RELATIONSHIP BETWEEN THE SOLUTION OF BBGKY-HIERARCHY OF KINETIC EQUATIONS AND THE PARTICLE SOLUTION OF VLASOV EQUATION

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ABSTRACT. It is shown that the solution of BBGKY hierarchy of kinetic equations can be obtained through the particle method solution of Vlasov equation.

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Suppose we are given a system of monoatomic molecules. Suppose that the molecules interact through a two-body potential ϕ. In the framework of classical statistical physics, we look for the solution of the hierarchy of BBGKY kinetic equations [2]:

$$(1) \qquad \frac{\partial}{\partial t} f_n(t) = [H_n, f_n(t)] + \frac{1}{v} \int \sum_{1 \le i \le n} [\phi(q_i - q), f_{n+1}(t)] dx,$$

where f_n is the probability density of the gas ensemble of time $t \in \mathbb{R}_+$ at position $q_1 \in \Lambda, q_2 \in \Lambda, \ldots, q_n \in \Lambda$ with the velocities $v_1 \in R_+^3, \ldots, v_n \in \mathbb{R}_+^3$ of particles. Therefore, $f : \mathbb{R}_+ \times F \to \mathbb{R}_+$ with the phase space $F = (\Lambda + \mathbb{R}_+^3)^n$.

Here,

$$H_n = \sum_{1 \le i \le n} T_i + \sum_{1 \le i < j \le n} \phi(q_i - q_j), \; T_i = \frac{p_i^2}{2m},$$

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$m = 1$ is the mass of a molecule, p the momentum of a molecule, $n \in N$, N is the number of molecules, V is the volume of the system; $N \to \infty, V \to \infty, v = \frac{V}{N} =$ const is volume per molecule [,] denotes the Poisson brackets.

Introducing the notation

$$
\begin{aligned}
(2) \qquad (\mathcal{H}f)_n &= [H_n, f_n]; \quad (\mathcal{D}_x f)_n(x_1, \ldots, x_n) = f_{n+1}(x_1, \ldots, x_n, x); \\
(\mathcal{A}_x f)_n &= \frac{1}{v} \sum_{1 \le i \le n} [\phi(q_i - q), f_n]; \\
f(t) &= \{f_1(t_1 x_1), \ldots, f_n(t, x_1, \ldots, x_n), \ldots, \}, \quad n = 1, 2, \ldots,
\end{aligned}
$$

we can write Equation (1) in the form

$$
(3) \qquad \frac{\partial}{\partial t} f(t) = \mathcal{H} f(t) + \int \mathcal{A}_x \mathcal{D}_x f(t) dx.
$$

Derivation of Hierarchy of Kinetic Equations for Correlation Functions

Theorem 1. *The hierarchy of kinetic equations for the correlation functions has the form*

$$
(4) \quad \frac{\partial}{\partial t}\varphi(t) = \mathcal{H}\varphi(t) + \frac{1}{2}\mathcal{W}(\varphi(t), \varphi(t)) + \int \mathcal{A}_x \mathcal{D}_x \varphi(t) dx + \int \mathcal{A}_x \varphi(t) * \mathcal{D}_x \varphi(t) dx,
$$

where

$$
\begin{aligned}
(5) \qquad f(t) &= \Gamma\varphi(t) = I + \varphi(t) + \frac{\varphi(t) * \varphi(t)}{2!} + \cdots + \frac{(*\varphi(t))^n}{n!} + \cdots, \\
\varphi(t) &= \{\varphi_1(t, x_1), \ldots, \varphi(t, x_1, \ldots, x_n), \ldots\}; \\
(6) \quad (\varphi * \varphi)(x) &= \sum_{Y C X} \varphi(Y)\varphi(X \setminus Y); \quad I * \varphi = \varphi; \\
(7) \qquad (*\varphi)^n &= \underbrace{\varphi * \varphi * \cdots * \varphi}\ n \text{ times} \\
X &= (x_1, \ldots, x_n) = (x_{(n)}); \quad Y = (x_{n'}), \quad n' \in n \cdot n' = 1, 2, \ldots; \\
(8) \qquad (\mathcal{U}\varphi_n) &= \left[\sum_{1 \le i < j \le n} \phi(q_i - q_j), \varphi_n \right], \\
(9) \quad \mathcal{W}(\varphi, \varphi) &= \sum_{Y C X} \mathcal{U}(Y; X \setminus Y)\varphi(Y)\varphi(X \setminus Y).
\end{aligned}
$$

Proof. To obtain (3), we substitute (4) in (2):

We have

$$\mathcal{D}_x\Gamma\varphi(t) = \mathcal{D}_x\varphi(t) * \Gamma\varphi(t), \tag{11}$$

$$\mathcal{A}_x\Gamma\varphi(t) = \mathcal{A}_x\varphi(t) * \Gamma\varphi(t), \tag{12}$$

$$\mathcal{A}_x\mathcal{D}_x\Gamma\varphi(t) = \mathcal{A}_x\mathcal{D}_x\varphi(t) * \Gamma\varphi(t) + \mathcal{A}_x\varphi(t) * \mathcal{D}_x\varphi(t) * \Gamma\varphi(t), \tag{13}$$

$$T\Gamma\varphi(t) = T\varphi(t) * \Gamma\varphi(t), \tag{14}$$

$$\mathcal{U}\Gamma\varphi(t) = \mathcal{U}\varphi(t) * \Gamma\varphi(t) + \frac{1}{2}\mathcal{W}(\varphi(t), \varphi(t) * \Gamma\varphi(t)), \tag{15}$$

$$\frac{\partial}{\partial t}\Gamma\varphi(t) = \frac{\partial}{\partial t}\varphi(t) * \Gamma\varphi(t). \tag{16}$$

Substituting (6) – (11) in (5), and multiplying both sides by $\Gamma(-\varphi(t))$, we obtain (3). □

See [8–10] for relevant discussion.

To investigate our system on the basis of arguments similar to those in [2], we can choose as expansion parameter v, setting

$$\phi(q_i - q_j) = v\theta(q_i - q_j) \tag{17}$$

and making substitution similar to [1-4, 8], we get

$$\varphi_n(t) = v^{n-1}\psi_n(t). \tag{18}$$

On the basis of (12), (13), Eq. (3) for n takes the form

$$\begin{aligned} \frac{\partial}{\partial t}\psi_n(t,X) &= \left[\sum_{1\le i\le n} T_i, \psi_n(t,X)\right] + v(\mathcal{U}\psi(t))_n(X) \\ &+\frac{v}{2}(\mathcal{W}\psi(t), \psi(t))_n(X) + v^2\int(\mathcal{A}_x\mathcal{D}_x\psi(t))_n(X)dx \\ &+v\int(\mathcal{A}_x\psi(t) * \mathcal{D}_x\psi(t))_n(X)dx. \end{aligned} \tag{19}$$

To solve Eq. (14), we apply perturbation theory, we shall seek a solution in the form of the series

$$\psi_n(t,X) = \sum_\mu v^\mu \psi_n^\mu(t,X), \quad n = 1, 2, 3, \ldots, , \mu = 0, 1, 2, \ldots . \tag{20}$$

Substituting the series of (15) in Eq. (14) and equating the coefficients of equal powers of v, we obtain

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_1\right)\psi_1^0(t) = 0, \left(\frac{\partial}{\partial t} + \mathcal{L}_1 + \mathcal{L}_2\right)\psi_2^0(t) = S_2^0, \ldots\ldots \left(\frac{\partial}{\partial t} + \sum_{i=1}\mathcal{L}_i\right)\psi_n^\mu(t) = S_n^\mu, ,$$

where we have introduced the notation

$$\begin{aligned} \mathcal{L}_1(\psi_1^0(t) &= v_1\frac{\partial}{\partial q_1}\psi_1^0(t,x_1) - \int\frac{\partial\theta(q_1 - q)}{\partial q_1}\frac{\partial\psi_1^0(t,x)}{\partial p_1}\psi_1^0(t,x)dx, \\ \mathcal{L}_i\psi_n^\mu(t) &= v_i\frac{\partial}{\partial q_1}\psi_n^\mu(t,X) - v\int\left(\mathcal{A}_x\psi_{(t)}^0\right)(x_i)(\mathcal{D}_x\psi^\mu)_{n-1}(t, X\setminus x_i)dx, \end{aligned}$$

and

$$S_n^\mu = \left(\mathcal{U}\psi^{\mu-1}(t)\right)_n(X) + \frac{1}{2}\sum_{\delta_1+\delta_2=\mu}\left(\mathcal{W}(\psi^{\delta_1}(t),\psi^{\delta_2}(t)\right)_n(X)$$

$$(21) \qquad +v\int\left(\mathcal{A}_x\mathcal{D}_x\psi^{\mu-1}(t)\right)_n(X)dx + v\int\sum_{\delta_1+\delta_2=\mu}(\mathcal{A}_x\psi^{\delta_1}(t)\mathcal{D}_x\psi^{\delta_2}(t))_n(X)dx.$$

Thus, the solution of Eq. (14) reduces to the solution of the homogeneous (16) and inhomogeneous (17), (18) Vlasov's [12] equations for $\psi_1^0(t)$ and $\psi_n^\mu(t)$, accordingly.

Theorem 2. *The series (15), $\psi_n(t,X) = \sum_\mu v^\mu \psi_n^\mu(t,X)$, where ψ_1^0 is defined in accordance with solution of Vlasov's equation and the remaining ψ_n^μ on the basis of the formula*

$$(22) \quad \psi_n^\mu(t,X) = \int dx_1' \cdots \int dx_n' \int_{-\infty}^t dt' S_n^\mu(t,x_1',\ldots,x_n') \bigcap_{1\le i\le n} G(t-t,x_i,x_i'),$$

is a solution of Eq. (14), if G satisfies equation:
$(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial q_i})G(t-t';x_i,x_i') - \frac{\partial\psi(t,x_i)}{\partial v_i}\int\frac{\partial\theta(q_i-q)}{\partial q_i}G(t-t';x,x_i')dx-$
$\int\frac{\partial\theta(q_i-q)}{\partial q_i}\frac{\partial G(t-t';x_i',x_i')}{\partial v_i}\psi(t,x)dx = 0$
with the initial condition

$$G(0;x_i,x_i') = \delta(x_i - x_i').$$

Proof. We consider Eqs. (16) and (17) where (16) is the Vlasov equation. This system of coupled equations for the single-molecule and two-molecule perturbations can serve to determine the successive approximations $\psi_n^\mu(t)$. $\psi_1^0(t,X)$ is the solution of Vlasov's equation.

Substituting [3, 8]

$$(23) \qquad \psi_2^0(t,x_1,x_2) = \int dx_1' \int dx_2' \int_{-\infty}^t dt' S_2^0(t';x_1',x_2')$$
$$G(t-t';x_1,x_1')G(t-t';x_2,x_2')$$

in (17), we see that (21) is a solution of (17) if

$$S_2^0(t,x_1,x_2) = [\theta(q_1-q_2),\psi_1^0(t;x_1)\psi_1^0(t,x_2)] + \int_{1\le i\le 2}[\theta(q_i-q),\psi_1^0(t;x_1)\psi_1^0(t;x)]dx$$

and if G satisfies equation

$$(24) \qquad \left(\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial q_1}\right)G(t-t';x_1,x_1') - \frac{\partial\psi(t,x_1)}{\partial v_1}\int\frac{\partial\theta(q_1-q)}{\partial q_1}$$
$$G(t-t';x,x_1')dx - \int\frac{\partial\theta(q_1-q)}{\partial q_1}\partial\frac{G(t-t';x_1,x_1')}{\partial v_1}\psi(t,x)dx = 0$$

with the initial condition

$$(25) \qquad G(0;x_1,x_1') = \delta(x_1 - x_1').$$

The recursive system of Eq. (18) can, with allowance for the established structure of the solutions, serve to determine the successive approximations $\psi_n^\mu(t)$ and, therefore, formula (15). Indeed substituting again (20) directly in (18), we can see that (20) is a solution of (18) if S_n^μ is defined in accordance with (19) and if G satisfies Eq. (22) with the initial condition (23). $\square$

Existence and uniqueness of the solution of the following Vlasov equation is studied in [5-7] by the particle method:

$$\partial_t \psi_1^0(t_1 x_1) = -v_1 \nabla_x \psi_1^0(t_1 x_1) + \frac{e_s}{m_s} \nabla_x \mathcal{A}^{k-1} \nabla_{v_1} \psi_1^0(t_1, x_1),$$

$$\psi_1^0(T_k) = f_1^{k-1}(T_k) \tag{26}$$

$$-\Delta_x U^k = \frac{1}{\epsilon_0} \sum_s \int_{\Gamma_2} e_1 f_1^k dS \quad T = T_k, \tag{27}$$

where $T_k = \frac{k}{n} T, k = 1, \ldots, n, n \in \mathbb{N}$ of size $\frac{1}{n} T, U^0$, solution of (25) with $f^0(0, P) = f^0(P); \theta(|q_i - q_j|)$ is Coulomb potential; U-potential by $E = -\Delta U$ satisfies Poisson's equation. In [5, 11], it is shown that $\psi_1^0(t, x_1, v_1) = (\psi^0 \Phi_{0,t})(x_1, v_1)$ is solution of the Vlasov equation. Here, we assume that E is Lipschitz continuous, $\Phi_{t,\tau} : F \to F$ is a measure-preserving group homomorphism [6] and ψ^0 is continuous initial conditions.

A numerical scheme for the Vlasov equation is as follows [11]: For every time step $t_k = k\Delta t, k = 0, 1, \ldots$

$$\begin{aligned} v_i^N(t_{k-1}) &= v_i^N(t_k) + \Delta t E\left(q_i^N(t_k)\right) \\ q_i^N(t_{k-1}) &= q_i^N(t_k) + \Delta t v_i^N(t_{k+1}) \\ \alpha_i^N(t_{k+1}) &= \alpha_i^N(t_k). \end{aligned}$$

Solution (20) of two equations (16), (17) of hierarchy are in good agreement with results of [3] for plasma physics and this method is opening possibilities to calculate the solutions of the complex kinetic equations of BBGKY hierarchy.

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