$f \sim \sum_{\gamma}^{\infty} b_{j,\gamma} |\det A|^j \varphi(A^j x - \gamma) + \sum_{k=j,\gamma}^{\infty} a_{k,\gamma} |\det A|^k \psi(A^k x - \gamma) \tag{3.1}$$

where

$$a_{j,\gamma} = \int_{R^d} f(x) F_{j,\gamma}(x) dx \tag{3.2}$$

$$b_{j,\gamma} = \int_{R^d} f(x) |\det A|^j \varphi(A^j x - \gamma) dx, \tag{3.3}$$

and $f \in L_2(R^d)$.

The *wavelet expansion* associated with dilation matrix A or *multivariable wavelet expansion* of f is

$$f \sim \sum_{j,\gamma} a_{j,\gamma} F_{j,\gamma}(x) dx \tag{3.4}$$

where $a_{j,\gamma}$ is given in (3.2).

Considering convergence in the sense of $L_2(R^d)$, we may write

$$f(t) = \sum_j \sum_\gamma a_{j,\gamma} F_{j,\gamma}(t) \tag{3.5}$$

and

$$\begin{aligned} f(t) &= \sum_\gamma b_{j,\gamma} |\det A|^{j/2} \varphi(A^j t - \gamma) + \sum_{k=j}^{\infty} \sum_\gamma a_{k,\gamma} F_{j,\gamma}(t) \\ &= f_j(t) + r_m(t). \end{aligned} \tag{3.6}$$

The function $f_m \in V_m$ is, in fact, the projection f onto V_m. It can be written as

$$f_m(x) = \int_{R^d} q_m(x,t) f(t) dt \tag{3.7}$$

where

$$q_m(x,t) = |\det A|^{m/2} q(A^m x, A^m t) \tag{3.8}$$

and

$$q(x,t) = \sum_{\gamma} \varphi(x-\gamma)\varphi(t-\gamma), \qquad \gamma \in Z^d, \tag{3.9}$$

$q_m(x,t)$ will be called the *reproducing kernel* of V_m.

We will use mainly the following results for wavelet expansion given in (3.4).

For a scaling function φ associated with a dilation matrix considered in Section 1, the following results hold [8, p. 138].

$$\int_{R^d} \varphi(x)dx = 1 \tag{3.10}$$

$$\sum_{\gamma \in Z^d} \varphi(x-\gamma) = 1 \tag{3.11}$$

$$|\varphi(t)| \leq \frac{C_k}{(1+|t|)^k}, \quad k = 1,2,3,\ldots \tag{3.12}$$

A sequence $\delta_m(x,y)$ of functions in $L_1(R^d)$ is called a *quasi-positive delta sequence* if the following conditions are satisfied:

there exists a constant C such that

$$\int_{R^d} |\delta_m(x,y)| dx \leq C, \quad \text{for all } y \in R^d,\ m \in N \tag{3.13i}$$

there exists a vector $c = (c_1, c_2, \ldots, c_d) > 0$ such that

$$\int_{[y-c,y+c]} \delta_m(x,y)dx \to 1 \tag{3.13ii}$$

uniformly on compact subset of R^d as $m \to \infty$;

for each $r > 0$,

$$\sup_{|x-y|\geq r} |\delta_m(x,y)| \to 0 \text{ as } m \to \infty. \tag{3.13iii}$$

LEMMA 3.1. *The reproducing kernel $q_m(x,y)$ of V_m, the multiresolution analysis associated with a dilation matrix A, is a quasi-delta sequence.*

For the sake of convenience we write the proof for the two-dimensional case.

Proof of Lemma 3.1. We have

$$\begin{aligned}\int_{R^2} |q_m(x,y)|dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\det A|^{m/2}\, |q(A^m x, A^m y)| dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |q(x, A^m y)| dx \\ &\leq C_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+|x-A^m y|)^{-k} dx = C \qquad \text{by (3.12).}\end{aligned}$$

Thus (3.13i) holds.

We can write

$$\begin{aligned}
\int_{y-c}^{y+c}\int_{y-c}^{y+c} q_m(x,y)dx &= \int_{A^m(y_1-c_1)}^{A^m(y_1+c_1)}\int_{A^m(y_2-c_2)}^{A^m(y_2+c_2)} q(x,A^m y)dx \\
&\leq \int_{t-A^m c_1}^{t+A^m c_1}\int_{t-A^m c_2}^{t+A^m c_2} q(x,t)dx \\
&= 1-\int_{t+A^m c_1}^{\infty}\int_{t+A^m c_2}^{\infty} - \int_{-\infty}^{t-A^m c_1}\int_{-\infty}^{t-A^m c_2} \\
&= 1 - I_1 - I_2 \\
I_1 &\leq c\int_{t+A^m c_1}^{\infty}\int_{t+A^m c_2}^{\infty}\frac{1}{1+(t-x)^k}dx \\
&= c\int_{A^m c_1}^{\infty}\int_{A^m c_2}^{\infty}\frac{1}{1+x^k}dx \to 0, \quad k>1,
\end{aligned}$$

as $m \to \infty$. Similarly, $I_2 \to 0$ as $m \to \infty$. Hence (3.13ii) holds. (3.13iii) can also be verified by using (3.12).

4. CONVERGENCE THEOREM

In 1966, Carleson proved the famous Lusin conjecture that the Fourier series of an arbitrary $L_2(R)$ function f converges pointwise almost everywhere to f. This result was extended by Hunt to L_p functions when $1 < p < 2$. In 1971, C. Feffeman proved that spherically summed two-dimensional Fourier series of $L_p(R^2)$ functions do not converge in $L_p(R^2)$ for certain real p. Kelly, Kon, Raphael [4, 5] have studied the convergence of wavelet expansions indicating the inter-connection between classical results in this area and their results including those concerning such expansions by Meyer and Walter.

We prove here a theorem on the pointwise convergence of two-dimensional wavelet expansions associated with a dilation matrix. The proof is also valid for higher dimensions. More precisely, we prove that a wavelet expansion associated with a dilation matrix of a continuous function f belonging to $L_1(R^2) \cap L_2(R^2)$ converges uniformly on a compact subset. It may be observed that this result extends Lemma 1, Corollary 1 and Theorem 1 in Walter [7].

Relaxation of continuity and weakening of regularity condition on the wavelet require further investigation on the lines of Kelly, Kon and Raphael [5].

THEOREM 4.1. *Let $q_m(x,y)$ be a reproducing kernel of a multiresolution analysis associated with a dilation matrix A; and let $f \in L_1(R^2)$ be continuous on an open set U in R^2, then*

$$f_m(y) = \int_{[y-\eta,y+\eta]\times R} q_m(x,y)f(x)dx \to f(y) \tag{4.1}$$

as $m \to \infty$ uniformly on compact subsets of U.

COROLLARY 4.1. *Let $f \in L_1(R^2) \cap L_2(R^2)$ be continuous on a subset U and let f_m be the projection of f into V_m, then*

$$f_m \to f \text{ as } m \to \infty,$$

uniformly on compact subsets of U.

Proof of Theorem 4.1. Let $\eta > 0$, then

$$\begin{aligned} f_m(y) &= \int_{[y-\eta,y+\eta]\times R} q_m(x,y)f(x)dx \\ &\quad + \int_{[y+\eta,\infty]\times R} q_m(x,y)f(x)dx \\ &\quad + \int_{[-\infty,y-\eta]\times R} q_m(x,y)f(x)dx \\ &= f(y)\int_{[y-\eta,y+\eta]\times R} q_m(x,y)dx + \int_{[y-\eta,y+\eta]\times R} q_m(x,y)(f(x)-f(y))dx \\ &\quad + \left\{\int_{[y+\eta,\infty]\times R} + \int_{[-\infty,y-\eta]\times R}\right\} = I_1 + I_2 + I_3. \end{aligned} \tag{4.2}$$

Now let K be a compact subset of U, and let V be a closed subset contained in U containing K. For $y \in V$, choose η such that $0 < \eta < c$. Further, we restrict η such that $|f(x) - f(y)| < \epsilon$ for $y \in K$ and $|x - y| < \eta$. From this it follows that

$$|I_2| \le \epsilon \int_{[y-\eta,y+\eta]\times R} |q_m(x,y)|dx \tag{4.3}$$

and

$$|I_3| \le \sup_{\eta \le |x-y|} |q_m(x,y)|\, \|f\|_{L_1(R^2)} \qquad \text{whenever } m \ge M_1 \tag{4.4}$$

where M_1 is so large that

$$\sup_{\eta \le |x-y|} |q_m(x,y)| < \epsilon \quad \text{for } m \ge M_1.$$

We choose $M_2 \ge M_1$ so large that

$$\left|1 - \int_{[y-\eta,y+\eta]} q_m(x,y)\right| < \epsilon, \qquad \text{whenever } m \ge M_2. \tag{4.5}$$

This holds because $q_m(x,y)$ is a quasi-positive delta sequence and so it follows by (3.13ii).

By (4.3), (4.4) and (4.5), we get

$$\begin{aligned} |f(y) - f_m(y)| &\le |f(y) - I_1| + |I_2| + |I_3| \\ &\le |f(y)|\left|1 - \int_{y-\gamma}^{y+\gamma} q_m(x,y)dx\right| \\ &\quad + \epsilon\int_{-\infty}^{\infty} |q_m(x,y)|dx + \epsilon\|f\|_1 \\ &\le \sup_{y\in[\alpha,\beta]} |f(y)|\epsilon + \epsilon C + \epsilon\|f\|_1 \end{aligned}$$

for $m \geq M_2$, which gives us the desired uniform convergence on $[\alpha, \beta]$ and hence on K. This proves the theorem.

Corollary 4.1 follows from Theorem 4.1 and Lemma 3.1

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