Q1. $y + x^2 \frac{dy}{dx} = 1 , y(1) = 0$

② $\frac{dy}{dx} = \frac{1-y}{x^2}$ (separable)

① $\frac{dy}{1-y} = \frac{1}{x^2} dx$

① $\int \frac{dy}{1-y} = \int \frac{1}{x^2} dx$

$-\ln|1-y| = \frac{-1}{x} + c$ ②

$\ln|1-y| = \frac{1}{x} - c.$ ●

$|1-y| = e^{-c} \cdot e^{\frac{1}{x}}$ ②

$1-y = c_1 e^{\frac{1}{x}} ; c_1 = \mp e^{-c}$ ①

$\boxed{y = 1 - c_1 e^{\frac{1}{x}}.}$ ②

$y(1) = 0 \Rightarrow$ ① $\boxed{c = \frac{1}{e}}$ $\boxed{\therefore y = 1 - \frac{e^{\frac{1}{x}}}{e}.}$ ②

(b) $x^2 y' + 2xy = 5y^3$; $y(1) = 1$ ①

Bernoulli ; set $u = y^{1-3} = y^{-2} = \frac{1}{y^2}$.

$$\frac{du}{dx} = -2y^{-3} \cdot \frac{dy}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$$ ②

$$\frac{x^2}{y^3} y' + \frac{2x}{y^2} = 5$$

②

$$-\frac{x^2}{2}\frac{du}{dx} + 2x \cdot u = 5$$

$$\frac{du}{dx} \overset{P(x)}{- \frac{4}{x}} u = \overset{Q(x)}{\frac{-10}{x^2}}$$ (linear) ①

$$\rho(x) = e^{\int P(x)dx} = e^{-4\int \frac{1}{x}dx} = e^{-4\ln(x)} = x^{-4}$$ ②

$$u(x) = \frac{\int Q(x)\rho(x)\,dx + C}{\rho(x)}$$ ④

$$= \frac{\int -10x^{-6}\,dx + C}{x^{-4}} = x^4(2x^{-5} + C)$$
$$= \frac{2}{x} + Cx^4.$$

$u(1)=1 \Rightarrow C = -1.$ ①

$\therefore y = 1/\sqrt{\frac{2}{x} - x^4}$. ② $y = \sqrt{\frac{x}{2 - x^5}}$

Q2 ⓐ $(x^2+y^3)dx + xy^2 dy = 0$

$$\frac{dy}{dx} = -\frac{(x^3+y^3)}{xy^2}$$ ①

$$= -\frac{1+(\frac{y}{x})^3}{(\frac{y}{x})^2}$$ ② (homogenous)

set ① $v = \frac{y}{x}$; $y = vx$; $\frac{dy}{dx} = \frac{dv}{dx}x + v$ ②

$$\therefore \quad \frac{dv}{dx}x + v = -\frac{1+v^3}{v^2} = -\frac{1}{v^2} - v$$

$$\frac{dv}{dx}x = -\frac{1}{v^2} - 2v = -\frac{1+2v^3}{v^2}$$

$$\frac{v^2\,dv}{1+2v^3} = -\frac{dx}{x}$$ ② (separable)

$$\int \frac{v^2}{1+2v^3} = -\int \frac{1}{x}dx$$

$$\frac{1}{6}\ln(1+2v^3) = -\ln(x) + c = \ln(\frac{c}{x})$$ ②

$$1 + 2\frac{y^3}{x^3} = \frac{C}{x^6}$$ ①

$$2\frac{y^3}{x^3} = \frac{C - x^6}{x^6}$$ ①

$$y^3 = \frac{C - x^6}{2x^3}$$ ①

②

$$\boxed{y = \sqrt[3]{\frac{9 - x^6}{2x^3}}}$$

ⓑ $x^2 y'' + 3x y' = 2$

$\boxed{u = y'}$ $\cancel{x}$ ①

$x^2 \frac{du}{dx} + 3x\, u = 2$ ②

$$\frac{du}{dx} + \overset{P(x)}{\boxed{\frac{3}{x}}}\, u = \overset{Q(x)}{\boxed{\frac{2}{x^2}}}$$ ①

linear

$$\rho(x) = e^{\int P(x)\,dx} = e^{\int \frac{3}{x} dx} = e^{3\ln x} = x^3.$$ ②

$$u = \frac{\int Q(x)\rho(x)\,dx + c}{\rho(x)} = \frac{\int \frac{2}{x^2}\cdot x^3 dx + c}{x^3}$$

$$\therefore\ u = \frac{x^2 + c}{x^3} = \frac{1}{x} + c x^{-3}$$ ②

$$\frac{dy}{dx} = \frac{1}{x} + c x^{-3}$$

$$y = \int \left(\frac{1}{x} + c x^{-3}\right) dx$$ ②

$$\boxed{\therefore\ y = \ln x - \frac{c}{2x^2} + D}$$

⑨ $$\underbrace{(xe^{2y}+1)}_{M}dx+\underbrace{x^2e^{2y}}_{N}dy=0$$ exact ①

$$\frac{\partial M}{\partial y} = x(2)e^{2y} = 2xe^{2y}$$ ③

equal

$$\frac{\partial N}{\partial x} = 2xe^{2y}.$$ ③

$$F(x,y) = \int M dx + g(y)$$ ③

$$= \int (xe^{2y}+1)dx + g(y)$$

$$= \frac{x^2}{2}e^{2y} + x + g(y)$$ ①

$$N = \frac{\partial F}{\partial y}$$

$$x^2e^{2y} = x^2e^{2y} + g'(y) \Rightarrow g'(y)=0$$

$g(y)=c$ ②

$F(x,y)=D$

$$\therefore \frac{x^2}{2}e^{2y} + x = E$$ ② $(E=D-C)$

Ⓠ3 Using Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - A)$$ ④

where $T(t) :=$ tempreature of the soup at time t.

$A = 25$ is the tempreature of the room.

$$\frac{dT}{T-25} = -k\,dt$$

$$\ln|T-25| = -kt + C$$ ②

$$T = 25 + C_1 e^{-kt}.$$

$$T(0) = 100 \Rightarrow 75 = C_1.$$

$$\boxed{T = 25 + 75e^{-kt}}$$ ③

$$T(10) = 60 \Rightarrow 35 = 75e^{-10k}$$

$$k = -\frac{1}{10}\ln\left(\frac{35}{75}\right) = -\frac{1}{10}\ln\left(\frac{7}{15}\right)$$ ②

$$k = -\frac{1}{10}\left[\ln(7) - \ln(15)\right]$$

$$= -\frac{1}{10}\left[1.946 - 2.708\right]$$

$$= 0.0762.$$ ①

We want to find $\tilde{t}$ such that

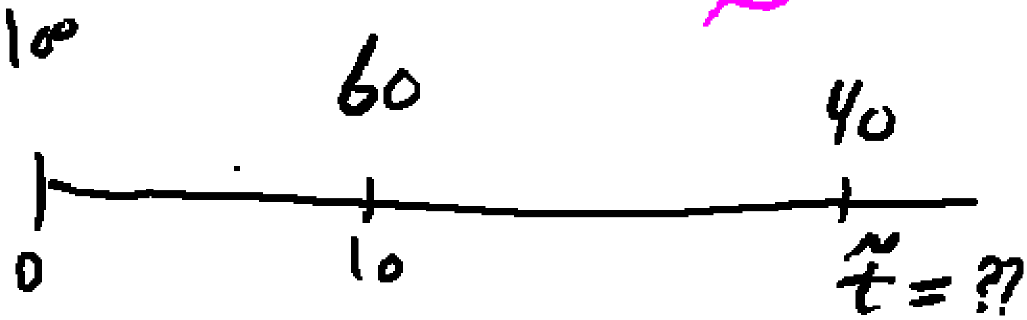
$$T(\tilde{t}) = 40.$$

So $$40 = 25 + 75e^{-k\tilde{t}}$$ ②

$$\tilde{t} = \frac{\ln\left(\frac{15}{75}\right)}{-k} = \frac{\ln(5)}{k}$$ ④

② $$= \frac{1.609}{0.0762}\text{ min}$$

$\approx$ 21 minutes.



$$\begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 2 & -1 & 2 & -1 & -1 \\ 1 & 2 & 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

①

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 3 \\ 2 & -1 & 2 & -1 & -1 & -1 \\ 1 & 2 & 1 & -2 & -2 & 2 \end{array}\right]$$

$R_2 \Longrightarrow -2R_1 + R_2$

$R_3 \Longrightarrow -R_1 + R_3$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 3 \\ 0 & -3 & 4 & -3 & -3 & -7 \\ 0 & 1 & 2 & -3 & -3 & -1 \end{array}\right]$$

②

$R_3 \leftrightarrows R_2$ (switch) ①

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 & -3 & -1 \\ 0 & -3 & 4 & -3 & -3 & -7 \end{array}\right]$$

$R_3 \Longrightarrow -3R_2 + R_3$ ②

$$\left[\begin{array}{ccccc|c} \boxed{1} & 1 & -1 & 1 & 1 & 3 \\ 0 & \boxed{1} & 2 & -3 & -3 & -1 \\ 0 & 0 & \boxed{10} & -12 & -12 & -10 \end{array}\right]$$

Echelon Form

free ②

$$10z - 12\boxed{u} - 12\boxed{w} = -10$$

$$z = \frac{12u + 12w - 10}{10}$$

By back substitution:

$$\boxed{z = \frac{6}{5}u + \frac{6}{5}w - 1}$$ ③

$$y + 2z - 3u - 3w = -1.$$

$$y = 3u + 3w - 1 - 2z$$

$$= 3u + 3w - 1 - 2\left[\frac{6}{5}u + \frac{6}{5}w - 1\right]$$

$$\boxed{y = \frac{3}{5}u + \frac{3}{5}w + 1}$$

③

$$x + y - z + u + w = 3$$

$$x = -y + z - u - w + 3$$

$$x = -\left(\frac{3}{5}u + \frac{3}{5}w + 1\right) + \left(\frac{6}{5}u + \frac{6}{5}w - 1\right) - u - w + 3$$

$$\boxed{x = -\frac{2}{5}u - \frac{2}{5}w + 1.}$$

③

solution set:

$$\left\{\left(-\frac{2}{5}u - \frac{2}{5}w + 1, \frac{3}{5}u + \frac{3}{5}w + 1, \frac{6}{5}u + \frac{6}{5}w - 1, u, w\right) \middle| \; u, w \in \mathbb{R}\right\}.$$

②