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# Proximal point algorithm for generalized multivalued nonlinear quasi-variational-like inclusions in Banach spaces ☆

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**Abstract**

In this paper, we define a new notion of $J^\eta$-proximal mapping for a nonconvex, lower semicontinuous, $\eta$-subdifferentiable proper functional in Banach spaces. The existence and Lipschitz continuity of $J^\eta$-proximal mapping of a lower semicontinuous, $\eta$-subdifferentiable proper functional are proved. By applying this notion, we introduce and study generalized multivalued nonlinear quasi-variational-like inclusions in reflexive Banach spaces and propose a proximal point algorithm for finding the approximate solutions of this class of variational inclusions. The convergence criteria of the iterative sequences generated by our algorithm is discussed. Several special cases are also given.


*Keywords:* Variational-like inclusions; $\eta$-subdifferentiable; $J^\eta$-proximal mapping; Reflexive Banach spaces; Algorithm

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## 1. Introduction

In 1994, Hassouni and Moudafi [1] introduced a perturbed method for solving a new class of variational inequalities, known as variational inclusions. A useful and important generalization of the variational inclusions is called the quasi-variational inclusion. Quasi-variational inclusions are being used as mathematical programming models to study a large number of equilibrium problems arising in finance, economics, transportation, optimization, operation research, and engineering sciences.

Adly [2], Huang [3], Ding [4,5], Ahmad and Ansari [6] have obtained some important extensions of the results in [1] in various different directions. Recently, Cohen [7] and Ding [8,9] have extended the auxiliary principle technique to suggest and analyze an innovative and novel iterative algorithm for computing the solution of mixed variational inequalities in reflexive Banach spaces. Chang et al. [10] and Chang [11] have studied some classes of set-valued variational inclusions with $m$-accretive operator and $\phi$-strongly accretive operators in uniformly Banach spaces.

Iterative algorithms have played a central role in the approximation solvability, especially of nonlinear variational inequalities as well as nonlinear equations in several fields such as applied mathematics, mathematical programming, mathematical finance, control theory and optimization, engineering sciences and others. In general we cannot use resolvent operator or proximal mapping technique for studying a perturbed algorithm for finding the approximate solutions of variational-like inequalities.

In this paper, we define a new notion of $J^\eta$-proximal mapping for a lower semicontinuous, $\eta$-subdifferentiable, proper (may not be convex) functional on Banach spaces. The existence and Lipschitz continuity of the $J^\eta$-proximal mapping of the functional are proved under suitable conditions in reflexive Banach spaces. By using this notion, we introduce and study generalized multivalued nonlinear quasi-variational-like inclusions in reflexive Banach spaces and propose a proximal point algorithm for finding the approximate solutions of our inclusions. The convergence of the iterative sequences generated by our algorithm is discussed. Several special cases are also discussed.

## 2. Preliminaries

Let $E$ be a Banach space with the dual space $E^\star$, $\langle u,x\rangle$ be the dual pairing between $u \in E^\star$ and $x \in E$ and $\mathrm{CB}(E^\star)$ be the family of all nonempty closed bounded subset of $E^\star$. $H(.,.)$ is the Hausdörff metric on $\mathrm{CB}(E^\star)$ defined by

$$H(A,B) = \max\left\{\sup_{u\in A} d(u,B), \sup_{v\in B} d(A,v)\right\} \quad \text{for all } A,B \in \mathrm{CB}(E^\star),$$

where $d(u,B) = \inf_{v\in B} d(u,v)$ and $d(A,v) = \inf_{u\in A} d(u,v)$.

We extend the concept of $\eta$-subdifferentiability of a functional defined by Lee et al. [12] in Hilbert spaces to a Banach space setting.

Let $\eta : E \times E \to E$ and $\phi : E \to R \cup \{+\infty\}$. A vector $w^\star \in E^\star$ is called an $\eta$-subgradient of $\phi$ at $x \in \mathrm{dom}\phi$ if

$$\langle w^\star, \eta(y,x)\rangle \leqslant \phi(y) - \phi(x) \quad \text{for all } y \in E.$$

Each $\phi$ can be associated with the following $\eta$-subdifferential map $\partial_n\phi$ defined by

$$\partial_n\phi(x) = \begin{cases} \{w^\star \in E^\star : \langle w^\star, \eta(y,x)\rangle \leqslant \phi(y) - \phi(x)\ \forall y \in E\}, & x \in \mathrm{dom}\phi, \\ \emptyset, & x \notin \mathrm{dom}\phi. \end{cases}$$

Let us recall the following definitions.

**Definition 2.1.** Let $A : E \to \mathrm{CB}(E^\star)$ be a set-valued mapping, $J : E \to E^\star$, $g : E \to E$, $\eta : E \times E \to E$ be the single-valued mappings.

(1) $A$ is said to be Lipschitz continuous with constant $\lambda_A \geqslant 0$ if

$$H(Ax, Ay) \leqslant \lambda_A\|x - y\| \quad \text{for all } x, y \in E.$$

(2) $J$ is said to be $\eta$-strongly monotone with constant $\alpha > 0$ if

$$\langle Jx - Jy, \eta(x,y)\rangle \geqslant \alpha\|x - y\|^2 \quad \text{for all } x, y \in E.$$

(3) $\eta$ is said to be Lipschitz continuous with constant $\tau > 0$ if

$$\|\eta(x,y)\| \leqslant \tau\|x - y\| \quad \text{for all } x, y \in E.$$

(4) $g$ is said to be $k$-strongly accretive ($k \in (0,1)$) if for any $x, y \in E$, there exists $j(x-y) \in \mathscr{F}(x-y)$ such that

$$\langle j(x-y), gx - gy\rangle \geqslant k\|x - y\|^2,$$

where $\mathscr{F} : E \to 2^{E^\star}$ is normalized duality mapping defined by

$$\mathscr{F}(x) = \{f \in E^\star : \langle f, x\rangle = \|x\|.\|f\|, \|f\| = \|x\|\} \quad \text{for all } x \in E.$$

Some examples and properties of the mapping $\mathscr{F}$ can be found in [13].

**Definition 2.2.** Let $E$ be a Banach space with the dual space $E^*$, $\phi : E \to R \cup \{+\infty\}$ be a proper, $\eta$-subdifferentiable (may not be convex) functional, $\eta : E \times E \to E$ and $J : E \to E^*$ be the mappings. If for any given point $x^\star \in E^\star$ and $\rho > 0$, there is a unique point $x \in E$ satisfying

$$\langle Jx - x^\star, \eta(y,x)\rangle + \rho\phi(y) - \rho\phi(x) \geqslant 0 \quad \text{for all } y \in E.$$

The mapping $x^\star \to x$, denoted by $J_\rho^{\partial_\eta\phi}(x^\star)$ is said to be $J^\eta$-proximal mapping of $\phi$. We have $x^\star - Jx \in \rho\partial_\eta\phi(x)$, it follows that

$$J_\rho^{\partial_\eta\phi}(x^\star) = (J + \rho\partial_\eta\phi)^{-1}(x^\star).$$

**Remark 2.1.** If $\phi : E \to R \cup \{+\infty\}$ is proper subdifferentiable and $\eta(y,x) = y - x$ for all $x, y \in E$, then Definition 2.2 of $J^\eta$-proximal mapping coincides with the definition of $J$-proximal mapping (see [14]).

Let $T, A : E \to \mathrm{CB}(E^\star)$ be set-valued mappings. Let $N : E^\star \times E^\star \to E^\star$, $f : E \to E^\star$, $\eta : E \times E \to E$ and $g : E \to E$ be single-valued mappings. Let $\phi : E \times E \to R \cup \{+\infty\}$ be such that for each fixed $x \in E$, $\phi(.,x)$ is a lower semicontinuous, $\eta$-subdifferentiable functional on $E$ (may not be convex) satisfying $g(E) \cap \mathrm{dom}\partial_\eta\phi(.,x) \neq \phi$, where $\partial_\eta\phi(.,x)$ is the $\eta$-subdifferential of $\phi(.,x)$. We consider the following generalized multivalued nonlinear quasi-variational-like inclusion problem in Banach spaces (for short, GMNQVLIP):

Find $x \in E$, $u \in T(x)$, $v \in A(x)$ such that $g(x) \in \mathrm{dom}(\partial_\eta\phi(.,x))$ and

$$\langle f(x) - N(u,v), \eta(y, g(x))\rangle \geqslant \phi(g(x),x) - \phi(y,x) \quad \text{for all } y \in E. \tag{2.1}$$

Special cases:

(1) If $E = H$, is a Hilbert space and $f(x) \equiv 0$, then problem (2.1) reduces to the following generalized quasi-variational-like inclusion problem:

$$\begin{cases} \text{Find } x \in H, u \in T(x), v \in A(x) \text{ such that } g(x) \in \mathrm{dom}(\partial_\eta\phi(.,x)) \text{ and} \\ \langle N(u,v), \eta(y, g(x))\rangle \geqslant \phi(g(x),x) - \phi(y,x) \quad \text{for all } y \in H. \end{cases} \tag{2.2}$$

Problem (2.2) is a variant form of the problem considered by Ding [15].

(2) If $E = H$, is a Hilbert space and $f(x) \equiv g(x)$, then problem (2.1) reduces to the following generalized multivalued nonlinear quasi-variational-like inclusion problem:

$$\begin{cases} \text{Find } x \in H, u \in T(x), v \in A(x) \text{ such that } g(x) \in \mathrm{dom}(\partial_\eta\phi(.,x)) \text{ and} \\ \langle g(x) - N(u,v), \eta(y, g(x))\rangle \geqslant \phi(g(x),x) - \phi(y,x) \quad \text{for all } y \in H. \end{cases} \tag{2.3}$$

Problem (2.3) was introduced and studied by Salahuddin and Ahmad [16].

(3) If $E = H$, is a Hilbert space, $f(x) \equiv 0$, $N(u,v) = u - v$ for all $u, v \in H$ and $T, A : H \to H$ are both single-valued mappings, then problem (2.1) reduces to the following general quasi-variational-like problem:

$$\begin{cases} \text{Find } x \in H \text{ such that } g(x) \in \mathrm{dom}(\partial_\eta\phi(.,x)) \text{ and} \\ \langle T(x) - A(x), \eta(y, g(x))\rangle \geqslant \phi(g(x),x) - \phi(y,x) \quad \text{for all } y \in H. \end{cases} \tag{2.4}$$

Problem (2.4) was introduced by Ding and Lou [17].

(4) If $E = H$, is a Hilbert space, $\eta(y,x) = y - x$ for all $y, x \in H$, and $f(x) \equiv 0$, $N(u,v) = f(u) - P(v)$ for all $u, v \in H$, where $f, P : H \to H$ are single-valued mappings and $\phi(x,y) = \phi(x)$ for all $x, y \in H$, and for each $x \in H$, $\phi(.,x)$ is a proper convex lower semicontinuous functional, then problem (2.1) reduces to the following problem:

$$\begin{cases} \text{Find } x \in H, u \in T(x), v \in A(x) \text{ such that} \\ \langle f(u) - P(v), y - g(x) \rangle \geqslant \phi(g(x)) - \phi(y) \quad \text{for all } y \in H. \end{cases} \tag{2.5}$$

Problem (2.5) is called set-valued nonlinear generalized variational inclusion problem which was introduced by Huang [3].

It is clear from these special cases that our problem (2.1) is a more general unifying one, which is one of the main motivations for this paper.

**Definition 2.3.** A functional $f : E \times E \to R \cup \{+\infty\}$ is said to be 0-diagonally quasi-concave (in short 0-DQCV) in $y$, if for any finite subset $\{x_1, \ldots, x_n\} \subset E$ and for any $y = \sum_{i=1}^n \lambda_i x_i$ with $\lambda_i \geqslant 0$ and $\sum_{i=1}^n \lambda_i = 1$,

$$\min_{1 \leqslant i \leqslant n} f(x_i, y) \leqslant 0.$$

**Lemma 2.1** [18]. *Let $D$ be a nonempty convex subset of a topological vector space and $f : D \times D \to R \cup \{\pm\infty\}$ be such that*

(i) *for each $x \in D$, $y \to f(x,y)$ is lower semicontinuous on each compact subset of $D$,*
(ii) *for each finite set $\{x_1, \ldots, x_n\} \in D$ and for each $y = \sum_{i=1}^n \lambda_i x_i$ with $\lambda_i \geqslant 0$ and $\sum_{i=1}^m \lambda_i = 1$, $\min_{1 \leqslant i \leqslant m} f(x_i, y) \leqslant 0$,*
(iii) *there exists a nonempty compact convex subset $D_0$ of $D$ and a nonempty compact subset $K$ of $D$ such that for each $y \in D \setminus K$, there is an $x \in c_0(D_0 \cup \{y\})$ satisfying $f(x,y) > 0$.*

*Then there exists $\hat{y} \in D$ such that $f(x, \hat{y}) \leqslant 0$ for all $x \in D$.*

Now we give some sufficient conditions which guarantee the existence and Lipschitz continuity of the $J^\eta$-proximal mapping of a proper functional on reflexive Banach space.

**Theorem 2.1.** *Let $E$ be a reflexive Banach space with the dual space $E^\star$ and $\phi : E \to R \cup \{+\infty\}$ be a lower semicontinuous, $\eta$-subdifferentiable, proper functional which may not be convex. Let $J : E \to E^\star$ be $\eta$-strongly monotone with constant $\alpha > 0$. Let $\eta : E \times E \to E$ be Lipschitz continuous with constant $\tau > 0$*

*such that $\eta(x,y) = -\eta(y,x)$ for all $x, y \in E$ and for any $x \in E$, the function $h(y,x) = \langle x^\star - Jx, \eta(y,x)\rangle$ is 0-DQCV in y. Then for any $\rho > 0$, and any $x^\star \in E^\star$, there exists a unique $x \in E$ such that*

$$\langle Jx - x^\star, \eta(y,x)\rangle + \rho\phi(y) - \rho\phi(x) \geqslant 0 \quad \text{for all } y \in E. \tag{2.6}$$

*That is $x = J_\rho^{\partial_\eta \phi}(x^\star)$ and so the $J^\eta$-proximal mapping of $\phi$ is well defined.*

**Proof.** For any $J : E \to E^\star$, $\eta : E \times E \to E$, $\rho > 0$ and $x^\star \in E^\star$, define a functional $f : E \times E \to R \cup \{+\infty\}$ by

$$f(y,x) = \langle x^\star - Jx, \eta(y,x)\rangle + \rho\phi(x) - \rho\phi(y) \quad \text{for all } x, y \in E.$$

Since $J, \eta(.,.)$ are continuous mappings and $\phi$ is lower semicontinuous, we have that for any $y \in E$, $x \to f(y,x)$ is lower semicontinuous on $E$. We claim that $f(y,x)$ satisfies the condition (ii) of Lemma 2.1. If it is false, then there exists a finite subset $\{y_1, y_0, \ldots, y_m\} \in E$ and

$$x_0 = \sum_{i=1}^{m} \lambda_i y_i \quad \text{with } \lambda_i \geqslant 0, \ \sum_{i=1}^{m} \lambda_i = 1 \text{ such that}$$

$$\langle x^\star - Jx_0, \eta(y_i, x_0)\rangle + \rho\phi(x_0) - \rho\phi(y_i) > 0 \quad \text{for all } i = 1, 2, \ldots, m.$$

Since $\phi$ is $\eta$-subdifferentiable at $x_0$, there exists a point $f_{x_0}^\star \in E^\star$ such that

$$\rho\phi(y_i) - \rho\phi(x_0) \geqslant \rho\langle f_{x_0}^\star, \eta(y_i, x_0)\rangle \quad \text{for all } i = 1, 2, \ldots, m.$$

It follows that

$$\langle x^\star - Jx_0 - \rho f_{x_0}^\star, \eta(y_i, x_0)\rangle > 0 \quad \text{for all } i = 1, 2, \ldots, m. \tag{2.7}$$

On the other hand, by assumption $h(y,x) = \langle x^\star - Jx_0 - \rho f_{x_0}^\star, \eta(y,x)\rangle$ is 0-DQCV in $y$, we have

$$\sum_{1 \leqslant i \leqslant m} \langle x - Jx_0 - \rho f_{x_0}^\star, \eta(y_i, x)\rangle \leqslant 0$$

which contradicts the inequality (2.7). Hence $f(y,x)$ satisfies the condition (ii) of Lemma 2.1.

Now we take a fixed $\bar{y} \in \mathrm{dom}\phi$. Since $\phi$ is $\eta$-subdifferentiable at $\bar{y}$, there exist a point $f_{\bar{y}}^\star \in E^\star$ such that

$$\phi(x) - \phi(\bar{y}) \geqslant \langle f_{\bar{y}}^\star, \eta(x, \bar{y})\rangle \quad \text{for all } x \in E.$$

Hence we have

$$\begin{aligned} f(\bar{y},x) &= \langle x^\star - Jx, \eta(\bar{y},x)\rangle + \rho\phi(x) - \rho\phi(\bar{y}) \\ &\geqslant \langle J\bar{y} - Jx, \eta(\bar{y},x)\rangle + \langle x^\star - J\bar{y}, \eta(\bar{y},x)\rangle + \rho\langle f_{\bar{y}}^\star, \eta(x,\bar{y})\rangle \\ &\geqslant \alpha\|\bar{y}-x\|^2 - \tau(\|x^\star\| + \|J\bar{y}\| + \rho\|f_{\bar{y}}^\star\|)\|\bar{y}-x\| \\ &= \|\bar{y}-x\|[\alpha\|\bar{y}-x\| - \tau(\|x^\star\| + \|J\bar{y}\| + \rho\|f_{\bar{y}}^\star\|)]. \end{aligned}$$

Let $r = (\frac{1}{\alpha})\tau[\|x^\star\| + \|J\bar{y}\| + \rho\|f_{\bar{y}}^\star\|]$ and $K = \{x \in E : \|\bar{y} - x\| \leqslant r\}$. Then $D_0 = \{\bar{y}\}$ and $K$ are both weakly compact convex subset of $E$ and for each $x \in E \setminus K$, there exists a $\bar{y} \in C_0(D_0 \cup \{\bar{y}\})$ such that $f(\bar{y},x) > 0$. Hence all conditions of Lemma 2.1 are satisfied. By Lemma 2.1, there exists an $\bar{x} \in E$ such that $f(y,\bar{x}) \leqslant 0$ for all $y \in E$, that is

$$\langle J\bar{x} - x^\star, \eta(y,\bar{x})\rangle + \rho\phi(y) - \rho\phi(\bar{x}) \geqslant 0 \quad \text{for all } y \in E.$$

Now we show that $\bar{x}$ is a unique solution of problem (2.6). Suppose that $x_1, x_2 \in E$ are arbitrary two solutions of problem (2.6). Then we have

$$\langle Jx_1 - x^\star, \eta(y,x_1)\rangle + \rho\phi(y) - \rho\phi(x_1) \geqslant 0 \quad \text{for all } y \in E, \tag{2.8}$$

$$\langle Jx_2 - x^\star, \eta(y,x_2)\rangle + \rho\phi(y) - \rho\phi(x_2) \geqslant 0 \quad \text{for all } y \in E. \tag{2.9}$$

Taking $y = x_2$ in (2.8) and $y = x_1$ in (2.9) and adding these inequalities, we have

$$\langle Jx_1 - x^\star, \eta(x_2,x_1)\rangle + \langle Jx_2 - x^\star, \eta(x_1,x_2)\rangle \geqslant 0. \tag{2.10}$$

Since $\eta(x,y) = -\eta(y,x)$ for all $x, y \in E$ and $J$ is $\eta$-strongly monotone with constant $\alpha > 0$, we have

$$\alpha\|x_1 - x_2\|^2 \leqslant \langle Jx_1 - Jx_2, \eta(x_1,x_2)\rangle \leqslant 0,$$

and hence we must have $x_1 = x_2$. This completes the proof. □

**Theorem 2.2.** *Let $E$ be a reflexive Banach space with the dual space $E^\star$, $J : E \to E^\star$ is $\eta$-strongly monotone continuous mapping with constant $\alpha > 0$, $\phi : E \to R \cup \{+\infty\}$ be a lower semicontinuous, $\eta$-subdifferentiable proper functional. Let $\eta : E \times E \to E$ be Lipschitz continuous with constant $\tau > 0$ such that $\eta(x,y) = -\eta(y,x)$ for all $x, y \in E$ and for any given $x \in E$, the functional $h(y,x) = \langle x^\star - Jx, \eta(y,x)\rangle$ is 0-DQCV in $y$, $\rho > 0$ is an arbitrary constant. Then the $J^\eta$-proximal mapping $J_\rho^{\partial_\eta\phi}$ of $\phi$ is $\tau/\alpha$-Lipschitz continuous.*

**Proof.** By Theorem 2.1, the $J^\eta$-proximal mapping $J_\rho^{\partial_\eta\phi}$ of $\phi$ is well defined. For any given $x^\star, y^\star \in E^\star$, let $x = J_\rho^{\partial_\eta\phi}(x^\star)$, $y = J_\rho^{\partial_\eta\phi}(y^\star)$, then $x^\star - Jx \in \rho\partial_\eta\phi(x)$, $y^\star - Jy \in \rho\partial_\eta\phi(y)$.

Hence

$$\langle x^\star - Jx, \eta(u,x)\rangle \geqslant \rho\phi(x) - \rho\phi(u) \quad \text{for all } u \in E, \tag{2.11}$$

$$\langle y^\star - Jy, \eta(u,y)\rangle \geqslant \rho\phi(y) - \rho\phi(u) \quad \text{for all } u \in E. \tag{2.12}$$

Taking $u = y$ in (2.11) and $u = x$ in (2.12) and adding this inequalities

$$\langle x^\star - Jx, \eta(y,x)\rangle + \langle y^\star - Jy, \eta(x,y)\rangle \geqslant 0.$$

Since $\eta$ is Lipschitz continuous with constant $\tau > 0$, $\eta(x,y) = -\eta(y,x)$ and $J$ is $\eta$-strongly monotone with constant $\alpha > 0$, we have

$$\begin{aligned}\langle Jy - Jx, \eta(y,x)\rangle &\leqslant \langle \eta(y,x), y^\star - x^\star\rangle \\ \alpha\|y-x\|^2 &\leqslant \langle \eta(y,x), y^\star - x^\star\rangle \leqslant \|\eta(y,x)\|\|y^\star - x^\star\| \\ &\leqslant \tau\|y-x\|\|y^\star - x^\star\|\end{aligned}$$

which implies that $J_\rho^{\partial_\eta\phi}$ is $\tau/\alpha$ Lipschitz continuous. □

## 3. Proximal point algorithm

**Definition 3.1.** The mapping $N : E^\star \times E^\star \to E^\star$ is said to be

(i) Lipschitz continuous with respect to the first argument, if there exists a constant $\lambda_{N_1} > 0$ such that

$$\|N(u_1,.) - N(u_2,.)\| \leqslant \lambda_{N_1}\|u_1 - u_2\|$$

for all $u_1 \in T(x_1)$, $u_2 \in T(x_2)$ and $x_1, x_2 \in E$.

(ii) Lipschitz continuous with respect to the second argument, if there exists a constant $\lambda_{N_2} > 0$ such that

$$\|N(.,v_1) - N(.,v_2)\| \leqslant \lambda_{N_2}\|v_1 - v_2\|$$

for all $v_1 \in A(x_1)$, $v_2 \in A(x_2)$ and $x_1, x_2 \in E$.

The following Lemma plays an important role in proving our main result.

**Lemma 3.1** [19]. *Let $E$ be a real Banach space and $\mathscr{F} : E \to 2^{E^\star}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle$$

*for all $j(x+y) \in \mathscr{F}(x+y)$.*

**Proof.** For any $x, y \in E$ and $j(x+y) \in \mathscr{F}(x+y)$, we have

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, j(x+y)\rangle = \langle x, j(x+y)\rangle + \langle y, j(x+y)\rangle \\ &\leqslant \tfrac{1}{2}(\|x\|^2 + \|j(x+y)\|^2) + \langle y, j(x+y)\rangle \\ &= \tfrac{1}{2}(\|x\|^2 + \|x+y\|^2) + \langle y, j(x+y)\rangle\end{aligned}$$

it follows that

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in \mathscr{F}(x+y)$. This completes the proof. □

We first transfer the GMNQVLIP (2.1) into a fixed point problem.

**Theorem 3.1.** *$(x,u,v)$ is a solution of GMNQVLIP (2.1) if and only if $(x,u,v)$ satisfies the following relation*:

$$g(x) = J_\rho^{\partial_\eta\phi(.,x)}\{J(g(x)) - \rho f(x) + \rho N(u,v)\}, \tag{3.1}$$

*where $x \in E$, $u \in T(x)$, $v \in A(x)$, $\rho > 0$ and $J_\rho^{\partial_\eta(.,x)} = (J + \rho\partial_\eta\phi(.,x))^{-1}$ is the $J^\eta$-proximal mapping of $\phi(.,x)$.*

**Proof.** Assume that $x \in E$, $u \in T(x)$, $v \in A(x)$ satisfies relation (3.1), i.e.,

$$g(x) = J_\rho^{\partial_\eta\phi(.,x)}\{J(g(x)) - \rho f(x) + \rho N(u,v)\}.$$

Since $J_\rho^{\partial_\eta\phi(.,x)} = (J + \rho\partial_\eta\phi(.,x))^{-1}$, the above equality holds if and only if

$$J(g(x)) - \rho f(x) + \rho N(u,v) \in J(g(x)) + \rho\partial_\eta\phi(g(x),x).$$

By the definition of $\eta$-subdifferential of $\phi(.,x)$, the above relation holds if and only if

$$\phi(y,x) - \phi(g(x),x) \geqslant \langle N(u,v) - f(x), \eta(y,g(x))\rangle \quad \text{for all } y \in E.$$

Hence we have

$$\langle f(x) - N(u,v), \eta(y,g(x))\rangle \geqslant \phi(g(x),x) - \phi(y,x) \quad \text{for all } y \in E,$$

i.e. $(x,u,v)$ is a solution of the GMNQVLIP (2.1). This fixed point formulation enables us to suggest the following proximal point algorithm. □

**Algorithm 3.1.** Let $T, A : E \to CB(E^\star)$ be set-valued mappings, $f : E \to E^\star$, $N : E^\star \times E^\star \to E^\star$, $\eta : E \times E \to E$, $J : E \to E^\star$ be single-valued mappings and $g : E \to E$ be the single-valued mapping with $g(E) = E$. Let $\phi : E \times E \to R \cup \{+\infty\}$ be a lower semicontinuous, $\eta$-subdifferentiable proper functional on $E$ satisfying $g(E) \cap \mathrm{dom}\partial_\eta\phi(.,x) \neq \phi$. For any $x_0 \in E$, $u_0 \in T(x_0)$, $v_0 \in A(x_0)$. By $g(E) = E$, there exists a point $x_1 \in E$ such that

$$g(x_1) = J_\rho^{\partial_\eta \phi(.,x_0)}\{J(g(x_0)) - \rho f(x_0) + \rho N(u_0, v_0)\}.$$

By Nadler [20], there exists $u_1 \in T(x_1)$ and $v_1 \in A(x_1)$ such that

$$\|u_1 - u_0\| \leqslant (1+1)H(T(x_1), T(x_0)),$$

$$\|v_1 - v_0\| \leqslant (1+1)H(A(x_1), A(x_0)).$$

Let

$$g(x_2) = J_\rho^{\partial_\eta \phi(.,x_1)}\{J(g(x_1)) - \rho f(x_1) + \rho N(u_1, v_1)\}$$

continuing the above process inductively, we can define the following iterative sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ for solving GMNQVLIP (2.1) as follows:

$$\begin{aligned} &g(x_{n+1}) = J_\rho^{\partial_\eta \phi(.,x_n)}\{J(g(x_n)) - \rho f(x_n) + \rho N(u_n, v_n)\}, \\ &u_n \in T(x_n), \quad \|u_{n+1} - u_n\| \leqslant \left(1 + \frac{1}{n+1}\right) H(T(x_{n+1}), T(x_n)), \\ &v_n \in A(x_n), \quad \|v_{n+1} - v_n\| \leqslant \left(1 + \frac{1}{n+1}\right) H(A(x_{n-1}), A(x_n)), \\ &\quad n = 0, 1, 2, \ldots, \end{aligned} \tag{3.2}$$

where $\rho > 0$ is a constant.

**Theorem 3.2.** *Let $T, A : E \to \mathrm{CB}(E^\star)$ be Lipschitz continuous mappings with Lipschitz constants $\lambda_T$ and $\lambda_A$, respectively. Let $g : E \to E$ and $f : E \to E^\star$ be Lipschitz continuous mappings with Lipschitz constants $\lambda_g$ and $\lambda_f$, respectively and g is k-strongly accretive ($k \in (0,1)$) satisfying $g(E) = E$. Let $\eta : E \times E \to E$ be Lipschitz continuous with Lipschitz constant $\tau > 0$ such that $\eta(x,y) = -\eta(y,x)$ for all $x, y \in E$ and for each given $x \in E$, the function $h(y,x) = \langle x^\star - Jx, \eta(y,x)\rangle$ is 0-DQCV in y. Let $N : E^\star \times E^\star \to E^\star$ be $\lambda_{N_1}$-Lipschitz continuous in the first argument and $\lambda_{N_2}$-Lipschitz continuous in the second argument. Let $\phi : E \times E \to R \cup \{+\infty\}$ be such that for each $x \in E$, $\phi(.,x)$ is a lower semicontinuous, η-subdifferentiable, proper functional satisfying $g(x) \in dom(\partial_\eta \phi(.,x))$. Let $J : E \to E^\star$ is η-strongly monotone with constant $\alpha > 0$ and $\lambda_j$-Lipschitz continuous. Suppose that there exists a constant $\rho > 0$ such that for each $x, y \in E$, $x^\star \in E^\star$*

$$\|J_\rho^{\partial_\eta \phi(.,x_n)}(x^\star) - J_\rho^{\partial_\eta \phi(.,x_{n-1})}(x^\star)\| \leqslant \mu \|x_n - x_{n-1}\| \tag{3.3}$$

*and the following conditions are satisfied.*

$$\frac{2\tau^2(\lambda_j^2 \lambda_g^2)}{\alpha^2} - \frac{3-\mu^2}{2} < K < 1,$$

$$0 < \rho < \sqrt{\frac{(2k+3)\alpha^2 - 4\tau^2(\lambda_j^2\lambda_g^2) + 2\alpha^2\mu^2}{8\tau^2(\lambda_f^2) + (\lambda_{N_1}\lambda_T + \lambda_{N_2}\lambda_A)^2}}. \tag{3.4}$$

*Then the iterative sequences* $\{x_n\}$, $\{u_n\}$ *and* $\{v_n\}$ *generated by Algorithm* 3.1 *converge strongly to* $x$, $u$ *and* $v$, *respectively and* $(x,u,v)$ *is a solution of GMNQVLIP* (2.1).

**Proof.** We can write

$$\|x_{n+1} - x_n\|^2 = \|g(x_{n+1}) - g(x_n) - g(x_{n+1}) + g(x_n) - x_{n+1} + x_n\|^2.$$

By Lemma 3.1, we have

$$\|x_{n+1} - x_n\|^2 \leqslant \|g(x_{n+1}) - g(x_n)\|^2 - 2\langle g(x_{n+1}) - g(x_n) + x_{n+1} - x_n, j(x_{n+1} - x_n)\rangle. \tag{3.5}$$

By Algorithm 3.1, we have

$$g(x_{n+1}) = J_\rho^{\partial_n\phi(\cdot,x_n)}[J(g(x_n)) - \rho f(x_n) + \rho N(u_n, v_n)].$$

Hence we have

$$\|g(x_{n+1}) - g(x_n)\|^2 = \|J_\rho^{\partial_n\phi(\cdot,x_n)}[J(g(x_n)) - \rho f(x_n) + \rho N(u_n, v_n)] - \{J_\rho^{\partial_n\phi(\cdot,x_{n-1})}[J(g(x_{n-1})) - \rho f(x_{n-1}) + \rho N(u_{n-1}, v_{n-1})]\}\|^2.$$

Since $\|x+y\|^2 \leqslant 2(\|x\|^2 + \|y\|^2)$, by the assumptions and Theorem 2.2, we have

$$\begin{aligned}
&\frac{1}{2}\|g(x_{n+1}) - g(x_n)\|^2 \\
&\leqslant \|J_\rho^{\partial_n\phi(\cdot,x_n)}[J(g(x_n)) - \rho f(x_n) + \rho N(u_n, v_n)] \\
&\quad - J_\rho^{\partial_n\phi(\cdot,x_n)}[J(g(x_{n-1})) - \rho f(x_{n-1}) + \rho N(u_{n-1}, v_{n-1})]\|^2 \\
&\quad + \|J_\rho^{\partial_n\phi(\cdot,x_n)}[J(g(x_{n-1})) - \rho f(x_{n-1}) + \rho N(u_{n-1}, v_{n-1})] \\
&\quad - J_\rho^{\partial_n\phi(\cdot,x_{n-1})}[J(g(x_{n-1})) - \rho f(x_{n-1}) + \rho N(u_{n-1}, v_{n-1})]\|^2 \\
&\leqslant \frac{\tau^2}{\alpha^2}\|\{J(g(x_n)) - \rho f(x_n) + \rho N(u_n, v_n)\} - \{J(g(x_{n-1})) - \rho f(x_{n-1}) \\
&\quad + \rho N(u_{n-1}, v_{n-1})\}\|^2 + \|J_\rho^{\partial_n\phi(\cdot,x_n)}(J(g(x_{n-1})) - \rho f(x_{n-1}) \\
&\quad + \rho N(u_{n-1}, v_{n-1})) - J_\rho^{\partial_n\phi(\cdot,x_{n-1})}(J(g(x_{n-1})) - \rho f(x_{n-1}) \\
&\quad + \rho N(u_{n-1}, v_{n-1}))\|^2 \\
&\leqslant \frac{\tau^2}{\alpha^2}\|\{J(g(x_n)) - \rho f(x_n) + \rho N(u_n, v_n)\}
\end{aligned}$$

$$
\begin{aligned}
&- \{J(g(x_{n-1})) - \rho f(x_{n-1}) + \rho N(u_{n-1}, v_{n-1})\}\|^2 + \mu^2\|x_n - x_{n-1}\|^2 \\
&\leqslant \frac{2\tau^2}{\alpha^2}\|J(g(x_n)) - J(g(x_{n-1}))\|^2 + 4\rho^2\frac{\tau^2}{\alpha^2}\|f(x_n) - f(x_{n-1})\|^2 \\
&+ 4\rho^2\frac{\tau^2}{\alpha^2}\|N(u_n, v_n) - N(u_n, v_{n-1}) + N(u_n, v_{n-1}) \\
&- N(u_{n-1}, v_{n-1})\|^2 + \mu^2\|x_n - x_{n-1}\|^2. \qquad (3.6)
\end{aligned}
$$

By the Lipschitz continuity of $J$ and $g$, we have

$$
\|J(g(x_n)) - J(g(x_{n-1}))\| \leqslant \lambda_j(\|g(x_n) - g(x_{n-1})\|) \leqslant \lambda_j\lambda_g\|x_n - x_{n-1}\|. \qquad (3.7)
$$

By the Lipschitz continuity of $f$, we have

$$
\|f(x_n) - f(x_{n-1})\| \leqslant \lambda_f\|x_n - x_{n-1}\|. \qquad (3.8)
$$

By the Lipschitz continuity of $N(.,.)$ in both the arguments, Algorithm 3.1, Lipschitz continuity of $T$ and $A$, we have

$$
\begin{aligned}
&\|N(u_n, v_n) - N(u_n, v_{n-1}) + N(u_n, v_{n-1}) - N(u_{n-1}, v_{n-1})\| \\
&\leqslant \|N(u_n, v_n) - N(u_n, v_{n-1})\| + \|N(u_n, v_{n-1}) - N(u_{n-1}, v_{n-1})\| \\
&\leqslant \lambda_{N_2}\|v_n - v_{n-1}\| + \lambda_{N_1}\|u_n - u_{n-1}\| \\
&\leqslant \lambda_{N_2}\left(1 + \frac{1}{n}\right)H(A(x_n), A(x_{n-1})) + \lambda_{N_1}\left(1 + \frac{1}{n}\right)H(T(x_n), T(x_{n-1})) \\
&\leqslant (\lambda_{N_1}\lambda_T + \lambda_{N_2}\lambda_A)\left(1 + \frac{1}{n}\right)\|x_n - x_{n-1}\|. \qquad (3.9)
\end{aligned}
$$

By (3.6)–(3.9), we obtain

$$
\begin{aligned}
\|g(x_{n+1}) - g(x_n)\|^2 &\leqslant \left[\frac{4\tau^2}{\alpha^2}(\lambda_j^2\lambda_g^2) + 8\rho^2\frac{\tau^2}{\alpha^2}\lambda_f^2 + 8\rho^2\frac{\tau^2}{\alpha^2}(\lambda_{N_1}\lambda_T + \lambda_{N_2}\lambda_A)^2 \right. \\
&\qquad \left. \times\left(1 + \frac{1}{n}\right)^2 + 2\mu^2\right]\|x_n - x_{n-1}\|^2 \\
&= \left[\frac{4\tau^2}{\alpha^2}(\lambda_j^2\lambda_g^2) + 8\rho^2\frac{\tau^2}{\alpha^2}\left(\lambda_f^2 + (\lambda_{N_1}\lambda_T + \lambda_{N_2}\lambda_A)^2\left(1 + \frac{1}{n}\right)^2\right) + 2\mu^2\right] \\
&\quad \times \|x_n - x_{n-1}\|^2. \qquad (3.10)
\end{aligned}
$$

Since $g : E \to E$ is $k$-strongly accretive, by (3.5), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\|^2 &\leqslant \|g(x_{n+1}) - g(x_n)\|^2 - 2\langle g(x_{n+1}) - g(x_n) + x_{n+1} - x_n, j(x_{n-1} - x_n)\rangle \\
&\leqslant \left[\frac{4\tau^2}{\alpha^2}(\lambda_j^2\lambda_g^2) + 8\rho^2\frac{\tau^2}{\alpha^2}\left(\lambda_f^2 + (\lambda_{N_1}\lambda_T + \lambda_{N_2}\lambda_A)^2\left(1 + \frac{1}{n}\right)^2\right) + 2\mu^2\right] \\
&\quad \times \|x_n - x_{n-1}\|^2 - 2(K + 2)\|x_{n+1} - x_n\|^2.
\end{aligned}
$$

It follows that

$$\|x_{n+1}-x_n\|^2 \leqslant \left[\frac{4\tau^2(\lambda_j^2\lambda_g^2)}{(2K+3)\alpha^2}+\frac{8\rho^2\tau^2(\lambda_f^2+(\lambda_{N_1}\lambda_T+\lambda_{N_2}\lambda_A)^2(1+1/n)^2)}{(2K+3)\alpha^2} + \frac{2\mu^2\alpha^2}{(2K+3)\alpha^2}\right]\|x_n-x_{n-1}\|^2 = \theta_n^2\|x_n-x_{n-1}\|^2, \qquad (3.11)$$

where

$$\theta_n = \sqrt{\left[\frac{4\tau^2(\lambda_j^2\lambda_g^2)+8\rho^2\tau^2(\lambda_f^2+(\lambda_{N_1}\lambda_T+\lambda_{N_2}\lambda_A)^2(1+1/n)^2)+2\alpha^2\mu^2}{(2K+3)\alpha^2}\right]}.$$

Let

$$\theta = \sqrt{\frac{4\tau^2(\lambda_j^2\lambda_g^2)+8\rho^2\tau^2(\lambda_f^2+(\lambda_{N_1}\lambda_T+\lambda_{N_2}\lambda_A)^2)+2\alpha^2\mu^2}{(2K+3)\alpha^2}}$$

clearly $\theta_n \to \theta$ as $n \to \infty$. Condition (3.4) implies that $0<\theta<1$, and so $0<\theta_n<1$ when $n$ is sufficiently large. It follows from (3.11) that $\{x_n\}$ is a cauchy sequence in $E$. Let $x_n \to x$. Since the mappings $T$ and $A$ are Lipchitz continuous, it follows from (3.2) that $\{u_n\}$ and $\{v_n\}$ are also cauchy sequences, we can assume that $u_n \to u$ and $v_n \to v$. Since $J, g, f$ and $N(.,.)$ are Lipschitz continuous mappings and by Algorithm 3.1, we have

$$g(x) = J_\rho^{\partial_n\phi(.,x)}\{J(g(x)) - \rho f(x) + \rho N(u,v)\}.$$

Now we will prove that $u \in T(x)$ and $v \in A(x)$. Infact, since $u_n \in T(x_n)$ and

$$\begin{aligned} d(u_n, T(x)) &\leqslant \max\left\{d(u_n,T(x)), \sup_{y\in T(x)} d(T(x_n),y)\right\} \\ &\leqslant \max\left\{\sup_{z\in T(x_n)} d(z,T(x)), \sup_{y\in T(x)} d(T(x_n),y)\right\} \\ &= D(T(x_n),T(x)). \end{aligned}$$

We have

$$\begin{aligned} d(u,T(x)) &\leqslant \|u-u_n\| + d(u_n,T(x)) \leqslant \|u-u_n\| + D(T(x_n),T(x)) \\ &\leqslant \|u-u_n\| + \lambda_T\|x_n - x\| \to 0 (n\to\infty), \end{aligned}$$

which implies that $d(u,T(x)) = 0$. Since $T(x) \in \mathrm{CB}(E)$, it follows that $u \in T(x)$. Similarly, we can prove that $v \in A(x)$. By Theorem 3.1, $(x,u,v)$ is a solution of GMNQVLIP (2.1). This completes the proof. □

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