KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematics and Statistics

MATH 260-(091)

Major Exam 2

Time: 90 Minutes

Name: Solution I.D. # Ser.#

Section: ___________

Show All Necessary Work

Question	Points
1	/11
2	/11
3	/11
4	/11
5	/11
6	/11
7	/11
8	/11
9	/11
10	/11
Total	/110

1. Express y in terms of t if $\begin{bmatrix} 2 & -3 & 4 \\ 3 & 5 & -1 \\ 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} t \\ 2t-1 \\ -t \end{bmatrix} + \begin{bmatrix} -t \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 7 \\ -6 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$= \begin{bmatrix} 2t - 6t + 3 - 4t \\ 3t + 10t - 5 + t \\ -8t + 4 + 2t \end{bmatrix} + \begin{bmatrix} -t \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 7 \\ -6 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \quad 3t + 10t - 5 + t + 1 - 7 = y$$

$$\therefore \quad y = 14t - 11$$

2. If $A = \begin{bmatrix} -a & -b & -c \\ 5d & 5e & 5f \\ g-a & h-b & i-c \end{bmatrix}$ and $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$, find det($A$) ?

$$\det(A) = \begin{vmatrix} -a & -b & -c \\ 5d & 5e & 5f \\ g-a & h-b & i-c \end{vmatrix} = - \begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g-a & h-b & i-c \end{vmatrix} = -5 \begin{vmatrix} a & b & c \\ d & e & f \\ g-a & h-b & i-c \end{vmatrix}$$

$$= -5 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= -5(7)$$

$$= -35$$

3. Find the inverse, if it exists, for the matrix: $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

$$[B \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{3}R_3]{-\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

$$\xrightarrow[-\frac{3}{2}R_3+R_2]{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] = [I \mid B^{-1}]$$

$$\therefore B^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

4. Express the vector $w = (5,4,2)$ as a linear combination of the vectors v_1, v_2, v_3, where $v_1 = (3,1,4)$, $v_2 = (1,2,0)$, $v_3 = (1,0,3)$.

Need to write w as $w = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$(5,4,2) = c_1(3,1,4) + c_2(1,2,0) + c_3(1,0,3)$$

$$\Rightarrow \begin{cases} 3c_1 + c_2 + c_3 = 5 & \text{---}(1) \\ c_1 + 2c_2 = 4 & \text{---}(2) \\ 4c_1 + 3c_3 = 2 & \text{---}(3) \end{cases}$$

solving this system:

$-3(1) + (3) \Rightarrow -5c_1 - 3c_2 = -13$ ——— (4)

$5(2) + (4) \Rightarrow 7c_2 = 7 \Rightarrow \boxed{c_2 = 1}$

substitute for c_2 in (2) $\Rightarrow \boxed{c_1 = 2}$

substitute for c_1 in (3) $\Rightarrow \boxed{c_3 = -2}$

$$\therefore (5,4,2) = 2(3,1,4) + 1(1,2,0) - 2(1,0,3)$$

i.e. $w = 2v_1 + v_2 - 2v_3$

5. Determine whether the vectors $(8,3,-4)$, $(5,4,-6)$, $(3,-1,2)$ form a basis for the vector space R^3.

we check for linear independence:

$$\begin{vmatrix} 8 & 5 & 3 \\ 3 & 4 & -1 \\ -4 & -6 & 2 \end{vmatrix} = 8\begin{vmatrix} 4 & -1 \\ -6 & 2 \end{vmatrix} - 5\begin{vmatrix} 3 & -1 \\ -4 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & 4 \\ -4 & -6 \end{vmatrix}$$

$$= 8(2) - 5(2) + 3(-2)$$

$$= 16 - 10 - 6$$

$$= 0$$

$\therefore$ The given vectors are linearly dependent, and hence they do not form a basis.

6. Show that the set W consisting of all vectors of the form $(a, 3a, 0)$ is a subspace of the vector space R^3.

Let $u, v \in W$ so that $u = (a, 3a, 0)$ & $v = (b, 3b, 0)$. Then

(i) $u + v = (a, 3a, 0) + (b, 3b, 0) = (a+b, 3a+3b, 0)$

$= ((a+b), 3(a+b), 0) \in W$

$\Rightarrow u + v \in W$.

Also, for any constant c, we have

(ii) $cu = c(a, 3a, 0) = (ca, 3ca, 0) \in W$

$\Rightarrow cu \in W$

Hence, W is a subspace of $\mathbb{R}^3$.

7. Are the functions: $f_1 = x^2 + 1$, $f_2 = 3x - 1$, $f_3 = -4x + 1$ linearly independent? Justify.

Using the Wronskian test:

$$W(f_1, f_2, f_3) = \begin{vmatrix} x^2+1 & 3x-1 & -4x+1 \\ 2x & 3 & -4 \\ 2 & 0 & 0 \end{vmatrix} = 2 \neq 0$$

$\Rightarrow f_1, f_2, f_3$ are linearly independent.

Another way, Consider $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$

$\Rightarrow c_1(x^2+1) + c_2(3x-1) + c_3(-4x+1) = 0$

$\Rightarrow c_1 x^2 + (3c_2 - 4c_3)x + (c_1 - c_2 + c_3) = 0$

$\Rightarrow c_1 = 0$, $3c_2 - 4c_3 = 0$, $c_1 - c_2 + c_3 = 0$

So, $c_1 = c_2 = c_3 = 0$ and therefore f_1, f_2, f_3 are linearly indep.

8. Find a basis and the dimension of the solution space of the system:

$$\begin{aligned} x_1+2x_2+x_3-3x_4&=0\\ 2x_1+4x_2+4x_3-x_4&=0\\ 3x_1+6x_2+7x_3+x_4&=0 \end{aligned}$$

Solving the system:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 0\\ 2 & 4 & 4 & -1 & 0\\ 3 & 6 & 7 & 1 & 0 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 0\\ 0 & 0 & 2 & 5 & 0\\ 0 & 0 & 4 & 10 & 0 \end{array}\right]$$

$$\xrightarrow[-2R_2+R_3]{\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 0\\ 0 & 0 & 1 & \frac{5}{2} & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -\frac{11}{2} & 0\\ 0 & 0 & 1 & \frac{5}{2} & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\left.\begin{array}{l} x_1, x_3 \text{ leading} \\ x_2, x_4 \text{ free} \end{array}\right\}$

$$\Rightarrow \begin{cases} x_4 = t \\ x_2 = s \\ x_3 = -\frac{5}{2}t \\ x_1 = -2s+\frac{11}{2}t \end{cases}$$

$$X = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix} = \begin{bmatrix} -2s+\frac{11}{2}t\\ s\\ -\frac{5}{2}t\\ t \end{bmatrix} = \begin{bmatrix} -2s\\ s\\ 0\\ 0 \end{bmatrix} + \begin{bmatrix} \frac{11}{2}t\\ 0\\ -\frac{5}{2}t\\ t \end{bmatrix} = s\begin{bmatrix} -2\\ 1\\ 0\\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{11}{2}\\ 0\\ -\frac{5}{2}\\ 1 \end{bmatrix}$$

$\therefore$ A basis for the solution space is $\{(-2,1,0,0),(11,0,-5,2)\}$

The dimension is 2.

9. Find the general solution of the DE: $2y'' - 7y' + 3y = 0$

The characteristic equation is $2\lambda^2 - 7\lambda + 3 = 0$

$$(2\lambda - 1)(\lambda - 3) = 0$$

$$\Rightarrow \quad \lambda = \frac{1}{2}, 3$$

The general solution is:

$$y = c_1 e^{x/2} + c_2 e^{3x}$$

10. Solve the IVP: $y'' + 2y' + y = 0, \qquad y(0) = 5, \; y'(0) = -3.$

The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$

$$\Rightarrow \quad (\lambda + 1)^2 = 1$$

$$\Rightarrow \quad \lambda = -1, -1 \text{ (repeated root)}$$

The general solution is: $y = c_1 e^{-x} + c_2 x e^{-x}$.

Now, $y' = -c_1 e^{-x} - c_2 x e^{-x} + c_2 e^{-x}$

$y(0) = 5 \Rightarrow \boxed{c_1 = 5}$

$y'(0) = -3 \Rightarrow -c_1 + c_2 = -3 \Rightarrow \boxed{c_2 = 2}$

The solution of the IVP is $y = 5e^{-x} + 2xe^{-x}$