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An iterative two-step algorithm for American option pricing

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In this paper we discuss the application of a very efficient algorithm proposed recently by Kočvara and Zowe to American option pricing. Modelling and numerical simulation of options depending on the history of underlying asset price, inflation and devaluation by evolution equations and inequalities with hysteresis are proposed.

1. Introduction

Wilmott *et al.* (1993) have extensively studied the applications of numerical methods in problems of banking and finance. Black & Scholes (1973) initiated the modelling of some equity derivatives by the parabolic partial differential equations with appropriate boundary conditions. Options are certain kinds of contracts, many of them have been named as European, American, Asian and Russian but they have nothing to do with the continent of origin; they refer to a technicality in the option contract. Nowadays, the main priority of financial institutions is to manage risk instead of dealing with cash and securities. In view of the unparalleled growth of financial derivatives in the last two decades, the proper modelling and studies of inter-element relationship is a challenging problem. The main task before successful financial institution is to understand these instruments and to develop risk-free strategies to yield maximum benefit. Some of the prevalent practical market methods and strategies are eloquently presented in Alexander (1996); see also Runggaldier (1996) for a more mathematical treatment.

The main objective of this paper is to discuss the application of a very efficient algorithm for a numerical solution of American options. A study of option pricing depending on the history of asset price and caring for inflation and devaluation is proposed. In Section 2, basic concepts are introduced, while Section 3 deals with the numerical simulation of American options. Modelling of loop-back option pricing and option pricing caring inflation and devaluation by evolution equations and inequalities with hysteresis is proposed in Section 4.

2. Basic concepts related to European and American options

Before introducing European and American options, we briefly mention the commonly used terms like asset or underlying asset, equity, derivative, equity derivative, expiry, option pricing, call option, put option, strike price (exercise price), risk management, volatility.

By *underlying asset*, often called only *underlying* or *asset*, we mean commodity, exchange, shares, stocks and bonds, etc. *Equity* is a share in the ownership of a company which usually guarantees the right to vote at meetings and a share in the dividends (payment to shareholders as return for investment in the cooperation). *Derivative* refers to either a contract or a security whose pay-off or final value is dependent on one or more features of the underlying equity. In many cases it is the price of the underlying equity which determines to a large extent the value of the equity derivative or derivative based on equity, although other factors like interest rates, time to maturity and strike price can also play a significant role. The termination time of a derivatives contract, usually when the final pay-off value is calculated and paid, is called *expiry*. Option, pricing or options are some kind of contracts: the right to the holder (owner) and an obligation to the seller (writer) of a contract either to buy or to sell an underlying asset at a fixed price for a premium. In *call options* the holder has the right, but not the obligation to buy the underlying asset at the strike price. Options in which the right to sell for the holder and the obligation to buy for the writer at a strike price E for the payment of a premium is guaranteed, are called *put options*. *Strike price* or exercise price is the price at which the underlying asset is bought in options. *Risk management* is the process of establishing the type and magnitude of risk in a business enterprise and using derivatives to control and shape that risk to maximize the business objective. *Volatility* is a measure of the standard deviations of returns. In practice it is understood as the average daily range of the last few weeks or average absolute value of the daily net change of the last few weeks.

Modelling of European options

A *European call option* is a contract with the following conditions: at a prescribed time in the future, known as the expiry date, the owner of the option may purchase a prescribed asset, called underlying asset, for a prescribed amount (strike price or exercise price). Similarly, a *European put option* is a contract in which at a prescribed time in the future the owner (holder) of the option may sell an asset for a prescribed amount.

Let $V(S,t)$ denote the value of an option which is the function of the underlying asset S and time t. Black & Scholes (1973) proved that V is a solution of the parabolic partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2.1}$$

where σ and r are volatility and interest rate, respectively.

Let $C(S,t)$ and $P(S,t)$ denote the value of $V(S,t)$, respectively, when it is a call option and put option. It has been shown (see, for instance, Wilmott *et al.* (1993)) that a

European call option $C(S, t)$ is a solution of the following boundary value problem:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{2.2}$$

$$C(S, T) = \max(S - E, 0) \tag{2.3}$$

$$C(0, t) = 0 \tag{2.4}$$

$$C(S, t) \to S \text{ as } S \to \infty \tag{2.5}$$

where S, σ, r are as above, and E and T are the exercise price and expiry time, respectively.

On the other hand, a European put option $P(S, t)$ is a solution of the following boundary value problem:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0 \tag{2.6}$$

$$P(S, T) = \max(E - S, 0) \tag{2.7}$$

$$P(0, t) = E\mathrm{e}^{-r(T-t)} \tag{2.8}$$

if r is independent of time

$$P(0, t) = E\mathrm{e}^{-\int_t^T r(\tau)\,\mathrm{d}\tau}$$

if r is time dependent.

As $S \to \infty$, the option is unlikely to be exercised and so

$$P(S, t) \to 0 \quad \text{as} \quad S \to \infty. \tag{2.9}$$

Equations (2.2)–(2.5) and (2.6)–(2.9) are known as the Black–Scholes model for call and put options, respectively.

The Black–Scholes call option model can be transformed into the diffusion equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad \text{for } -\infty < x < \infty,\ \tau > 0 \tag{2.10}$$

with

$$u(x, 0) = \max(\mathrm{e}^{\frac{1}{2}(k+1)x} - \mathrm{e}^{\frac{1}{2}(k-1)x}, 0), \tag{2.11}$$

by putting $S = E\mathrm{e}^x, t = T - \tau/\frac{1}{2}\sigma^2$ and

$$C(S, t) = E\mathrm{e}^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau}u(x, \tau),$$

where $k = \frac{r}{\frac{1}{2}\sigma^2}$.

The Black–Scholes put option model can analogously be written in the form of the diffusion equation.

Modelling of American options

American options are those options which can be exercised by any time prior to expiry time. American call and put options are related to buying and selling, respectively. The valuation of American options leads to free boundary problems. Typically, at each time T, there is a valuation of S which marks the boundary between two regions; namely, on side one should hold the option and on the other side one should exercise it. Let us denote this boundary by $S_f(t)$ (generally, this critical asset value varies with time). Since we do not know $S_f(t)$ *a priori*, we are lacking one piece of information compared with the corresponding European option problem. Thus with American options we do not know *a priori* where to apply boundary conditions. This situation resembles the obstacle problem and can be effectively tackled by methods of variational inequalities (see, for instance, Kinderlehrer & Stampacchia (1980), Glowinski (1984) and Giannessi (1994)). Wilmott *et al.* (1993) have shown that American call option and put options can be formulated as the following boundary value problems and equivalent variational inequalities.
American call option is modelled by the following boundary value problem:

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geqslant 0 \tag{2.12}$$

$$u(x,\tau) - g(x,\tau) \geqslant 0 \tag{2.13}$$

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \cdot (u(x,\tau) - g(x,\tau)) = 0 \tag{2.14}$$

$$u(x,0) = g(x,0) \tag{2.15}$$

$$u(a,\tau) = g(a,\tau) = 0 \tag{2.16}$$

$$u(b,\tau) = g(b,\tau), \tag{2.17}$$

where

$$g(x,\tau) = \mathrm{e}^{\frac{1}{4}(k+1)^2\tau} \max(\mathrm{e}^{\frac{1}{2}(k+1)x} - \mathrm{e}^{\frac{1}{2}(k-1)x}, 0). \tag{2.18}$$

The financial variables S, t and the option value C are again computed by putting $S = E\mathrm{e}^x$, $t = T - \tau/\frac{1}{2}\sigma^2$ and

$$C(S,t) = E\mathrm{e}^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(x,\tau).$$

In order to avoid technical complications, the problem is restricted to a finite interval (a,b) with $-a$ and b large enough. In financial terms, we assume that we can replace the exact boundary conditions by the approximations for small values of S, $P = E - S$, while for large values, $P = 0$.

Let us denote by u_τ the function $x \mapsto u(x,\tau)$. The equivalent parabolic variational inequality is as follows: find $u = u_\tau \in K_\tau$ (τ runs over $[0, \frac{1}{2}\sigma^2 T]$) such that

$$\begin{gathered}\left(\frac{\partial u}{\partial \tau}, \varphi - u\right)_2 + a(u, \varphi - u) \geqslant 0 \\ \text{for all } \varphi \in K_\tau,\ \text{a.e. } \tau \in \left(0, \tfrac{1}{2}\sigma^2 T\right),\ u(x,0) = g(x,0),\end{gathered} \tag{2.19}$$

where

$$K_\tau := \{v \in H^1(a,b) \mid v(a) = g(a,\tau), v(b) = g(b,\tau), v(x) \geqslant g(x,\tau)\}$$

and $(\cdot,\cdot)_2$ denotes the inner product on $L^2(a,b)$. With

$$W\left(0, \tfrac{1}{2}\sigma^2 T\right) := \left\{v \mid v \in L^2\left(0, \tfrac{1}{2}\sigma^2 T; H^1(a,b)\right), \frac{\partial v}{\partial \tau} \in L^2\left(0, \tfrac{1}{2}\sigma^2 T; H^{-1}(a,b)\right)\right\}$$
$$W_0\left(0, \tfrac{1}{2}\sigma^2 T\right) := \left\{v \mid v \in W\left(0, \tfrac{1}{2}\sigma^2 T\right), v(0) = g(\cdot, 0)\right\}$$

and

$$\mathcal{K} := \left\{v \mid v \in W\left(0, \tfrac{1}{2}\sigma^2 T\right), v_\tau \in K_\tau \text{ for a.e. } \tau \in \left[0, \tfrac{1}{2}\sigma^2 T\right]\right\}$$
$$\mathcal{K}_0 := \left\{v \mid v \in W_0(0, \tfrac{1}{2}\sigma^2 T), v_\tau \in K_\tau \text{ for a.e. } \tau \in \left[0, \tfrac{1}{2}\sigma^2 T\right]\right\}$$

we can formulate an equivalent variational inequality.

Find $u \in \mathcal{K}_0$ such that

$$\int_0^{\frac{1}{2}\sigma^2 T} \left(\frac{\partial u}{\partial \tau}, \varphi - u\right)_2 \mathrm{d}\tau + \int_0^{\frac{1}{2}\sigma^2 T} a(u, \varphi - u)\,\mathrm{d}\tau \geqslant 0 \quad \text{for all } \varphi \in \mathcal{K}. \tag{2.20}$$

American put option is modelled by a boundary value problem that only differs in the boundary conditions and the (transformed) pay-off function g:

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \geqslant 0 \tag{2.21}$$
$$(u(x,\tau) - g(x,\tau)) \geqslant 0 \tag{2.22}$$
$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \cdot (u(x,\tau) - g(x,\tau)) = 0 \tag{2.23}$$
$$u(x,0) = g(x,0) \tag{2.24}$$
$$u(a,\tau) = g(a,\tau) \tag{2.25}$$
$$u(b,\tau) = g(b,\tau) = 0, \tag{2.26}$$

where

$$g(x,\tau) = \mathrm{e}^{\frac{1}{4}(k+1)^2\tau} \max(\mathrm{e}^{\frac{1}{2}(k-1)x} - \mathrm{e}^{\frac{1}{2}(k+1)x}, 0). \tag{2.27}$$

The equivalent variational inequality is formulated analogously to (2.19) or (2.20), with the only change in the boundary conditions and the function g.

It may be said that in American call option, $C(S,t)$ lies above the pay-off $\max(S - E, 0)$; in the transformed variables, this condition takes the form $(u(x,\tau) - g(x,\tau)) \geqslant 0$. The condition $\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \geqslant 0$ means that the return from the risk-free delta-hedged portfolio is less than the risk-free interest rate r.

The numerical method to solve the complementarity problems modelling American call and put options will be the same. In Wilmott *et al.* (1993) it is discussed in great detail how the projected successive overrelaxation (SOR) method can be employed for numerical simulation of American options. In the next section we describe a new, more efficient algorithm, its application to American option pricing and present numerical comparison with the projected SOR method. A similar approach to the numerical treatment of the problem, based on the LCP reformulation, can be found in a recent paper by Huang & Pang (1998).

3. Numerical simulation of American call option by a two-step algorithm

We discretize the boundary value problem (2.12)–(2.17) by the finite difference method and solve it using the Crank–Nicholson scheme; see, e.g. (Glowinski, 1984) or (Wilmott *et al.*, 1993). At each time step we need to solve a linear complementarity problem (LCP):

Find $u^{m+1} \in \mathbb{R}^n$ such that

$$\begin{aligned} Cu^{m+1} \geqslant b^m, \qquad u^{m+1} \geqslant g^{m+1} \\ (u^{m+1} - g^{m+1})^T (Cu^{m+1} - b) = 0. \end{aligned} \tag{3.1}$$

Here C is an $n \times n$ real symmetric positive definite matrix given below:

$$C = \begin{pmatrix} 1+\alpha & -\frac{1}{2}\alpha & 0 & \cdots & 0 \\ -\frac{1}{2}\alpha & 1+\alpha & -\frac{1}{2}\alpha & & \vdots \\ 0 & -\frac{1}{2}\alpha & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1+\alpha & -\frac{1}{2}\alpha \\ 0 & \cdots & 0 & -\frac{1}{2}\alpha & 1+\alpha \end{pmatrix}$$

with $\alpha = \frac{\delta\tau}{(\delta x)^2}$, $\delta\tau$ and δx being the time and space discretization parameters, respectively. Vectors u^{m+1} and g^{m+1} are the discrete counterparts of $u(x, \tau)$ and $g(x, \tau)$ from (2.12)–(2.17) at a time step $(m+1)\delta\tau$ and b^m is a 'right-hand side' vector containing information from the previous time step $m\delta\tau$.

Note that the matrix C is large and sparse and that the problem (3.1) has to be solved repeatedly in each time step. Thus we need a fast LCP solver (in this application, literally, time is money). Several algorithms were proposed for LCPs with large sparse symmetric positive definite matrix. These algorithms are either based on methods for solving linear systems, like the SOR method (Mangasarian, 1977) or the preconditioned conjugate gradient (PCG) method (Mittelmann, 1981; Leary, 1979), or on optimization methods for convex quadratic programs, like the gradient projection method (Moré & Toraldo, 1991) or interior-point methods (Kojima *et al.*, 1991).

Recently, a two-step algorithm has been proposed by Kočvara & Zowe (1994). This algorithm, based on ideas of multigrid methods, combines the efficiency of the PCG method for solving linear systems with the ability of SOR to smooth down the solution error. The smoothing property of a variant of SOR with a projection on a feasible set, called SORP, enables to detect fast the active set

$$I(u) := \{i \mid u_i = g_i\}.$$

In the above definition and in the rest of this section we skip the time step index and write (3.1) as

Find $u \in \mathbb{R}^n$ such that

$$\begin{aligned} Cu \geqslant b, \qquad u \geqslant g \\ (u-g)^T(Cu-b) = 0. \end{aligned} \tag{3.2}$$

Instead, we will use the upper index to denote the successive iterate; that is, u^k will be the kth iterate of a particular method. By u^* we denote the (unique) solution to (3.2). We further denote by C_{ij} the (i, j) component of the matrix C. Finally, let us define the feasible set of (3.2):

$$S := \{v \in \mathbb{R}^n \mid v_i \geqslant g_i,\ i = 1, 2, \dots, n\}.$$

The two-step algorithm mentioned above proved to be very efficient for large LCPs. The examples in Kočvara & Zowe (1994) even indicate that the algorithm, based on PCG, is asymptotically as fast as PCG for linear systems itself. It is our strong belief that the algorithm fits naturally to our LCP and significantly improves the efficiency of the overall time stepping procedure. In the following text we describe the algorithm in detail.

We first recall the definition of SORP.

SORP (Mangasarian, 1977)

Choose $x^0 \in \mathbb{R}^n$ and put for $k = 0, 1, 2, \dots$

$$x_i^{k+1} = \max\left\{x_i^k - \omega\frac{1}{A_{ii}}\left(\sum_{j<i} A_{ij}x_j^{k+1} + \sum_{j\geqslant i} A_{ij}x_j^k - b_i\right), c_i\right\}, \qquad i = 1, 2, \dots, n, \tag{3.3}$$

where $\omega \in (0, 2)$ is a relaxation parameter, and one backward SORP step:

$$\begin{aligned} \tilde{u}_i^{k+1} &= \max\left\{u_i^k - \omega\frac{1}{C_{ii}}\left(\sum_{j<i} C_{ij}\tilde{u}_j^{k+1} + \sum_{j\geqslant i} C_{ij}u_j^k - b_i\right), g_i\right\}, \quad i = 1, 2, \dots, n \\ u_i^{k+1} &= \max\left\{\tilde{u}_i^{k+1} - \omega\frac{1}{C_{ii}}\left(\sum_{j\leqslant i} C_{ij}\tilde{u}_j^{k+1} + \sum_{j>i} C_{ij}u_j^{k+1} - b_i\right), g_i\right\}, \quad i = n, n-1, \dots, 1. \end{aligned} \tag{3.4}$$

We shall denote by $SSORP^m(x; A, b, c)$ the value which we obtain in m steps (3.4). If $m = 1$ we skip the superscript.

The new algorithm can be viewed as a variant of the active set strategy. That means, at each iteration step one has to solve a linear system with a matrix of similar structure as that of C. This system, however, does not have to be solved exactly, particularly when the actual active set $I(u^k)$ is far away from $I(u^*)$. The idea is to perform just a few steps of a preconditioned conjugate gradient method. For completeness we give below the definition of the PCG algorithm for the solution of system

$$Au = b$$

with a symmetric and positive definite matrix A.

PCG (see, e.g., Barret *et al.*, 1993)

Let M be a symmetric positive definite matrix (the preconditioner).
Choose $x^0 \in \mathbb{R}^n$ and $\varepsilon > 0$.
Set $r^0 = b - Au^0$, $p^0 = z^0 = M^{-1}r^0$ and do for $k = 0, 1, 2, \dots$:

$$\begin{aligned}
&\alpha_k = \langle r^k, z^k\rangle / \langle p^k, Ap^k\rangle \\
&x^{k+1} = x^k + \alpha_k p^k \\
&r^{k+1} = r^k - \alpha_k A p^k \\
&z^{k+1} = M^{-1} r^{k+1} \\
&\text{if } \langle z^{k+1}, r^{k+1}\rangle \geqslant \varepsilon, \text{ then if } \|r^{k+1}\| \geqslant \varepsilon, \text{ continue} \\
&\beta_k = \langle r^{k+1}, z^{k+1}\rangle / \langle r^k, z^k\rangle \\
&p^{k+1} = z^{k+1} + \beta_k p^k.
\end{aligned}$$

We denote by $PCG^s(u; A, b)$ the point which we reach in s PCG steps starting from u.

We are now going to explain the new approach. Assume that we have an approximation u of the solution u^* to (3.2). We again denote by $I(u)$ the active set with respect to the constraint $u \geqslant g$, that is,

$$I(u) := \{i \mid u_i = g_i\}.$$

Let $p(u)$ be the cardinality of $I(u)$ and $P_{I(u)} : \mathbb{R}^n \to \mathbb{R}^{n-p(u)}$ the operator which assigns a vector $v \in \mathbb{R}^n$ the reduced vector in $\mathbb{R}^{n-p(u)}$ obtained by omitting the components of v with indices from $I(u)$. We skip u in this notation if this does not lead to confusion. Further we write I^* for $I(u^*)$.

The basic idea of the new method is the following: We try to identify the active set by a few steps of SSORP; we get an approximation of I^*. With this approximation, we perform several steps of PCG. Then, again some steps of SSORP to improve the approximation of I^*, and so on. Therefore we call the method SSORP-PCG. One iteration of the SSORP-PCG algorithm consists of two steps as given below.

SSORP-PCG (Kočvara & Zowe, 1994)

Choose $u^0 \in \mathbb{R}^n$, $m \in \mathbb{N}$, $s \in \mathbb{N}$ and do for $k = 0, 1, 2, \dots$:
Step 1: Perform m SSORP steps (3.4) and put

$$u^{k+1/2} := SSORP^m(u^k; C, b, g).$$

Step 2: Determine P_I with $I := I(u^{k+1/2})$ and compute

$$r^k = b - Cu^{k+1/2}.$$

Perform s PCG steps and define with

$$z^k := PCG^s(0, P_I C P_I^T, P_I r^k)$$

the next iterate

$$u^{k+1} := u^{k+1/2} + \gamma P_I^T z^k,$$

where γ is the largest real number such that $\gamma \leqslant 1$ and $u^{k+1} \in S$.

It was proved that the sequence of iterates $\{u^k\}_{k \in \mathbb{N}}$ produced by SSORP-PCG converges to the solution x^* of LCP (3.2).

Just as for multigrid methods, the number of SSORP steps (parameter m in Step 1) can be chosen small; already for $m = 2$ we obtained good results. The number of PCG steps (parameter s in Step 2) is more problem-dependent. Generally speaking, s should grow with the condition number of the matrix A. We recommend to take $s = 5$ for well-conditioned problems and $s = 10$ otherwise.

Concerning the preconditioner, it is well known that efficient preconditioning matrices for elliptic problems are those based on incomplete factorization (Axelsson & Barker, 1984; Gustafsson, 1990). We implemented the $MIC(0)^*$ algorithm (Gustafsson, 1990).

Below we will compare SSORP-PCG with SORP. To guarantee equal conditions, we have chosen the following stopping criteria:

- The SORP method with relaxation parameter $\omega = 1.9$ was stopped when $\|u_{k+1} - u_k\|_2$ became less than 10^{-9} the first time.
- The stopping criterion in SSORP-PCG guarantees an accuracy comparable to the one in the SORP implementation: if we applied SORP after stopping SSORP-PCG, then we typically had $\|u_{k+1} - u_k\|_2 \leqslant 10^{-9}$.

Numerical results

In this section we present results of an example computed by the new algorithm and, for comparison, also by the plain SORP method. We would like to emphasize that the data shown below should not be evaluated from the viewpoint of overall efficiency; the example is academic, the choice of time and space discretization parameters, as well as the parameter α, can be far from being optimal. Our goal is to demonstrate the efficiency of the SSORP-PCG algorithm for solving a particular subproblem (LCP) which, no doubt, is large and has to be solved repeatedly many times.

We have solved an example from Wilmott *et al.* (1995), in order to get comparable results. This is a problem of computing American put option with interest rate $r = 0{\cdot}10$, volatility $\sigma = 0{\cdot}4$ and exercise price $E = 10$. The calculation is carried out with $\alpha = 1$ and with the expiry time of three months. The space interval $[a, b]$ is chosen as $[-0.5, 0.5]$. We carried out the computation for three different space discretization steps: $\delta x = 0.01$, $\delta x = 0.001$ and $\delta x = 0.0001$.

Table 1 shows the corresponding values of the time step $\delta\tau$, the size n of the $n \times n$ matrix C and the number of time steps N saying how many LCPs we have to solve.

These three problems were solved using the two-step algorithm SSORP-PCG. Table 2 shows the overall CPU time needed to solve the problem, as well as the time needed to solve one LCP. These times are compared with the solution times of the plain SORP method. Table 2 also shows the (average) number of active constraints. Note that for a large number of active constraints (compared to the problem size n) the SORP methods becomes very efficient. This is observed on the number of SORP iterations, shown in the

TABLE 1

δx	$\delta\tau$	n	N
0.01	10^{-4}	99	200
0.001	10^{-6}	999	20 000
0.0001	10^{-8}	9999	2 000 000

TABLE 2

δx	S-P one step CPU	S-P overall CPU	SORP one step CPU	SORP overall CPU	#active constr.	#SORP iter.
0.01	0.0021	0.38	0.0089	1.45	20	75
0.001	0.016	390.4	0.041	783.9	400	32
0.0001	0.44	*	1.05	*	8500	28

TABLE 3

δx	S-P one step CPU	S-P overall CPU	SORP one step CPU	SORP overall CPU	#SORP iter.
0.01	0.0023	0.43	0.013	2.39	109
0.001	0.016	486.3	0.11	2221.1	87
0.0001	0.44	*	2.89	*	74

last column of Table 2; with an increasing number of variables, the number of iteration decreases. This is very untypical behaviour caused by the particular data of the problem.

The second factor which influences the behaviour of the algorithms is that in both, SORP and SSORP-PCG, the solution from the previous time step was taken as an initial approximation for the current time step. This technique, in fact, favours the SORP algorithm. This is clearly seen from Table 3 which shows the CPU times obtained with the initial approximation for each time step taken as zero vector. In this situation, SSORP-PCG is a clear winner.

Our results show that the new algorithm certainly outperforms the SORP method, even though the problem data favour the latter.

All numerical experiments were carried out on a Sun Ultra1 m140 computer running the operation system Solaris 2.6. The CPU times are in seconds.

4. Evolution equations and inequalities in option pricing

In this section we propose the possibility of modelling different kinds of options by evolution equations and inequalities with hysteresis. The phenomenon of hysteresis occurs in a large number of practical situations and plays an important role in areas like ferromagnetism, phase transitions solid to liquid or liquid to solid. In the past it has attracted the attention of physicists, engineers and mathematicians alike; see for instance Brokate & Sprekels (1996) for updated references. This phenomenon may also have a significant role in problems of banking and finance, but it seems to have escaped the attention of researchers.

Let D be a bounded domain in $\mathbb{R}^n$. Visintin (1986) has studied the following classes of parabolic differential equations with hysteresis:

$$\frac{\partial u}{\partial t} - \Delta u + w = f \quad \text{in } \Omega := D \times (0, T) \tag{4.1}$$

$$\frac{\partial u}{\partial t} - \Delta u = g \quad \text{in } \Omega \tag{4.2}$$

$$\frac{\partial u}{\partial \nu} + w = h \quad \text{on } \Sigma := \partial D \times (0, T) \tag{4.3}$$

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + w = f \quad \text{in } \Omega, \tag{4.4}$$

where f, g, h are data and ν denotes the outward normal. Here w may be in general a hysteresis operator and, in particular, delay and anticipation operator defined below.

Let ρ_1 and ρ_2 be fixed real numbers with $\rho_1 < \rho_2$. Further let u be a continuous function on $[0, T]$ and w^0 be given by

$$\begin{aligned} w^0 &= -1 \quad \text{if } u(0) \leqslant \rho_1 \\ w^0 &= 1 \quad \text{if } u(0) \geqslant \rho_1 \\ w^0 &= -1 \text{ or } 1 \quad \text{if } \rho_1 < u < \rho_2. \end{aligned}$$

A function $w : [0, T] \to \{1, -2\}$ is called *jump delay operator* or is said to fulfill *jump condition with delay* if it has the following properties:

$w(0) = w^0$

if $u(t) \leqslant \rho_1$ ($u(t) \geqslant \rho_2$, respectively) then $w(t) = -1$ ($w(t) = 1$, respectively) $\forall t \in [0, T]$

w can jump from -1 to 1 (from 1 to -1, respectively) at time t only if $u(t) = \rho_2$ (ρ_1, respectively); these are the only discontinuities of w.

Let

$$\alpha(\xi) = (\xi - \rho_2)^+ - (\xi - \rho_1)^- \tag{4.5}$$

$$\beta(\xi) = \xi - \alpha(\xi) \quad \text{for all } \xi \in \mathbb{R} \tag{4.6}$$

$$S(\eta) = \begin{cases} \{-1\} & \text{if } \eta < 0 \\ [-1, 1] & \text{if } \eta = 0 \\ \{1\} & \text{if } \eta > 0. \end{cases} \tag{4.7}$$

Let $u : [0, T] \to \mathbb{R}$ be absolutely continuous. A measurable function $w(t)$ is said to fulfill a *jump condition with anticipation* or is called *jump anticipation operator* if a.e. in $[0, T]$

$w(t) = -1$ ($w(t) = -1$, respectively) provided $u(t) < \rho_1$ ($u(t) > \rho_2$, respectively)

$w(t) \in S(u'(t))$ provided $\rho_1 \leqslant u(t) \leqslant \rho_2$

or, equivalently,

$$w \in S(\alpha(u)) \text{ a.e. in } (0, T)$$

$$w \in S(\beta(u)') \text{ a.e. in } (0, T).$$

Anticipation corresponds to the case in which in a certain range the output depends on the trend of the input u, and not on its value. Such situations arise in sociology and economics, points out Visintin. If the input is absolutely continuous in time, then the anticipation criterion allows to take advantage of the forecast offered to some extent by the inertia of the variable u. Uniqueness of the solution to (4.1)–(4.4) for fairly general w has been proved by Hilbertin; see, for example, Brokate & Sprekels (1996, pp. 134–136). Evolution variational inequalities with memory terms and hysteresis have also been studied by Kenmochi *et al.* (1992) and Kenmochi & Visintin (1993). A systematic study of the numerical solution of evolution problems with hysteresis is presented in Verdi (1994) with updated comprehensive literature.

The concept of loop back options is also discussed in Wilmott *et al.* (1993, pp. 156–217), which include Asian options, where the option value depends on the current asset price S and the history of the underlying asset price. It is clear from the cited literature that such situation can be modelled by hysteresis operators and in turn by evolution variational inequalities. Numerical techniques described by Verdi, Visintin, Kenmochi, etc could be used for simulation and visualization. It is pertinent to point out here that replacing the arithmetic average

$$\frac{1}{t}\int_0^1 S(\tau)\,\mathrm{d}\tau$$

in an Asian option by

$$\frac{1}{t}\int_{\xi}^{t+\xi} S(\tau)\,\mathrm{d}\tau \quad \text{uniformly in } \xi$$

may give better insight.

In modelling of options, inflation and devaluation of currency with respect to other currencies have not been taken into account. In our view, diffusion equations (4.1)–(4.4) will model European options where these two facts are taken into account, while inclusion of these two parameters into American options will lead to evolution variational inequalities with hysteresis of the type investigated by Kenmochi, Koyama and Visintin.

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