ON SOME RECENT DEVELOPMENTS CONCERNING MOREAU'S SWEEPING PROCESS

A.H. Siddiqi
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia
ahasan@kfupm.edu.sa

P. Manchanda
Department of Mathematics
GNDU, Amritsar-143005, India
kmanch@jla.vsnl.net.in

M. Brokate
Zentrum Mathematik
Technical University Munich
D 80290 Munich, Germany
brokate@appl-math.tu-muenchen.de

Abstract The main objective of this paper is to present an overview of Moreau's sweeping process $u'(t) \in N_{C(t)}(u(t))$ along with some of our results concerning new variants of this process. Several open problems are mentioned.

Keywords: Sweeping process, evolution variational inequality, evolution quasi-variational inequality, state-dependent sweeping process, degenerate sweeping process, variants of sweeping process, play and stop operators, sweeping process without convexity.

1. Introduction

A sweeping process comprises two important ingredients: one part that sweeps and the other that is swept. For example, imagine the Euclidean plane and consider a large ring with a small ball inside it; the ring starts to move at time $t = 0$. Depending on the motion of the ring, the ball will first stay where it is (in case it is not hit by the ring); otherwise it is swept towards the interior of the ring. In this latter case, the velocity of the ball has to point inward to the ring in order not to leave.

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We consider the case where the small ball has diameter zero, that is, it degenerates to a point. We replace the ring and its interior by an arbitrary closed convex set. In mathematical terms, the problem then becomes

$$-u'(t) \in N_{C(t)}\Big(u(t)\Big) \quad \text{a.e in } [0, T], \quad u(0) = u_0 \in C(0). \tag{1.1}$$

Here, for any closed convex set C subset of a Hilbert space H and $x \in C$, the set

$$N_C(x) = \Big\{y \in H, \langle y, v - x\rangle \leq 0 \text{ for all } v \in C\Big\} \tag{1.2}$$

denotes the outward normal cone to C at x. $u(t)$ denotes the position of the ball at time t and $C(t)$ is the ring at time t. The expression $N_{C(t)}\Big(u(t)\Big)$ denotes the outward normal cone to the set $C(t)$ at position $u(t)$ as defined in Equation (1.2). Thus, Equation (1.1) simply means that the velocity $u'(t)$ of the ball has to point inward to the ring at almost every (a.e.) time $t \in [0, T]$. The restriction is due to the fact that usually we will not have a smooth function $t \to u(t)$ satisfying (1.1), but functions satisfying (1.1) that are differentiable everywhere besides on some subset of $[0, T]$ of measure zero. The initial condition $u(0) \in C(0)$ states that the ball is initially contained in the ring. Equation (1.1) is the simplest instance of the sweeping process, introduced by Moreau [28] in the seventies.

In general, the time-dependent moving set at $t \to C(t)$ is given, and we want to prove the existence of a solution (preferably unique) $t \to u(t)$ that will take values in some Hilbert space (Here $H = \mathbb{R}^2$). It is allowed that $C(t)$ changes its shape while moving, whereas in the introductory example the ring simply moved by translation and maintained its original shape. The sweeping process plays an important role in elastoplasticity and dynamics for unilateral problems (see, for example, [4, 21, 25, 27, 30]).

In Section 2, we present a resumé of some important results for the sweeping process (1.1) concerning existence and uniqueness of solutions. Section 3 is devoted to a generalization of the sweeping process where the moving set depends on the current state $u(t)$; that is, $C = \Big(t, u(t)\Big)$ instead of $C = C(t)$. This has been studied by Kunze and Monteiro Marques [18]. We present in Section 4 a degenerate sweeping process studied by Kunze and Monteiro Marques [16, 17]. In Section 5, we discuss some unpublished results of Manchanda and Siddiqi [23] and Siddiqi, Manchanda and Brokate [36]. Section 6 deals with the existence of solutions to the nonconvex sweeping case, that is, the case when $C(t)$ is not a convex set in (1.1). These existence results have been obtained by Benabdellah [2] and Colombo and Goncharav [8]. Section 7 provides the relationship between the play operator, the stop operator and the sweeping process. In Section 8, we remark on several open problems.

2. Moreau's Sweeping Process

Let H be a separable Hilbert space, let $x \in H$ and $C \subset H$ be closed, convex and nonempty. Then there exists a unique $y \in C$ that minimizes the distance of

x to C. y is called the projection of x onto C and is written as $y = \mathrm{Proj}(x, C)$. $y = \mathrm{Proj}(x, C)$ if and only if (we denote the norm in H by $|\cdot|$)

$$|x - y| = d(x, C) \text{ where } d(x, C) = \inf_{z \in C} |x - z| .$$

Equivalently, $y = \mathrm{Proj}(x, C)$ if and only if

$$y \in C,\ \langle y - x, y - z\rangle \leq 0 \text{ for all } z \in C . \tag{2.1}$$

Let us denote by

$$d_H(C_1, C_2) = \max\left\{\sup_{x \in C_2} d(x, C_1),\ \sup_{x \in C_1} d(x, C_2)\right\} \tag{2.2}$$

the Hausdorff distance between the subsets C_1 and C_2 of the Hilbert space H.

The variation of a function $u : [0, T] \to H$ is defined as

$$\left.\begin{aligned} \mathrm{Var}(u) &= \mathrm{Var}(u, [0, T]) \\ &= \sup\left\{\sum_{i=0}^{N-1} |u(t_{i+1}) - u(t_i)| \;\middle|\; \begin{array}{l} 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T \\ \text{is a partition of } [0, T] \end{array}\right\} \end{aligned}\right\} \tag{2.3}$$

and u is called a function of bounded variation if $\mathrm{Var}(u) < \infty$. u is called Lipschitz continuous if there is a $K > 0$ such that

$$|u(t) - u(s)| \leq K|t - s| \quad t, s \in [0, T]. \tag{2.4}$$

It can be easily checked that every Lipschitz continuous function u is of bounded variation, and that $\mathrm{Var}(u) \leq KT$ if (2.4) holds. u is called absolutely continuous, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{N} (t_k - s_k) < \delta \quad \Rightarrow \quad \sum_{k=1}^{N} |u(t_k) - u(s_k)| < \varepsilon$$

holds for every finite collection $(I_k)_{1 \leq k \leq N}$, $N \in \mathbb{N}$, of pairwise disjoint nonempty subintervals $I_k = [s_k, t_k]$ of $[0, T]$. If u is absolutely continuous, it is differentiable almost everywhere and satisfies

$$u(t) = u(0) + \int_0^t u'(s)\, ds , \quad t \in [0, T] , \quad \mathrm{Var}(u) = \int_0^t |u'(s)|\, ds .$$

where the integral has the meaning of the Bochner-Lebesgue integral. The choice $\delta = \varepsilon / K$ shows that u is absolutely continuous if it satisfies (2.4).

In order to prove the existence of solutions to the sweeping process and its variants, we often construct a sequence of approximating solutions whose variations are uniformly bounded. Then the compactness result of Theorem 2.2 allows us to select a subsequence which converges towards some limit function, and then the task is to show that the limit function is indeed the desired solution.

Definition 2.1 . *An absolutely continuous function $u : [0,T] \to H$ is a solution of the sweeping process (1.1) if*

(i) $u(0) = u_0$,

(ii) $u(t) \in C(t)$ *for all* $t \in [0,T]$,

(iii) $-u'(t) \in N_{C(t)}\Big(u(t)\Big)$ *a.e. in* $[0,T]$.

Theorem 2.1 (Existence Theorem for Moreau's Process) . *Let $t \to C(t)$ be Lipschitz continuous, that is,*

$$d_H\Big(C(t), C(s)\Big) \le K|t-s|, \qquad t,s \subset [0,T], \tag{2.5}$$

and $C(t) \subset H$ be nonempty, closed, and convex for every $t \in [0,T]$. Let $u_0 \in C(0)$. Then there exists a solution $u : [0,T] \to H$ of (1.1) satisfying (2.4). In particular, $|u'(t)| \le K$ for almost every $t \in (0,T)$.

The following results are required for the proof of this theorem and for the subsequent discussions.

Theorem 2.2 *Let H be a Hilbert space and $\{u_n\}$ a sequence of functions $u_n : [0,T] \to H$ which are bounded uniformly in norm and variation, i.e.,*

$$|u_n(t)| \le M_1, \quad n \subset \mathbb{N}, \quad t \in [0,T], \quad \text{and } Var(u_n) \le M_2, \quad n \in \mathbb{N}. \tag{2.6}$$

for some constants $M_1, M_2 > 0$ independently of $n \in \mathbb{N}$ and $t \in [0,T]$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and a function $u : [0,T] \to H$ such that $Var(u) \le M_2$ and $u_{n_k}(t) \to u(t)$ weakly in H for all $t \in [0,T]$; i.e.,

$$\langle u_{n_k}(t), z\rangle \to \langle u(t), z\rangle \quad \text{for all } z \in H \text{ as } k \to \infty.$$

The following lemma summarizes some facts related to weak convergence.

Lemma 2.1 *Let $u_n \to u$ weakly in H.*

(a) $|u| \le \liminf_{n\to\infty} |u_n|$ *holds.*

(b) If $u_n \in C + \overline{B}_{\epsilon_n}(0)$ for some closed convex $C \subset H$ and some sequence $\epsilon_n \to 0$, then $u \in C$.

Lemma 2.2 *Let $u : [0, T] \to H$ be an absolutely continuous function. Then*

$$\int_0^T \langle u'(t), u(t)\rangle dt = \frac{1}{2}|u(T)|^2 - \frac{1}{2}|u(0)|^2.$$

For more details of the proof of Theorem 2.1 and the intermediate results above, we refer to [21, 25] or [28].

Theorem 2.3 (Uniqueness of solution of Moreau's sweeping process) *The solution of (1.1) is unique in the class of absolutely continuous functions.*

Corollary 2.1 (Dependence on data) *Under the assumptions of Theorems 2.1 and 2.3, if u and v are two solutions with $u(0) = u_0$ and $v(0) = v_0$, then*

$$|u(t) - v(t)| \le |u_0 - v_0|, \qquad t \in [0, T].$$

Theorem 2.4 (Dependence on the moving set) *Let $t \mapsto C(t)$ and $t \mapsto D(t)$ be two moving sets which satisfy (2.5) with Lipschitz constants L_C and L_D, respectively. Assume that $C(t)$ and $D(t)$ are nonempty, closed, and convex for every $t \in [0, T]$. Then, if u denotes the solution to the sweeping process with $t \mapsto C(t)$ and initial value $u(0) = u_0$, and if v denotes the solution to the sweeping process with $t \mapsto D(t)$ and initial value $v(0) = v_0$, the estimate*

$$|u(t) - v(t)|^2 \le |u_0 - v_0|^2 + 2(L_C + L_D)\int_0^t \Delta(s)ds, \quad t \in [0, T], \quad (2.7)$$

holds, where

$$\Delta(t) = d_H\Big(C(t), D(t)\Big), \quad t \in [0, T].$$

Proof. (See [21].) For fixed $t \in [0, T]$, we have $u(t) \in C(t) \subset D(t) + \overline{B}_{\Delta(t)}(0)$. Hence, there exist vectors $d(t) \in D(t)$ and $r(t) \in H$ such that $u(t) = d(t) + r(t)$ and $|r(t)| \le \Delta(t)$. It can be shown that it is possible to choose the maps $t \mapsto d(t)$ and $t \mapsto r(t)$ as being measurable. Similarly, we find $v(t) = c(t) + s(t)$ with $c(t) \in C(t)$ and $|s(t)| \le \Delta(t)$. We verify using Lemma 2.2 that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(|u(t) - v(t)|^2\Big) \\
&= \langle u'(t) - v'(t), u(t) - v(t)\rangle \\
&= \langle u'(t), u(t) - c(t)\rangle + \langle v'(t), v(t) - d(t)\rangle - \langle u'(t), s(t)\rangle - \langle v'(t), r(t)\rangle \\
&\le -\langle u'(t), s(t)\rangle - \langle v'(t), r(t)\rangle \le \Big(|u'(t)| + |v'(t)|\Big)\Delta(t).
\end{aligned}$$

According to Theorem 2.1, $|u'(t)| \le L_C$ and $|v'(t)| \le L_D$ almost everywhere, thus integration yields (2.7). ∎

It may be observed that the following schemes are of vital importance to many proofs for sweeping processes. We fix $n \in \mathbb{N}$ and choose a time discretization

$$0 = t_0^n < t_1^n < \cdots < t_{N-1}^n < t_N^n = T, \text{ with } (t_{i+1}^n - t_i^n) \leq \frac{1}{n}, \quad 0 \leq i \leq N-1; \tag{2.8}$$

For example, we can set $t_i^n = i/n$, but we need not fix the discretization explicitly. The value of $N \in \mathbb{N}$ will depend on n, and $N \to \infty$ for $n \to \infty$. We define the step approximation $u^n : [0, T] \to H$ as follows. Let

$$u_0^n = u_0, \quad u_{i+1}^n = \text{proj}(u_i^n, C(t_{i+1}^n)) \in C(t_{i+1}^n), \quad 0 \leq i \leq N-1. \tag{2.9}$$

This is the "catching up" algorithm, since the approximation u_{i+1}^n is made to catch up with the set $C(t_{i+1}^n)$ through projection. Recall that we have to achieve $u(t) \in C(t)$ for the solution.

The u_n are defined via linear interpolation

$$u_n(t) = u_i^n + \left(\frac{t - t_i^n}{t_{i+1}^n - t_i^n}\right)(u_{i+1}^n - u_i^n), \quad t \in [t_i^n, t_{i+1}^n]. \tag{2.10}$$

In order to prove existence of a solution, we want to find a subsequence of $(u_n)_{n \in \mathbb{N}}$ that converges to a solution of (1.1) or one of its variants. To this end, we wish to apply Theorem 2.2 and we have to derive the uniform bound in norm and variation in (2.6).

3. The State-Dependent Sweeping Process

In this section we discuss a generalization of the classical sweeping process given by (1.1) where we allow the underlying set $C(t)$ to depend also on the current state $u = u(t)$, so the moving set now becomes $C(t, u(t))$. Thus, our new problem is to find u such that

$$-u'(t) \in N_{C(t,u(t))}(u(t)) \quad \text{a.e. in } [0, T], \; u(0) = u_0 \in C(0, u_0). \tag{3.1}$$

Similarly as before a solution u of (3.1) must satisfy $u(t) \in C(t, u(t))$ for $t \in [0, T]$. In order to prove its existence, we need the following property instead of (2.5)

$$d_H(C(t, u), C(t, v)) \leq L_1|t - s| + L_2|u - v|_H \quad t, s \in [0, T]. \tag{3.2}$$

An important special case of (3.1) is given by the following evolution quasi-variational inequality.

Find $v : [0, T] \to H$ with $v(t) \in \Gamma(v(t))$ such that

$$\langle v'(t) + f(t), w - v(t)\rangle \geq 0 \text{ for all } w \in \Gamma(v(t)), \; v(0) = v_0 \in \Gamma(v_0), \tag{3.3}$$

where $f : [0, T] \to H$ is some inhomogeneity, and $\Gamma(v) \subset H$ is a set of constraints.

(3.3) can be written in the form

$$-v'(t) \in N_{\Gamma(v(t))}(v(t)) + f(t) \text{ a.e. in } [0, T], v(0) = v_0 \in \Gamma(v_0). \qquad (3.4)$$

Thus if v is a solution of (3.4) and if we define $u(t) = v(t) + \int_0^t f(s)ds$ and

$$C(t, u) = \Gamma\left(u - \int_0^t f(s)ds\right) + \int_0^t f(s)ds, \qquad (3.5)$$

then u is a solution of (3.1) with the initial condition $u_0 = v_0 \subset C(0, u_0)$. While dealing with (3.3), we always assume that

$$d_H(\Gamma(v), \Gamma(w)) \leq L|v - w| \quad v, w \in H. \qquad (3.6)$$

It may be remarked that elliptic and evolution (in particular, parabolic) quasi-variational inequalities have been studied independently by several authors such as Baiochhi, Benssoussan, Lions, Kočvara, Kunze, Mosco, Outrata, Prigozhin, Rodrigues, Zowe (see for more references [1, 3, 9, 11, 32, 33, 34, 35]). Prigozhin [32] has modelled sandpile growth by a parabolic quasivariational inequality. Kočvara, Outrata and Zowe have studied algorithms for quasivariational inequalities. Recently, J.L. Lions has indicated that parallel algorithms for evolution quasivariational inequalities could be studied. Kunze and Rodrigues [22] consider a class of quasi-variational inequalities for a second-order elliptic operator and apply it to stationary problems arising in superconductivity, thermoplasticity, and in electrostatics with implicit ionization threshold. Siddiqi and Manchanda [35] have proved two existence theorems, one for evolution quasivariational inequalities and the other for a time-dependent quasivariational inequality modeling the quasistatic problem of elastoplasticity with combined kinematic-isotropic hardening.

In general (3.1) may not have a solution. However, if $L_2 < 1$ in (3.2), then (3.1) has a solution. Consequently, the quasivariational inequality (3.3) or (3.4) with the restriction $L < 1$ in (3.6) has a solution. More precisely, Kunze and Monteiro Marques [18] have proved the following theorem:

Theorem 3.1 *Let (3.2) hold for $0 \leq L_2 < 1$ and let $C(t, u) \subset H$ be nonempty, closed, and convex for $t \in [0, T]$ and $u \in H$. Assume that*

$$\left(\bigcup_{u \in A} C(t, u)\right) \cap \overline{B}_R(0) \qquad (3.7)$$

is a relatively compact subset of H for all bounded $A \subset H$ and all $R > 0$. If $u_0 \in C(0, u_0)$, then (3.1) has a solution on $[0, T]$.

Obviously, the compactness condition above is always satisfied if H is finite-dimensional.

The proof by time discretization now leads to the implicit discrete equation

$$u_i^n = \text{proj}(u_{i-1}^n, C(t_i^n, u_i^n)), \qquad i = 1, 2, 3, \ldots, N.$$

For this the following lemma plays the key role.

Lemma 3.1 *If $t \in [0,T]$ and $u \in C(s,u)$ for some $s \in [0,T]$, then there exists $v \in H$ such that $v = \text{proj}(u, C(t,v))$ and $|v - u| \leq L_1|t-s|/(1-L_2)$.*

The proof of this lemma is based on Schauder's fixed point theorem and an inequality due to Moreau concerning projections, for details see [21].

4. Degenerate Sweeping Processes

Sweeping processes of the following type

$$-u'(t) \in N_{C(t)}(Au(t)) \text{ a.e. in } [0,T], u(0) = u_0 \in \text{dom}(A), \; Au_0 \in C(0) \tag{4.1}$$

are known as degenerate sweeping process; they may fail to have solutions even in the case where A is linear, bounded, selfadjoint and satisfies $\langle Au, u\rangle \geq 0$. For example, let $H = \mathbb{R}^2$, $[0,T] = [0,1]$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $C(t) = [0,1] \times [t,1]$ for $t \in [0,1]$. (4.1) has no solution in this case with initial condition $u_0 = (0,0) \in \text{dom}(A)$. Degenerate sweeping processes have been discussed in references [16, 17, 21] and references therein. Kunze and Monteiro Marques [17] have proved the following theorem.

Theorem 4.1 *Let $A : H \to H$ be linear, bounded and selfadjoint such that*

$$\langle Au, u\rangle \geq \alpha\|u\|, \qquad \alpha > 0. \tag{4.2}$$

(2.5) holds for $t \to C(t)$ and $Au_0 \in C(0)$. Then (4.1) has a unique solution, which is Lipschitz continuous.

It may be observed that Theorem 2.1 is obtained by choosing $A = I$ in Theorem 4.1.

5. Variants of Sweeping Processes

The following variant of the sweeping process in (2.1) has been studied by Siddiqi, Manchanda and Brokate [36]:

Find $u : [0,T] \to H$, where H is a separable Hilbert space such that

$$-u(t) \in N_{C(t)}(u'(t)), \quad \text{a.e. in } [0,T] \qquad u(0) = u_0. \tag{5.1}$$

Theorem 5.1 *Assume that $t \in [0,T] \to C(t)$ satisfies (2.5) and $C(t) \subset H$ is closed and convex for every $t \in [t_0,T]$. Moreover, assume that C is uniformly bounded, that is, there exists $K > 0$ such that $C(t) \subset B_K(0)$ for all t. Then (5.1) has a unique solution which also satisfies (2.4).*

Proof. For any $n \in \mathbb{N}$, set $h_n = T/n$ and let $(t_i^n)_{0 \leq i \leq n}$, $t_i^n = ih_n$ denote the corresponding equidistant partition of $[0,T]$. We want to define a discrete solution (u_i^n), $0 \le i \le n$, by

$$-u_i^n \in N_{C(t_i^n)}\left(\frac{u_i^n - u_{i-1}^n}{t_i^n - t_{i-1}^n}\right), \qquad u_0^n = u_0. \tag{5.2}$$

Introduce (instead of u_i^n) a new unknown

$$z = \frac{u_i^n - u_{i-1}^n}{t_i^n - t_{i-1}^n}$$

Then (5.2) is equivalent to

$$-u_{i-1}^n - (t_i^n - t_{i-1}^n)z \in N_{C(t_i^n)}(z)$$

which is equivalent to

$$(I + N_{C(t_i^n)})z \ni -\frac{u_{i-1}^n}{t_i^n - t_{i-1}^n}. \tag{5.3}$$

Because $N_{C(t_i^n)}$ is maximal monotone,

$$\text{Range}(I + N_{C(t_i^n)}) = H$$

Therefore (5.3) has a solution $z \in H$. Since z belongs to the domain of $N_{C(t_i^n)}$, we must have $z \in C(t_i^n) \subset B_K(0)$, thus

$$\left|\frac{u_i^n - u_{i-1}^n}{t_i^n - t_{i-1}^n}\right| \le K. \quad \forall i$$

We now define the piecewise linear interpolate

$$u_n : [0, T] \to H$$

by

$$u_n(t) = u_{i-1}^n + (t - t_{i-1}^n)\frac{u_i^n - u_{i-1}^n}{t_i^n - t_{i-1}^n}, \quad t \in (t_{i-1}^n, t_i^n)$$

Since

$$u_n'(t) = \frac{u_i^n - u_{i-1}^n}{t_i^n - t_{i-1}^n}, \qquad t \in (t_{i-1}^n, t_i^n)$$

we have

$$\|u_n'\|_\infty \le K.$$

We now perform the passage to the limit. Due to Theorem 2.2, there exists a function $u : [0, T] \to H$ such that, for a suitable subsequence, $u_{n_k}(t) \to u(t)$

weakly in H. On the other hand, the sequence (u'_{n_k}) is bounded in $L^2(0,T;H)$. Therefore, for a suitable subsequence again denoted by u_{n_k} we have $u'_{n_k} \to w$ weakly in $L_2(0,T;H)$. By passing to the limit in

$$u_{n_k}(t) = u_0 + \int_0^t u'_{n_k}(s)ds\,,$$

we see that

$$u(t) = u_0 + \int_0^t w(s)ds$$

holds for all $t \in [0,T]$, thus $u' = w$ a.e. in $[0,T]$. For the remainder of the convergence argument, we write u_n instead of u_{n_k}. We have

$$u'_n(t) \in C(t_i^n) \quad \text{ if } \quad t \in (t_{i-1}^n, t_i^n),$$

so

$$\operatorname{dist}(u'_n(t), C(t)) \le d_H(C(t_i^n)), C(t)) \le K|t_i^n - t| \le Lh_n, \quad \text{ a.e. in } t.$$

Because of this estimate, we can apply Lemma 2.1 to the closed convex subset

$$\mathcal{C} = \{v : v \in L^2(0,T;H),\ v(t) \in C(t) \text{ a.e.}\}$$

of the Hilbert space $L^2(0,T;H)$ to conclude that $u' \in \mathcal{C}$, thus $u'(t) \in C(t)$ a.e. in $[0,T]$. It remains to prove that

$$\langle -u(t), u'(t) - z\rangle \ge 0 \quad \forall z \in C(t) \tag{5.4}$$

holds a.e. in $[0,T]$. Fix $t \in (t_{i-1}^n, t_i^n)$ and $z \in C(t)$. We have for any $\tilde{z} \in H$

$$\begin{aligned}\langle -u_n(t), u'_n(t) - z\rangle &= \langle -u_n(t) + u_i^n, u'_n(t) - z\rangle \\ &\quad + \langle -u_i^n, u'_n(t) - \tilde{z}\rangle + \langle -u_i^n, \tilde{z} - z\rangle.\end{aligned}$$

Choose $\tilde{z} \in C(t_i^n)$ such that $|z - \tilde{z}| \le Kh_n$, then by (5.2)

$$\langle -u_i^n, u'_n(t) - \tilde{z}\rangle \ge 0\,.$$

and

$$\langle -u_n(t), u'_n(t) - z\rangle \ge \langle -u_n(t) + u_i^n, u'_n(t) - z\rangle - Kh_n\|u_n\|_\infty.$$

Now, for every $v \in L^2(0,T;H)$ with $v(t) \in C(t)$ a.e. we have

$$\lim_{n\to\infty} \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \langle -u_n(t) + u_i^n, u'_n(t) - v(t)\rangle\, dt = 0.$$

so

$$\int_0^T \langle -u(t), u'(t) - v(t)\rangle \, dt = \lim_{n\to\infty} \int_0^T \langle -u_n(t), u_n'(t) - v(t)\rangle \, dt \geq 0$$

holds for all v with $v(t) \in C(t)$ a.e. Passing to (5.4) in the standard manner, we conclude the proof of existence.

To prove uniqueness, let $u_1, u_2 : [0, T] \to H$ be solutions of (5.1) with initial values $u_1(0) = u_0^1$, $u_2(0) = u_0^2$. Then

$$\langle -u_1(t), u_2'(t) - u_1'(t)\rangle \leq 0 \,, \quad \langle -u_2(t), u_1'(t) - u_2'(t)\rangle \leq 0 \,.$$

so

$$\frac{1}{2}\frac{d}{dt}\left|u_2(t) - u_1(t)\right|^2 \leq 0$$

and

$$\left|u_2(t) - u_1(t)\right| \leq \left|u_0^1 - u_0^2\right| .$$

This implies uniqueness. ∎

Manchanda and Siddiqi have studied the following variant of the state-dependent sweeping process

$$\left.\begin{aligned} -u(t) &\in N_{C(t,u(t))}(u'(t)) \quad \text{a.e. in } [0,T] \\ u'(0) &= u_0 \in C(0, u_0). \end{aligned}\right\} \tag{5.5}$$

Theorem 5.2 *Let $C(t, u)$ be a nonempty, closed, and convex set for $t \in [0, T]$, $u \in H$ and $(t, u) \to C(t, u)$ satisfy (3.2) for $0 \leq L_2 < 1$ and (3.7) hold. Then (5.5) has a solution.*

6. The Sweeping Process Without Convexity

In recent years, some efforts have been made to study Moreau's sweeping process in the setting of nonconvex sets $C(t)$ (see, for example, [2, 8]). Benabdellah [2] has proved the following theorem extending Theorem 2.1 for nonconvex subsets of a finite-dimensional normed space.

Theorem 6.1 *Let $C : I \to cl(\mathbb{R}^n)$ be a multi-function such that there exists a constant $L > 0$ and $d_H(C(t), C(t')) \leq L|t' - t|$ holds for all $t, t' \in I$. Let $u_0 \in C(0)$. Then there exists an absolutely continuous function $u : I \to \mathbb{R}^n$ such that*

$$u'(t) \in N_{C(t)}(u(t)) \quad \textit{a.e. in } I \tag{6.1}$$

$$u(t) \in C(t) \quad \textit{for all } t \in I \tag{6.2}$$

$$u(0) = u_0. \tag{6.3}$$

Theorem 6.1 has been extended to infinite-dimensional spaces by Colombo and Goncharov [8].

7. The Play and Stop Operator

Let us come back to the sweeping process in its original form,

$$\dot{u}(t) \in N_{C(t)}(u(t)) \,. \tag{7.1}$$

Let us consider the special case of a purely translational motion

$$C(t) = v(t) - Z \,,$$

where Z is a fixed closed, convex and nonempty subset of H and $v : [0, T] \to H$ is a given function, which we now call the input function. The evolution variational inequality corresponding to (7.1), namely

$$-\langle -\dot{u}(t), x - u(t) \rangle \leq 0 \,, \quad \forall x \in C(t) \,.$$

can be equivalently written as

$$\langle \dot{u}(t), v(t) - u(t) - \zeta \rangle \geq 0 \,, \quad \forall \zeta \in Z \,. \tag{7.2}$$

The initial condition must have the form

$$u(0) - v(0) = z_0 \,, \quad z_0 \in Z \,.$$

If we additionally introduce the function $z = v - u$, we see that the sweeping process takes on the equivalent form

$$u(t) + z(t) = v(t) \,, \quad z(0) = z_0 \,, \tag{7.3}$$

$$z(t) \in Z \,, \quad \langle \dot{u}(t), z(t) - \zeta \rangle \geq 0 \quad \forall \zeta \in Z \,. \tag{7.4}$$

The existence and uniqueness theorem for the sweeping process yields for every input function v and every initial value z_0 a unique pair of functions (u, z) which solve (7.3), (7.4). The corresponding solution operators

$$u = \mathcal{P}(v; z_0) \,, \quad z = \mathcal{S}(v; z_0) \,, \tag{7.5}$$

are called the *play operator* and the *stop operator*, respectively. They constitute basic elements of the mathematical theory of rate independent hysteretic processes; for example, the celebrated Preisach model in ferromagnetism can be written as a nonlinear superposition of a continuous one-parameter family of play operators.

As with the sweeping process, there is a direct geometric interpretation of the play and the stop operator. Let us consider the translational movement defined by

$$Z(t) = u(t) + Z \,. \tag{7.6}$$

Now the input function v governs the movement of the convex set Z which is required to follow v as $v(t) \in Z(t)$. Moreover according to (7.4) its velocity vector $\dot{u}(t)$ lies within the normal cone $N_Z(z(t))$, where $z(t) = v(t) - u(t)$

represents the position of the input relative to Z. In particular, $Z(t)$ does not move as long as $v(t) \in \text{int}Z(t)$; if $v(t) \in \partial Z(t)$ and $\dot{v}(t)$ points outward in a nontangential direction, (7.4) and (7.5) force $Z(t)$ to move in the direction of an outward normal.

The properties of the play and the stop operator have been studied extensively, see Krejčí [14] for an extensive survey which also includes the results presented in Brokate [4] and Desch [10].

8. Relationship between Variational Inequalities and a Few Open Problems

W. Han, B.D. Reddy, and G.C. Schroeder [12] have studied the following abstract variational problem:

Problem 8.1 Find $w : [0, T] \to H$, $w(0) = 0$, such that for almost all $t \in (0, T)$, $\dot{w}(t) \in K$ and

$$\left.\begin{array}{l} a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) - \langle \ell(t), \\ \quad z - \dot{w}(t)\rangle \geq 0 \ \ \forall\, z \in K, \end{array}\right\} \tag{8.1}$$

where H denotes a Hilbert space, K a nonempty, closed, convex and convex cone in H; $a(\cdot,\cdot)$ denotes a real bilinear, symmetric, bounded and H-elliptic form on $H \times H$; $\ell \in H^{1,2}(0, T, H^*)$ and $j(\cdot)$ denotes non-negative, convex, positively homogeneous and Lipschitz continuous functional on K into R.

Siddiqi and Manchanda [35] have studied the following quasi-variational problem:

Problem 8.2 Find $u \in K(u) \cap C$, $u(0) = 0$ such that for almost all $t \in [0, T]$,

$$a(u(t), v - \dot{u}(t)) \geq \langle F(t), v - \dot{u}(t)\rangle, \quad \forall\, v \in K(u). \tag{8.2}$$

Natural questions are:

(1) What is the relationship between Problem 8.1 and (5.1)?

(2) Is it possible to find a variant of the result of Kunze and Monteiro Marques ((3.1) and Theorem 3.1) which will include Problem 8.2 as a special case?

The following classes of variational inequalities are discussed in Duvaut and Lions [9] and Glowinski, Lions and Tremolieres [11, pp. 454–474].

Problem 8.3 It is worthwhile to investigate a class of sweeping process which include these evolution variational inequalities.

Find $u \in K$:

$$\left.\begin{array}{l} a(u'(t), v - u'(t)) + j(v) - j(u'(t)) \geq \langle \ell(t), v - u'(t)\rangle \\ a(u''(t), v - u'(t)) + j(v) - j(u''(t)) \geq \langle \ell(t), v - u'(t)\rangle. \end{array}\right\} \tag{8.3}$$

Problem 8.4 Raymond [34] has generalized the Lax-Milgram lemma in the following form:

Theorem 8.1 *Let H be a real Hilbert space and A a linear operator on H. If*

$$\inf_{\|x\|=1} (\langle Ax, x\rangle + \|Ax\|) > 0, \tag{8.4}$$

the operator A is continuous and invertible.

In this theorem, the coercivity of A has been relaxed in the form of (8.4).

An interesting problem could be to explore the possibility of replacing condition (4.2) in Theorem 4.1 by a weaker condition (8.4).

Problem 8.5 Obtain an analogous result to Theorem 4.1 for the state-dependent sweeping process given by (3.1).

Problem 8.6 Could we prove a result analogous to Theorem 6.1 for the state-dependent sweeping process, that is to say, could we prove existence and uniqueness of solution of state-dependent sweeping process (3.1) under appropriate conditions?

Problem 8.7 On the lines of Benabdellah [2] and Colombo and Goncharov [8] one may try to prove existence and regularity of play and stop operators similar to Theorem 7.1 (relaxing the convexity of the underlying set) and Theorem 1.1 in [10].

Problem 8.8 In recent years, parallel algorithms for evolution variational inequalities have been studied by Lions (see, for example, reference in Siddiqi and Manchanda [35]). Proceeding along the lines of Lions one may introduce N Hilbert spaces H_i and a family of linear, bounded operators $r_i \in \mathcal{L}(H; H_i)$, $i = 1, 2, \ldots, N$. For a given family of Hilbert space H_{ij} such that $H_{ij} = H_{ji}\ \ \forall\ i, j = 1, 2, \ldots, N$ and a family of operators r_{ij} such that $r_{ij} \in \mathcal{L}(H_j, H_{ij})$, one may decompose (1.1) into N inclusions $-u_i'(t) \in N^i_{C_i(t)}(u_i(t))$ plus appropriate terms containing r_{ij} and u_i a.e. in $[0, T]$, $u_i(0) = u_0^i \in C_i(0)$, $i = 1, 2, \ldots, N$.

Does this system of inclusion has a unique solution/solutions; whether this solution/solutions converge(s) to the solution of (1.1).

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