Department of Mathematics

Math 201 Summer (053)

First Major Exam | Mon 10 / 7 / 2006 | Time $1\frac{1}{4}$ hours

Name: Key I.D.#: ____________ Serial #: ______

Answer all the questions

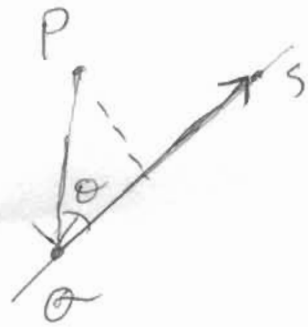
Show all of your work

Question #	1	2	3	4	5	6	7	8	Total
Grade	/5	/5	/5	/5	/5	/5	/5	/5	/40

1. Let the points $P(2,1,0)$, $Q(1,-1,2)$, and $S(3,-2,-1)$. Find the distance from the point P to the line joining the points Q and S.

Method 1 by cross product

$\overrightarrow{QS} = \langle 2, -1, -3 \rangle$, $\overrightarrow{PQ} = \langle -1, -2, 2 \rangle$



$$\text{distance} = \frac{\|\overrightarrow{PQ} \times \overrightarrow{QS}\|}{\|\overrightarrow{QS}\|}$$

$$\overrightarrow{PQ} \times \overrightarrow{QS} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & 2 \\ 2 & -1 & -3 \end{vmatrix} = \vec{i}(8) - \vec{j}(-1) + \vec{k}(5)$$

$$= 8\vec{i} + \vec{j} + 5\vec{k}$$

$$d = \frac{\sqrt{64+1+25}}{\sqrt{4+1+9}} = \frac{\sqrt{90}}{\sqrt{14}} = \frac{\sqrt{45}}{\sqrt{7}}$$

Method 2 by Dot product

$$\underset{\overrightarrow{QS}}{\text{proj}}\, \overrightarrow{PQ} = \frac{\overrightarrow{PQ} \cdot \overrightarrow{QS}}{\|\overrightarrow{QS}\|^2}\, \overrightarrow{QS} = \frac{-2+2-6}{14} \langle 2, -1, -3 \rangle$$

$$= \frac{-3}{7} \langle 2, -1, -3 \rangle$$

$$d = \|\overrightarrow{PQ} - \underset{\overrightarrow{QS}}{\text{proj}}\, \overrightarrow{PQ}\| = \| \langle -1, -2, 2 \rangle - \langle \tfrac{-6}{7}, \tfrac{3}{7}, \tfrac{9}{7} \rangle \|$$

$$= \| \langle \tfrac{-1}{7}, \tfrac{-17}{7}, \tfrac{5}{7} \rangle \| = \sqrt{\frac{1+289+25}{49}} = \sqrt{\frac{315}{49}} = \sqrt{\frac{45}{7}}$$

Key

2. Let $\mathbf{u} = \langle 3, 1, -2 \rangle$ and $\mathbf{v} = \langle 0, 2, 1 \rangle$ be two vectors , find parametric equations of the line containing the point $P(1, -5, 2)$ and parallel to the vector $\mathbf{u} \times \mathbf{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -2 \\ 0 & 2 & 1 \end{vmatrix} = \vec{i}(1+4) - \vec{j}(3-0) + \vec{k}(6-0)$$
$$= 5\vec{i} - 3\vec{j} + 6\vec{k}$$

A line parallel to $\vec{u} \times \vec{v}$ and through the point $(1, -5, 2)$ is :

$x = 1 + 5t$, $y = -5 - 3t$, $z = 2 + 6t$

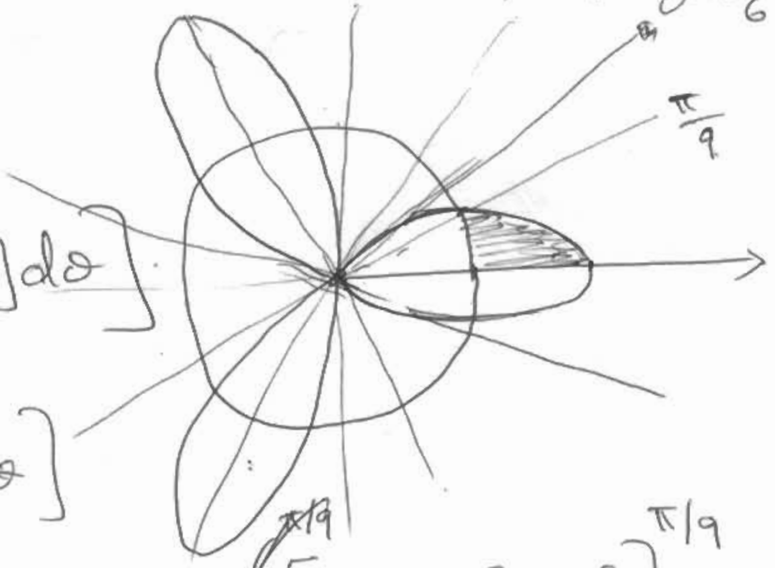
3. Find the area inside the polar curve $r = 2\cos 3\theta$ and outside the polar curve $r = 1$.

$$2\cos 3\theta = 1 \Rightarrow \cos 3\theta = \frac{1}{2}$$

$$\therefore 3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}$$

$$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$$

3-leave rose



By symmetry,

$$A = 6 \cdot \frac{1}{2}\left[\int_0^{\pi/9} \left[(2\cos 3\theta)^2 - 1\right] d\theta\right]$$

$$= 3\left[\int_0^{\pi/9} (4\cos^2 3\theta - 1)\, d\theta\right]$$

$$= 3\int_0^{\pi/9} (2 + 2\cos 6\theta - 1)\, d\theta = 3\left[\theta + \frac{1}{3}\sin 6\theta\right]_0^{\pi/9}$$

$$= 3 \cdot \frac{\pi}{9} + \frac{\sqrt{3}}{2} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$ ✓

Key

4. Find the values of t at which the parametric curve $x = 2t^2 + 4t$, $y = t^3 - 3t + 2$ has horizontal tangent line.

horizontal tangent line if $\frac{dy}{dt} = 0$ + $\frac{dx}{dt} \neq 0$.

$\frac{dy}{dt} = 3t^2 - 3 = 3(t^2 - 1)$, if $\frac{dy}{dt} = 0 \Rightarrow t = 1$ or -1

$\frac{dx}{dt} = 4t + 4$, $\left.\frac{dx}{dt}\right|_{t=1} = 8 \neq 0$ ✓ , $\left.\frac{dx}{dt}\right|_{t=-1} = 0$ N.

So the only value is at $\boxed{t = 1}$

5. Determine whether the two lines L_1 and L_2 are parallel , intersecting or skew, where the parametric equations of the lines

$L_1 : x = 1 + 2t \; , y = 3 - t \; , \; z = -t$

$L_2 : x = -2 - 4t \; , y = 2 - 3t \; , \; z = 1 + t$

$V_1 = \langle 2, -1, -1 \rangle$, $\vec{V}_2 = \langle -4, -3, 1 \rangle$

clearly $\vec{v}_1$ + $\vec{v}_2$ are not parallel. So the lines are not parallel

check for intersecting:

$1 + 2t = -2 - 4s$ -- (1)

$3 - t = 2 - 3s$ -- (2)

$-t = 1 + s$ -- (3)

from (3) $t = -1 - s$, substitute in (2)

$3 + 1 + s = 2 - 3s \Rightarrow -4s = 2 \Rightarrow s = \frac{-1}{2}$, $t = -1 - \frac{-1}{2} = \frac{-1}{2}$

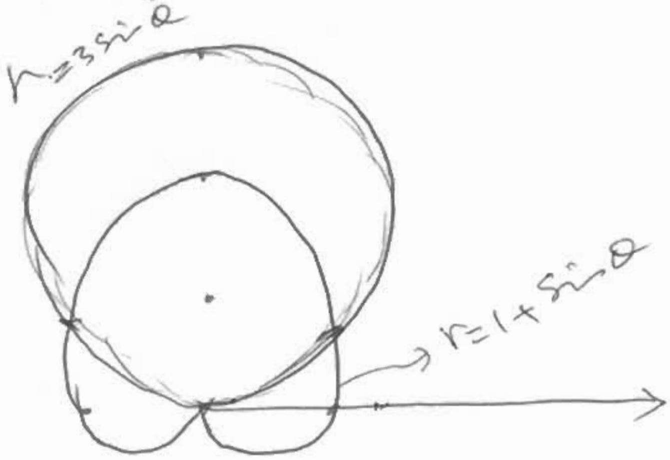
check in eq. (1) $1 + 2\frac{-1}{2} = -2 - 4(\frac{-1}{2})$

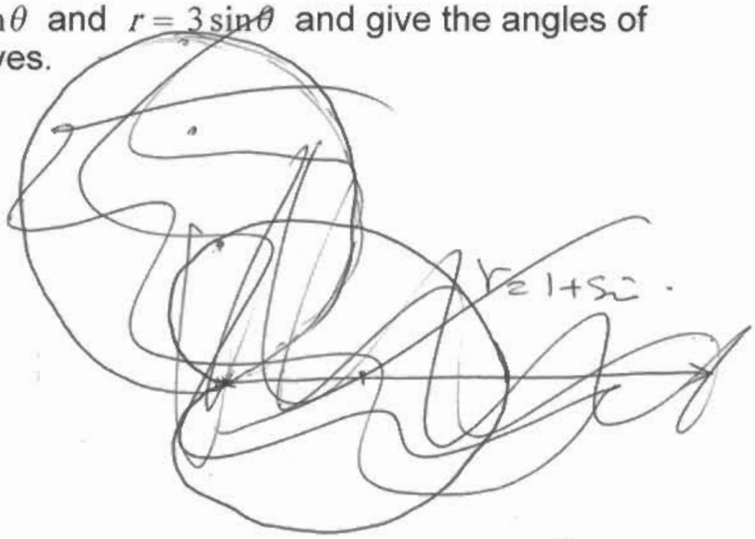
$0 = 0$ o.k.

So the two lines are intersecting at the point $(0, \frac{7}{2}, \frac{1}{2})$

Key

6. Sketch the two polar curves $r = 1 + \sin\theta$ and $r = 3\sin\theta$ and give the angles of all points of intersection between the curves.





$1+\sin\theta = 3\sin\theta \Longrightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2}$

$\Longrightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6},$

The two curves intersects at the origin also: where $r=0$,

So there are three points at $\theta = \frac{\pi}{6}, r = \frac{3}{2}$, & $\theta = \frac{5\pi}{6}, r = \frac{3}{2}$ and at $r=0$ (θ any value)

7. Find equation of the sphere centered at $C(1,3,-2)$ and passes through the point $Q(0,1,2)$.

$r =$ distance from C to Q $= \sqrt{(1-0)^2 + (3-1)^2 + (-2-2)^2}$

$= \sqrt{1+4+16} = \sqrt{21}$

$(x-1)^2 + (y-3)^2 + (z+2)^2 = 21$ is the equation.

Key

8. For each of the following give a short answer in the assigned space:

a. Let the vectors $\mathbf{u} = \langle 1,-1,2 \rangle$ and $\mathbf{v} = \langle 0,1,4 \rangle$. Find a unit vector in the direction of $\mathbf{u}-\mathbf{v}$.

$\vec{u}-\vec{v} = \langle 1,-2,-2 \rangle$, $\|\vec{u}-\vec{v}\| = 3$

$\dfrac{\vec{u}-\vec{v}}{\|\vec{u}-\vec{v}\|} = \langle \frac{1}{3}, \frac{-2}{3}, \frac{-2}{3} \rangle$

b. Give the center and the radius of the sphere with the given equation $x^2+y^2+z^2+2x-4y+6z+10=0$.

$(x+1)^2+(y-2)^2+(z+3)^2 = -10+1+4+9 = 4$

$C(-1,2,-3)$, $r=2$

c. Let $P(2, \frac{\pi}{6})$ with polar coordinates , find equivalent rectangular coordinates.

$x = r\cos\theta = 2\cos\frac{\pi}{6} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

$y = r\sin\theta = 2\sin\frac{\pi}{6} = 2\cdot\frac{1}{2} = 1$

$(\sqrt{3}, 1)$

d. Let the vectors $\mathbf{a} = \langle -1,1,0 \rangle$ and $\mathbf{b} = \langle 0,-1,2 \rangle$, find $\mathrm{Proj}_{\mathbf{a}}\,\mathbf{b}$.

$\mathrm{Proj}_{\vec{a}}\vec{b} = \dfrac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|^2}\cdot\vec{a} = \dfrac{0-1+0}{2}\langle -1,1,$

$\langle \frac{1}{2}, \frac{-1}{2}, 0 \rangle$

e. Find a so that $\mathbf{u}\cdot\mathbf{v} = 1$, where $\mathbf{u} = \langle a,-1,3 \rangle$ and $\mathbf{v} = \langle 2,0,-1 \rangle$.

$\vec{u}\cdot\vec{v} = 2a-3 = 1 \Rightarrow 2a = 4$

$a = 2$

$a = 2$