King Fahd University of Petroleum and Minerals
Department of Mathematics
Math 201 Sem I 2004

First Major Exam　　Sat 16 / 10 / 2004　　Time $1\frac{1}{4}$ hours

Name: Key　　I.D.#: ____________　　Serial #: ____

Section #: 13　　16

Answer all the questions

Show all of your work

All the questions have equal mark

Question #	1	2	3	4	5	6	7	8	Total
Grade									

1. Find the area of the triangle with vertices $P(2,-3,1)$, $Q(1,0,2)$, and $R(0,1,-2)$.

$\vec{PQ} = \langle -1, 3, 1 \rangle$, $\vec{PR} = \langle -2, 4, -3 \rangle$, $\vec{QR} = \langle -1, 1, -4 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -1 & 3 & 1 \\ -2 & 4 & -3 \end{vmatrix} = i(-9-4) - j(3+2) + k(-4+6)$$

$$= -13\vec{i} - 5\vec{j} + 2\vec{k}$$

$$\text{Area of triangle} = \frac{\|\vec{PQ} \times \vec{PR}\|}{2} = \frac{\sqrt{13^2+5^2+2^2}}{2}$$

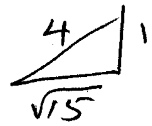
$$= \frac{\sqrt{169+25+4}}{2} = \frac{\sqrt{198}}{2} \text{ cm}$$

Solution key

2. **Find the cosine of the angles at which the the polar curve $r = 1 - 2\sin\theta$ has horizontal tangent line**

horizontal Tangent if $\frac{dy}{d\theta} = 0$ & $\frac{dx}{d\theta} \neq 0$

$y = r\sin\theta = (1-2\sin\theta)\sin\theta = \sin\theta - 2\sin^2\theta$



$\frac{dy}{d\theta} = \cos\theta - 2\cdot 2\sin\theta\cos\theta$, if $\frac{dy}{d\theta} = 0 \Rightarrow$

$\cos\theta = 0$ or $\sin\theta = \frac{1}{4} \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \theta_1 \;\&\; \theta_2$, $\cos\theta_1 = \frac{\sqrt{15}}{4}$

$x = r\cos\theta = (1-2\sin\theta)\cos\theta = \cos\theta - \underbrace{2\sin\theta\cos\theta}_{\sin 2\theta}$

$\frac{dx}{d\theta} = -\sin\theta - 2\cos 2\theta$

θ	$\frac{dx}{d\theta}$
$\frac{\pi}{2}$	$\neq 0$
$\frac{3\pi}{2}$	$\neq 0$
θ_1	$\neq 0$
θ_2	$\neq 0$

The cosine of the angles are

$\cos\frac{\pi}{2} = 0 = \cos\frac{3\pi}{2}$,

$\cos\theta_1 = \frac{\sqrt{15}}{4}$, $\cos\theta_2 = \frac{-\sqrt{15}}{4}$

3. **Determine whether the two lines are parallel , intersecting , or skew;**

$L_1 : x = 2 + t \quad y = 1 - t \quad z = 2t$, $L_2 : x = 2t \quad y = 1 - 2t \quad z = 2 + t$

$\vec{V}_1 = \langle 1, -1, 2 \rangle$, $\vec{V}_2 = \langle 2, -2, 1 \rangle$

Clearly $\vec{V}_1$ is not parallel to $\vec{V}_2$ so the lines are not parallel.

L_1: $x = 2+s$, $y = 1-s$, $z = 2s$, L_2: $x = 2t$, $y = 1-2t$, $z = 2+t$

$2+s = 2t$ -- ①

$1-s = 1-2t$ -- ②

$2s = 2+t$

① ② $\Rightarrow s = 2t-2$

$1-(2t-2) = 1-2t$

$3-2t = 1-2t \Rightarrow 2 = 1$

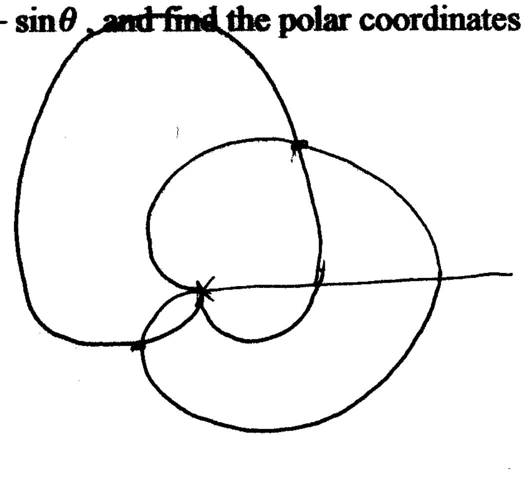
not good

no solution

So the two lines are $\underline{\text{skew}}$ lines

Solution key

4. **Sketch the polar curves $r = 1+\cos\theta$, and $r = 1+\sin\theta$, ~~and find~~ the polar coordinates of the points of intersection.**



$1+\cos\theta = 1+\sin\theta$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4},$

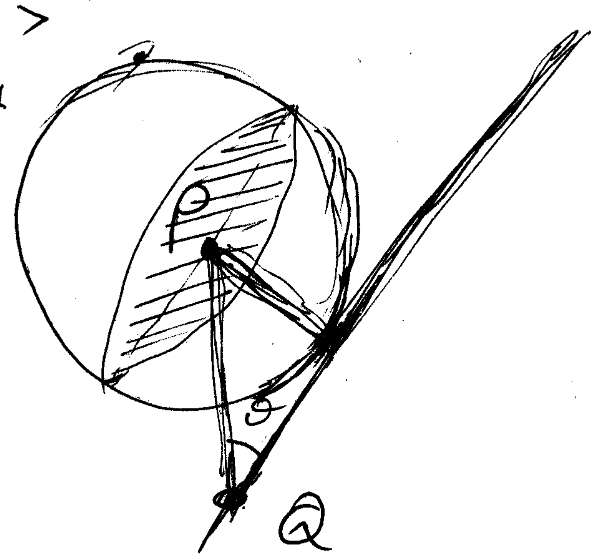
Also the pole is a common point of both graphs

$P_1\left(1+\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right), P_2\left(1-\frac{\sqrt{2}}{2}, \frac{5\pi}{4}\right), P_3(0, \pi)$

5. **Find the equation of the sphere centered at $P(1,-2,0)$ and tangent to the line**
$L: x = 2-t \quad y = 1+2t \quad z = -2t.$

$P(1,-2,0), \quad \vec{V} = \langle -1, 2, -2 \rangle$

Let $Q(2,1,0)$ be a point on the line.



$\vec{PQ} = \langle 1, 3, 0 \rangle$

$h = \|\vec{PQ}\| \sin\theta = \frac{\|\vec{PQ}\| \cdot \|\vec{PQ} \times \vec{V}\|}{\|\vec{PQ}\| \cdot \|\vec{V}\|}$

$$\vec{PQ} \times \vec{V} = \begin{vmatrix} i & j & k \\ 1 & 3 & 0 \\ -1 & 2 & -2 \end{vmatrix} = i(-6) - j(-2) + k(2+3) = -6\vec{i} + 2j + 5k$$

$h = \frac{\sqrt{36+4+25}}{\sqrt{1+4+4}} = \frac{\sqrt{65}}{3}$ is the radius

$\therefore$ The equation of the sphere is:

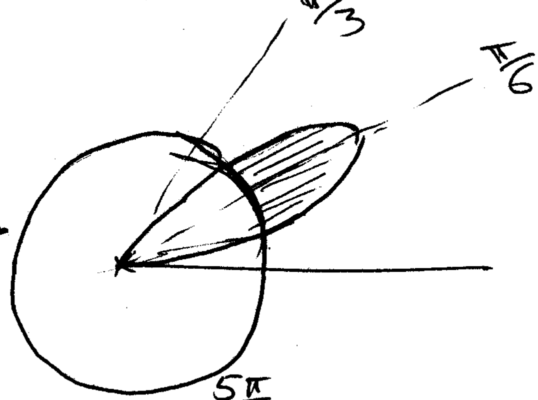
Solution Key

6. Find the area inside one leave of the rose $r = 2\sin 3\theta$ and outside $r = 1$

$2\sin 3\theta = 1 \Rightarrow \sin 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\theta = \frac{\pi}{18}, \frac{5\pi}{18}$ $\quad \frac{1}{2} - \frac{1}{2}\cos 6\theta$



$\cancel{\frac{1}{2} - \frac{1}{2}\cos 6\theta}$

$$A = \frac{1}{2}\int_{\pi/18}^{5\pi/18} \underbrace{[4\sin^2 3\theta - 1]}\, d\theta = \frac{1}{2}\int_{\pi/18}^{5\pi/18} (2 - 2\cos 6\theta - 1)$$

$$= \frac{1}{2}\int_{\pi/18}^{5\pi/18} [1 - 2\cos 6\theta]\, d\theta = \frac{1}{2}\left[\theta - \tfrac{1}{3}\sin 6\theta\right]_{\pi/18}^{5\pi/18}$$

$$= \frac{1}{2}\left[\frac{2\pi}{9} - \frac{1}{3}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right] = \frac{1}{2}\cdot\frac{2\pi}{9} + \frac{1}{6}\cdot\sqrt{3} = \frac{\pi}{9} + \frac{\sqrt{3}}{6}$$

7. Let **u** and **v** be vectors in 3-space, show that $\mathbf{u} \bullet \mathbf{v} = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$

$$R.H.S. = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 =$$

$$= \frac{1}{4}(u+v)\cdot(u+v) - \frac{1}{4}(u-v)\cdot(u-v)$$

$$= \frac{1}{4}\left[\cancel{u\cdot u} + 2u\cdot v + \cancel{v\cdot v} - \cancel{u\cdot u} + 2u\cdot v - \cancel{v\cdot v}\right]$$

$$= \frac{1}{4}\cdot 4\, u\cdot v = u\cdot v = L.H.S.$$