King Fahd University of Petroleum & Minerals
Department of Mathematical Sciences
Math654 "Advanced Topics in Algebra"
Semester 061 (Fall 2006)
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Take Home Exam (Due: 20.01.2007, 12:30 pm)

Remark: Give self-contained proofs and arguments.

* **SET 1 (Solve the following 2 questions)**

Q1. Let P be a module over a **commutative ring** R. Show that:

1. P is a $*$-module if and only if ${}_RP$ is a *quasi-progenerator*.
2. P is a *classical* 1*-tilting module* if and only if ${}_RP$ is a *progenerator*.

Q2. Give **examples** of

1. a $*$-modules that is neither a quasi-progenerator nor classical 1-tilting.
2. a classical 1-tilting module that is not a progenerator.

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* **SET 2 (Solve any 10 of the following questions)**

Q3. Let $(R, \mathfrak{m})$ be a Noetherian local commutative ring, $M \neq 0$ a finitely generated R-module, $R_1, .., R_n \in \mathfrak{m}$ an M-sequence and set $M' := M/(R_1, .., R_n)M$. Show that $\dim(M') = \dim(M) - n$.

Q4. Show that every zero-dimensional Noetherian commutative ring is Cohen-Macaulay.

Q5. every one-dimensional reduced commutative ring (i.e. with no non-zero nilpotent elements) is Cohen-Macaulay. Construct an example of a one-dimensional commutative ring, which is not Cohen-Macaulay.

Q6. Let $(R, \mathfrak{m})$ be a Noetherian local commutative ring, $x_1, ..., x_n$ an R-sequence. Show that R is Gorenstein if and only if $R/(x_1, ..., x_n)$ is Gorenstein.

Q7. Show that if R is a Gorenstein commutative ring, then so is the polynomial ring $R[X_1, .., X_n]$.

Q8. Let R, S be ring, ${}_SW_R$ a bimodule and Q_R an injective module. Show that for every module N_S we have

$$\mathrm{Ext}^i_S(N, \mathrm{Hom}_R(W, Q)) \simeq \mathrm{Hom}_R(\mathrm{Tor}^S_i(N, W), Q) \text{ for all } i \geq 0.$$

Q9. Let R, S be rings, ${}_SV_R$ a bimodule, C_R an injective cogenerator and $V^* := \mathrm{Hom}_R(V, C)$. Consider for each M_R and N_S the *canonical* maps

$$\nu_M : \mathrm{Hom}_R(V, M) \otimes_S V \to M \text{ and } \eta_N : N \to \mathrm{Hom}_R(V, N \otimes_S V).$$

Show that:

1. $M \in \mathrm{Gen}(V_R)$ if and only if ν_M is surjective.
2. $N \in \mathrm{Cogen}(V^*_S)$ if and only if η_N is injective.

Q10. Let R be a commutative ring and T a f.g. R-module. Show that the following statements are equivalent:

1. ${}_RT$ generates all injective R-modules;
2. $\sigma[T] := \overline{\mathrm{Gen}(T)} = R\text{-Mod}$;
3. ${}_RT$ is faithful.

Q11. Let R be a domain, $S \subseteq R \backslash \{0\}$ a multiplicatively closed subset and consider the associated Fuchs-Salce tilting module δ_S. Show that δ_S is a generator in the class of S-divisible R-modules.

Q12. Let R be a domain, $\delta := \delta_{R\backslash\{0\}}$ be the Fuchs tilting module, $\mathcal{P}_1$ is the class of R-modules with projective dimension at most 1 and $\mathcal{DI}$ is the class of divisible R-modules. Show that the 1-tilting *cotorsion pair* induced by δ is $(\mathcal{P}_1, \mathcal{DI})$ in case:

1. R is a Prüfer domain;
2. R is a Matlis domain.

Q13. Show that every n-tilting module over a von Neumann regular ring is projective.

Q14. Give an example of a direct summand of a tilting module that is not partial tilting (clarify the defect).

Q15. Give a complete description of the tilting modules over $\mathbb{Z}$.

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