4.1 Inverse Functions

Definition *Let f and g be two functions, such that*

$$g(f(x)) = x \text{ for every } x \in \text{Domain}(f)$$
$$f(g(y)) = y \text{ for every } y \in \text{Domian}(g).$$

*Then we say f and g are **inverses**. Moreover we set $f^{-1} := g$ and $g^{-1} := f$.*

Example $f(x) = 2x - 1$ *and* $g(x) = \frac{x+1}{2}$ *are inverses. For all* $x \in R = \text{Domain}(f)$ *we have*

$$g(f(x)) = \frac{f(x)+1}{2} = \frac{2x-1+1}{2} = x$$

and for all $y \in R = \text{Domain}(g)$ *we have*

$$f(g(y)) = 2g(y) - 1 = 2\left(\frac{y+1}{2}\right) - 1 = y.$$

Example $f(x) = x^3$ *and* $g(x) = \sqrt[3]{x}$ *are inverses. For all* $x \in R = \text{Domain}(f)$ *we have*

$$g(f(x)) = \sqrt[3]{x^3} = x$$

and for all $y \in R = \text{Domain}(g)$ *we have*

$$f(g(y)) = (g(y))^3 = (\sqrt[3]{y})^3 = y.$$

Example $f(x) = x^2$ *and* $g(x) = \sqrt{x}$ *are <u>not inverses</u>. For each* $x \in R = \text{Domain}(f)$ *we have*

$$g(f(x)) = \sqrt{f(x)} = \sqrt{x^2} = |x| \neq x \ (\text{if } x < 0).$$

Example *If we restrict the domain of* $f(x) = x^2$ *to be* $[0, \infty)$*, then*

$$f(x) = x^2, [0, \infty) \to [0, \infty)$$

and

$$g(x) = \sqrt{x} : [0, \infty) \to [0, \infty)$$

become inverses: for all $x \in [0, \infty)$ *we have*

$$g(f(x)) = \sqrt{g(x)} = \sqrt{x^2} = |x| = x \ (\text{since } x \geq 0)$$

and for all $y \in [0, \infty)$*, we have* $f(g(y)) = (\sqrt{y})^2 = y$.

Example *If we restrict the domain of* $f(x)$ *to be* $(-\infty, 0]$*, then*

$$f(x) = x^2 : (-\infty, 0] \to [0, \infty)$$

has an inverse, which can be easily seen to be:

$$g(x) = -\sqrt{x} : [0, \infty) \to (-\infty, 0].$$

How to find Inverse Functions?

If an equation $y = f(x)$ can be solved for x as a function $x = g(y)$, then f and g are inverse functions and $f^{-1}(x) = g(x)$.

Example *Consider* $f(x) = \frac{2x-5}{3}$. *Solving* $y = \frac{2x-5}{3}$ *for* x*, we get* $2x - 5 = 3y$*, hence* $x = \frac{3y+5}{2} = g(y)$.

So f has inverse given by $f^{-1}(x) = \frac{3x-5}{2}$.

Definition *A function $f : D \to R$ is called* **1-1** (***injective***), *if*

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \text{ for all } x_1, x_2 \in D,$$

or equivalently

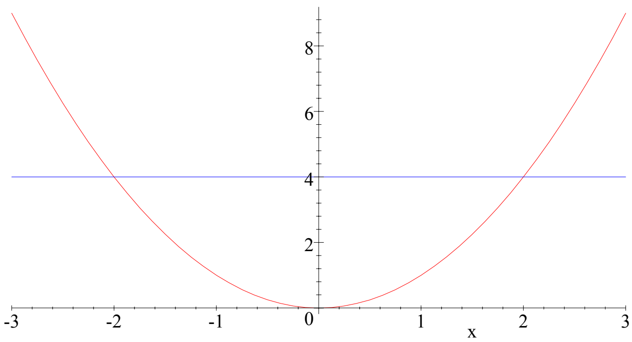
$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ for all } x_1, x_2 \in D.$$

Summary *If there exists (at least one) horizontal line that intersects the graph of $y = f(x)$ more than once, then $f(x)$ is not 1-1.*

Example *The function*

$$f(x) = x^2 : (-\infty, \infty) \to [0, \infty)$$

is not 1-1, since $f(-2) = 4 = f(2)$. Notice that the line $y = 4$ intersect the graph of $y = x^2$ twice.



$f(x) = x^2;\ y = 4$

Notice that $f(x) = x^2$ becomes 1-1, if we restrict D to $(-\infty, 0]$ or to $[0, \infty)$.

Summary *If a function $f(x)$ is 1-1 on D and has range R, then it has inverse $f^{-1} : R \to D$, and moreover $f^{-1}(x) = g(x)$, where $x = g(y)$.*

Example *Consider $f(x) = 2x^3 - 5$. If $f(x_1) = f(x_2)$ for some $x_1, x_2 \in (-\infty, \infty) = D(f)$, then*

$$2x_1^3 - 5 = 2x_2^3 - 5 \Rightarrow 2x_1^3 = 2x_2^3$$
$$\Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2.$$

Hence $f(x)$ is 1-1 with range $(-\infty, \infty)$. Solving $y = 2x^3 - 5$ for x, we find $x = \sqrt[3]{\frac{y+5}{2}} = g(y)$ and so $f^{-1}(x) = g(x) = \sqrt[3]{\frac{x+5}{2}}$.

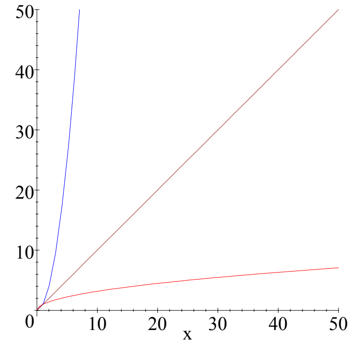
Theorem *If f and g are inverse functions, then the graph of $y = g(x)$ is the mirror image of the graph of $y = f(x)$ in the line $y = x$.*

Example *The function*

$$f(x) = x^2 : [0, \infty) \to [0, \infty)$$

has inverse

$$g(x) = \sqrt{x} : [0, \infty) \to [0, \infty).$$



$y = x^2,\ y = x,\ y = \sqrt{x}$

Definition *A function* $f : D \to R$ *is* ***increasing*** *on* $I \subseteq D$, *if for all* $x_1, x_2 \in I$:

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$

It's ***decreasing*** *on* $I \subseteq D$, *if for all* $x_1, x_2 \in I$:

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

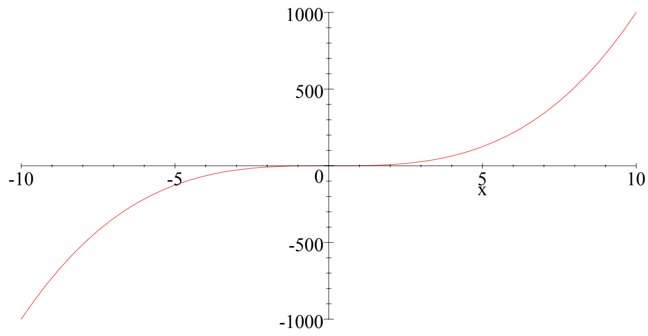
Remark *If a function is increasing or decreasing on its domain, then it's* 1-1 *on that domain.*

Lemma *Let* D *be an interval and* $f : D \to R$ *be differentiable.*

1. *If* $f'(x) > 0$ *for all* $x \in D$, *then* f *is increasing on* D.
2. *If* $f'(x) < 0$ *for all* $x \in D$, *then* f *is decreasing on* D.

Theorem *If* D *is an interval and* $f : D \to R$ *(where* $R = \text{Range}(f)$*) is differentiable with* $f'(x) > 0$ *or* $f'(x) < 0$ *for all* $x \in D$, *then* f *has an inverse function* $f^{-1} : R \to D$.

Example $f(x) = x^3$ *is increasing on* $(-\infty, \infty)$. *So* $f(x)$ *is* 1-1, *and has inverse* $f^{-1}(x) = \sqrt[3]{x}$.



$f(x) = x^3$

Theorem *Let* D *be an interval and*

$$f : D \to R \text{ (where } R = \text{Range}(f)\text{)}$$

be a 1-1 *continuous function. Then* R *is an interval and* $f^{-1} : R \to D$ *is also continuous.*

Derivatives of Inverse Functions

Theorem *Let D be an open interval and*

$$f : D \to R \ (\textit{where } R = \textit{Range } f)$$

be differentiable and 1-1 on D. If $f'(f^{-1}(x)) \neq 0$ for some $x \in R$, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{dx/dy}.$$

Proof Let $y = f^{-1}(x)$ with $f'(y) \neq 0$. Then $f(y) = x$ and we get by implicit differentiation $f'(y) . \frac{dy}{dx} = 1$, hence

$$(f^{-1})'(x) = \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

Note also we get that $\frac{dx}{dy} = f'(y)$, hence

$$(f^{-1})'(x) = \frac{1}{f'(y)} = \frac{1}{dx/dy}.$$

Corollary *Let D be an interval and*

$$f : D \to R \ (\textit{where } R = \textit{Range } f)$$

be differentiable with $f'(x) > 0$ or $f'(x) < 0$ for all $x \in D$. Then f has inverse $f^{-1} : R \to D$ and f^{-1} is differentiable on R with

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

Example *Consider $f(x) = x^3 + x + 1$. Then $f'(x) = 3x^2 + 1 > 0$ for all $x \in (-\infty, \infty)$. So $f(x)$ is increasing on $(-\infty, \infty)$ (hence 1-1) and therefore has inverse. Let $y = f^{-1}(x)$, so that $x = f(y) = y^3 + y + 1$. Then $\frac{dx}{dy} = 3y^2 + 1$ and*

$$\frac{d}{dx} f^{-1}(x) = \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{3y^2 + 1}.$$

One may also differentiate

$$x = y^3 + y + 1$$

implicitly to get

$$1 = (3y^2 + 1)\frac{dy}{dx} \textit{ and so } \frac{dy}{dx} = \frac{1}{3y^2 + 1}.$$

In particular

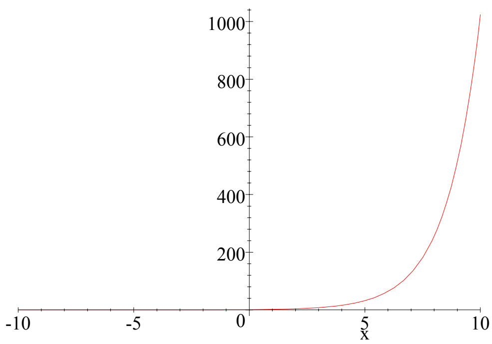
$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{3(0)^2 + 1} = 1.$$

4.2 Exponential and Logarithmic Functions

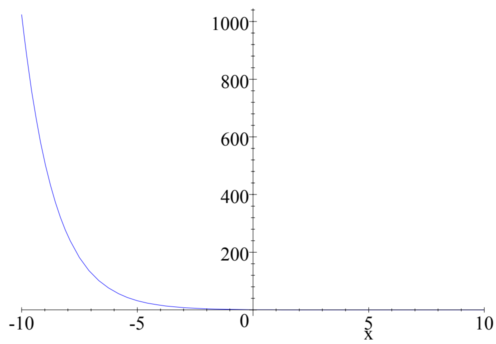
Definition *An **exponential function** is one of the form $f\ x \quad b^x$, where $b \quad 0, \quad \backslash\ 1$.*

	$b \quad 1$	$0 \quad b \quad 1$
Domian b^x	,	,
Range b^x	$0,$	$0,$
y-intercept	1	1
Monotonicity	*increasing*	*decreasing*
Far-Right Behaviour	$\lim_{x} b^x$	$\lim_{x} b^x \quad 0$
Far-Left Behaviour	$\lim_{x} b^x \quad 0$	$\lim_{x} b^x$
Asymptotes	$y \quad 0$	$y \quad 0$

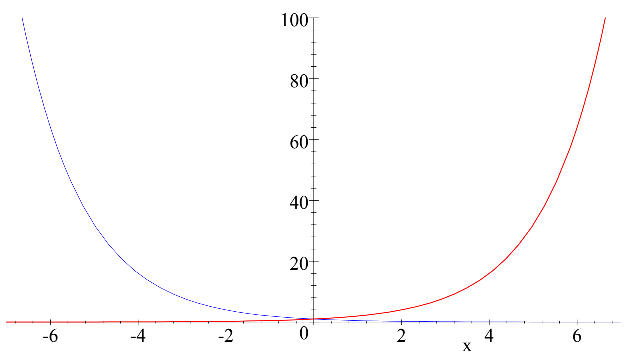
Remark *Let $b \quad 0, \quad \backslash\ 1$. Then the graph of $f\ x \quad b^x$ is the reflection (mirror image) of the graph of $g\ x \quad b^{\ x} \quad \frac{1}{b}^{\ x}$ in the y-axis.*



$f\ x \quad 2^x$



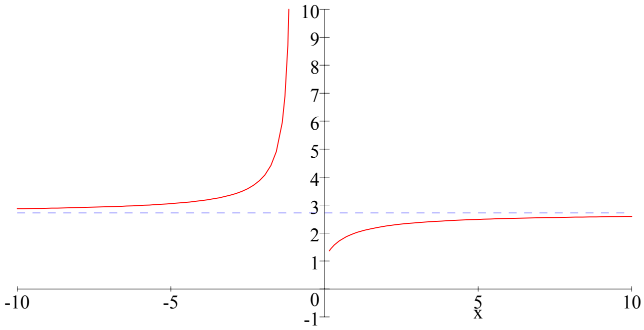
$g\ x \quad \frac{1}{2}^{\ x}$



$f\ x \quad 2^x,\ g\ x \quad \frac{1}{2}^{\ x}$

The Irrational Number e

Consider the function $f(x) = \left(1 + \frac{1}{x}\right)^x$.



$$f(x) = \left(1 + \frac{1}{x}\right)^x$$

Summary *It's clear that $f(x)$ has a horizontal asymptote. We define*

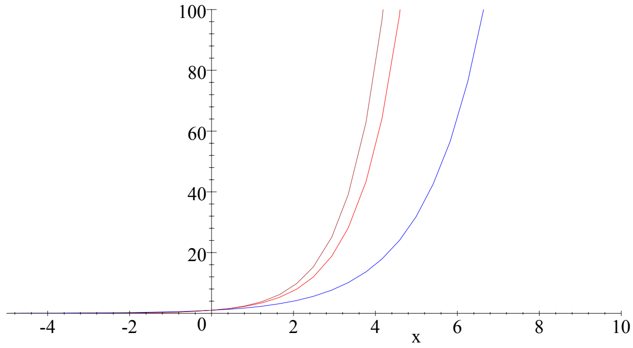
$$e := \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x.$$

With the substitution $v = \frac{1}{x}$, we see that as $x \to \pm\infty$, $v \to 0$ and so

$$\lim_{v \to 0} (1 + v)^{\frac{1}{v}} = e.$$

In fact "e" can be proved to be an irrational real number and can be approximated as follows:

x	$\left(1 + \frac{1}{x}\right)^x$
1	2
10	2.593742
100	2.704814
1000	2.716924
10,000	2.718156
100,000	2.718268
1,000,000	2.718280



$y \quad 3^x, y \quad e^x, y \quad 2^x$

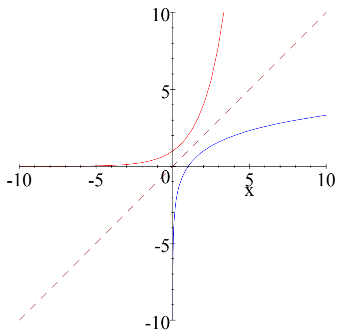
The Logarithmic Functions

Let $b \quad 0, \quad \backslash\ 1$. The function

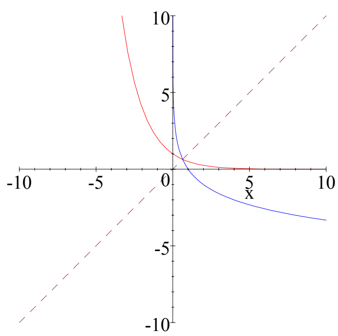
$$f\,x \quad b^x : \quad , \quad 0,$$

is **monotone** (increasing, if b 0 and decreasing, if b 0). So it's 1-1 and has inverse

$$g\,x \quad \log_b x : \ 0, \quad , \quad .$$



$f\,x \quad b^x, f^{\ 1}\,x \quad \log_b x;\ b \quad 1$



$f\,x \quad b^x, f^{\ 1}\,x \quad \log_b x;\ b \quad 1$

In fact for $b \quad 0, \quad \backslash\ 1$ we have

$y \quad b^x \quad x \quad \log_b y$ for every x R and $y \quad 0$

and

$y \quad \log_b x \quad x \quad b^y$ for every $x \quad 0$ and y R.

Summary *Let* $b \quad 1$.

	$f\,x \quad b^x$	$f^{\ 1}\ x \quad log_b x$
Domain	,	0,
Range	0,	,
intercepts	$b^0 \quad 1$	$log_b 1 \quad 0$
	$b^1 \quad b$	$log_b b \quad 1$
Monotonicity	*increasing*	*increasing*
Injectivity	1-1 (*injective*)	1-1 (*injective*)
Far-Right Behaviour	$\lim_{x} b^x$	$\lim_{x} log_b x$
Far-Left Behaviour	$\lim_{x} b^x \quad 0$	$\lim_{x\ 0} log_b x$

Let $0 \quad b \quad 1$.

	$f\,x \quad b^x$	$f^{\ 1}\ x \quad log_b x$
Domain	,	0,
Range	0,	,
intercepts	$b^0 \quad 1$	$log_b 1 \quad 0$
	$b^1 \quad b$	$log_b b \quad 1$
Monotonicity	*decreasing*	*decreasing*
Injectivity	1-1 (*injective*)	1-1 (*injective*)
Far-Right Behaviour	$\lim_{x} b^x \quad 0$	$\lim_{x} log_b x$
Far-Left Behaviour	$\lim_{x} b^x$	$\lim_{x\ 0} log_b x$

Theorem (***Algebraic Properties of*** $log_b x$).

Let $b \quad 0, \quad \backslash\ 1\ ,\ a \quad 0,\ c \quad 0$ *and* r R. *Then*

$$log_b\ a \quad c \qquad log_b a \quad log_b c \qquad \textit{Product Property}$$
$$log_b\ \tfrac{a}{c} \qquad log_b a \quad log_b c \qquad \textit{Quotient Property}$$
$$log_b\ a^r \qquad r\,log_b a. \qquad \textit{Power Property}$$
$$log_b\ \tfrac{1}{c} \qquad log_b c \qquad \textit{Reciprocal Property}$$

Remark *Let* $b \quad 0, \quad \backslash\ 1\ ,\ a \quad 0,\ c \quad 0$. *Then*

$$log_b\ a \quad c \qquad log_b a \quad log_b c;$$
$$log_b\ a \quad c \qquad log_b a \quad log_b c.$$

Example *Consider the equation*

$$\frac{e^{x} \quad e^{\;x}}{2} \quad 1.$$

Then we get after multiplying with e^x *and rearranging the equation*

$$e^{2x} \quad 2e^{x} \quad 1 \quad 0.$$

With the substitution $u \quad e^x$, *we get the quadratic equation*

$$u^2 \quad 2u \quad 1 \quad 0,$$

which has two solutions

$$u \quad \frac{2 \quad \sqrt{2^{\;2} \quad 4\,1 \quad 1}}{2\,1}$$

$$\frac{2 \quad 2\sqrt{2}}{2} \quad 1 \quad \sqrt{2}\,.$$

Since $u \quad e^x \quad 0$, *we ignore* $u \quad 1 \quad \sqrt{2} \quad 0$ *and consider*

$$e^x \quad u \quad 1 \quad \sqrt{2}\,,$$

which yields

$$x \quad ln\;u \quad ln\;1 \quad \sqrt{2}\;\;.$$

Remark *Let* $b, c \quad 0, \quad \backslash\; 1$. *Let* $x \quad 0$ *and* $y:\; log_b x$. *Then* $b^y \quad x$ *and we get*

$$log_c\; b^y \quad log_c x$$

$$y\, log_c b \quad log_c x,$$

which yields

$$log_b x \quad \frac{log_c x}{log_c b}.$$

In particular for any $x \quad 0$, *we have*

$$log_b x \quad \frac{log\,x}{log\,b} \quad \frac{ln\,x}{ln\,b}.$$

Example *To find* $log_3 \quad$, *we use the calculator and the formula*

$$log_3 \quad \frac{ln}{ln\,3} \quad \frac{1.1447}{1.0986} \quad 1.042$$

4.3 Derivatives of Logarithmic and Exponential Functions

Theorem *Let* $b \in (0, \infty) \setminus \{1\}$*. Then*

$$\frac{d}{dx}\left(\log_b x\right) = \frac{1}{x \ln(b)}, \text{ where } x > 0.$$

In particular

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x \ln(e)} = \frac{1}{x}, \text{ where } x > 0.$$

Example

$$\frac{d}{dx}\left(\ln\left(x^2+1\right)\right) = \frac{1}{x^2+1}\,(2x)$$

Example *Let*

$$f(x) = \ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0. \end{cases}$$

Then

$$f'(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x}(-1), & x < 0. \end{cases}$$

$$= \frac{1}{x}, \ x \neq 0.$$

Example *We can use* ***logarithmic differentiation*** *to find the derivative of functions like*

$$y = \frac{x^3 \sqrt[5]{x-3}}{\left(1+x^4\right)^2}.$$

Taking the absolute value of both sides, we get

$$|y| = \frac{|x|^3 \left|x-3\right|^{\frac{1}{5}}}{\left(1+x^4\right)^2}.$$

Applying the "*ln*" *to both sides we get*

$$\ln|y| = 3\ln|x| + \frac{1}{5}\ln|x-3| - 2\ln\left(1+x^4\right).$$

Differentiating the new equation implicitly yields:

$$\frac{1}{y}\,y' = \frac{3}{x} + \frac{1}{5(x-3)} - 2\,\frac{1}{1+x^4}\left(4x^3\right),$$

hence

$$y' = \frac{x^3 \sqrt[5]{x-3}}{\left(1+x^4\right)^2}\left(\frac{3}{x} + \frac{1}{5(x-3)} - \frac{8x^3}{1+x^4}\right), \ x \notin \{0, 3\}.$$

Theorem *Let* r *be an arbitrary real number and* $y = x^r$*. Then* $y' = rx^{r-1}$.

Proof When $x = 0$ the result is obvious. If $x \neq 0$, then applying the "ln" to

$$|y| = |x|^r, \ x \neq 0$$

yields

$$\ln|y| \quad \ln\ |x|^r \quad r\ln\ |x|\ .$$

Differentiating the new equation implicitly results in

$$\frac{1}{y}\quad y \quad r\frac{1}{x},$$

hence

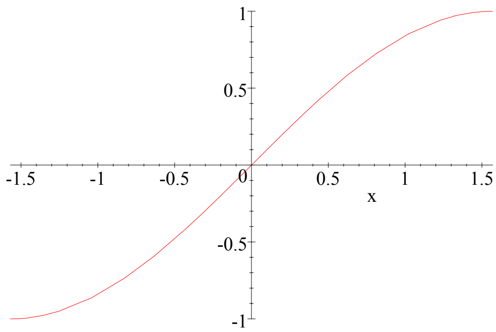
$$y \quad rx^{r\ \ 1}.$$

Example *Let* $f\ x \quad x^2 \quad x^{1\ \ 2e}$. *Then* $f\ \ x \quad 2\ \ x^{2\ \ \ 1} \quad 1 \quad 2e\ \ x^{\ 2e}$.

4.4 Inverse Trigonometric Functions

Definition *Consider the* 1-1 *function*

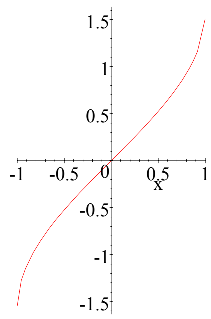
$$f(x) = \sin x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1].$$



$f(x) = \sin x$

Then $f(x)$ *has inverse the* ***inverse sine function***:

$$g(x) = \sin^{-1} x : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$



$g(x) = \sin^{-1} x$

Remarks From the graph of $f(x) = \sin x$ and $g(x) = \sin^{-1} x$ we have

$f(x) = \sin x$	$g(x) = \sin^{-1} x$
$\lim_{x \to \frac{\pi}{2}^-} \sin x = 1$	$\lim_{x \to 1^-} \sin^{-1} x = \frac{\pi}{2}$
$\lim_{x \to -\frac{\pi}{2}^+} \sin x = -1$	$\lim_{x \to -1^+} \sin^{-1} x = -\frac{\pi}{2}$
$\sin 0 = 0$	$\sin^{-1} 0 = 0$
$\sin(-x) = -\sin x$ "odd"	$\sin^{-1}(-x) = -\sin^{-1} x$ "odd"
increasing	increasing
continuous	continuous

Theorem *Let* $g(x) = \sin^{-1}x$. *Then*

$$g'(x) = \frac{1}{\sqrt{1-x^2}}, \quad x \in (-1, 1).$$

Proof Let $y = g(x) = \sin^{-1}x$. Then $\sin y = x$ and we get with implicit differentiation

$$\cos y \, \frac{dy}{dx} = 1,$$

hence

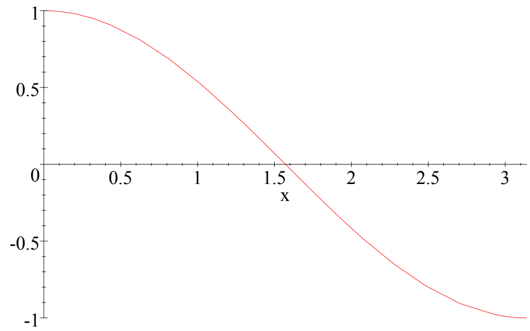
$$\begin{aligned} g'(x) = \frac{dy}{dx} &= \frac{1}{\cos x} \\ &= \frac{1}{\sqrt{1-\sin^2 x}} \\ &= \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1. \end{aligned}$$

Example

$$\begin{aligned} \frac{d}{dx}\left(\sin^{-1}x^7\right) &= \frac{1}{\sqrt{1-\left(x^7\right)^2}} \cdot 7x^6 \\ &= \frac{7x^6}{\sqrt{1-x^{14}}}. \end{aligned}$$

Definition *Consider the* 1-1 *function*

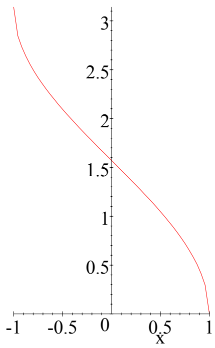
$$\cos x : [0, \pi] \to [-1, 1].$$



$$f(x) = \cos x$$

Then $f(x)$ *has inverse the* ***inverse cosine function***:

$$g(x) = \cos^{-1} x : [-1, 1] \to [0, \pi].$$



$$g(x) = \cos^{-1} x$$

Remarks From the graph of $f(x) = \cos x$ and $g(x) = \cos^{-1} x$ we have

$f(x) = \cos x$	$g(x) = \cos^{-1} x$
$\lim_{x \to 0^+} \cos x = 1$	$\lim_{x \to 1^-} \cos^{-1} x = 0$
$\lim_{x \to \pi^-} \cos x = -1$	$\lim_{x \to -1^+} \cos^{-1} x = \pi$
$\cos \frac{\pi}{2} = 0$	$\cos^{-1} 0 = \frac{\pi}{2}$
decreasing	decreasing
continuous	continuous

Theorem *Let $g(x) = \cos^{-1}x$. Then*

$$g'(x) = -\frac{1}{\sqrt{1-x^2}},\ x \in (-1,1).$$

Proof Let $y = g(x) = \cos^{-1}x$. Then $\cos y = x$ and we get with implicit differentiation $-\sin y \cdot \frac{dy}{dx} = 1$, hence

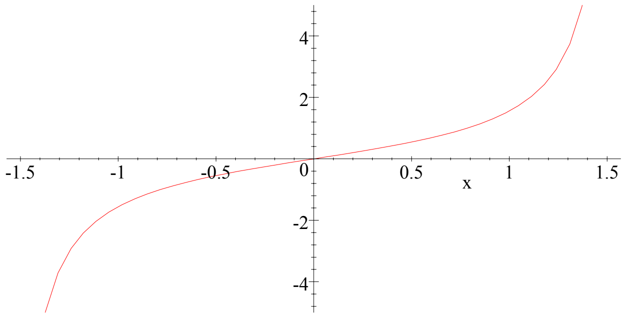
$$\begin{aligned} g'(x) = \frac{dy}{dx} &= -\frac{1}{\sin y} \\ &= -\frac{1}{\sqrt{1-\cos^2 y}} \\ &= -\frac{1}{\sqrt{1-x^2}},\quad -1 < x < 1. \end{aligned}$$

Example

$$\begin{aligned} \frac{d}{dx}\cos^{-1}(e^{-x}) &= -\frac{1}{\sqrt{1-(e^{-x})^2}}\cdot(-e^{-x}) \\ &= \frac{e^{-x}}{\sqrt{1-e^{-2x}}}. \end{aligned}$$

Definition *Consider the* 1-1 *function*

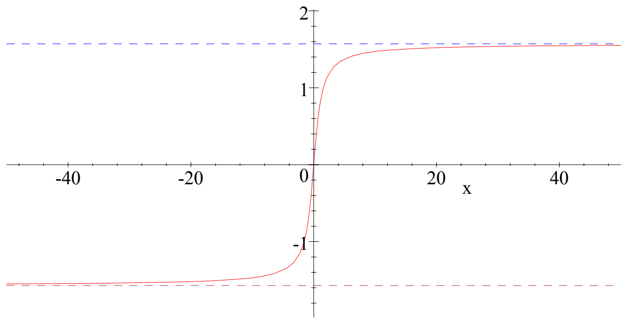
$$f\ x \qquad \tan x : \quad \overline{\ \ }\!\!\!\!\!\!\ 2, \overline{\ \ }\!\!\!\!\!\!\ 2 \qquad ,$$



$$f\ x \qquad \tan x$$

Then $f\ x$ *is invertible with inverse the* ***inverse tan function***:

$$g\ x \qquad \tan\ ^{1}x : \qquad , \qquad \overline{\ \ }\!\!\!\!\!\!\ 2, \overline{\ \ }\!\!\!\!\!\!\ 2$$



$$g\ x \qquad \tan\ ^{1}\ x\ ,\ y \quad \overline{\ \ }\!\!\!\!\!\!\ 2,\ y \qquad \overline{\ \ }\!\!\!\!\!\!\ 2$$

Remarks From the graph of $f\ x$ $\tan x$ and $g\ x$ $\tan\ ^{1}x$ we have

$f\ x \quad \tan x$	$g\ x \quad \tan\ ^{1}x$
$\lim_{x \quad \overline{2}} \tan x$	$\lim_{x} \tan\ ^{1}x \quad \overline{2}$
$\lim_{x \quad \overline{2}} \tan x$	$\lim_{x} \tan\ ^{1}x \quad \overline{2}$
vertical asymptotes $x \quad \overline{2}$	horizontal asymptotes $y \quad \overline{2}$
$\tan\ 0 \qquad 0$	$\tan\ ^{1}\ 0 \qquad 1$
increasing	increasing
continuous	continuous

Theorem *Let $g(x) = \tan^{-1} x$. Then*

$$g'(x) = \frac{1}{1+x^2} \text{ for all } x \in \mathbb{R}.$$

Proof Let $y = g(x) = \tan^{-1} x$. Then $\tan y = x$ and we get with implicit differentiation $\sec^2 y \, \frac{dy}{dx} = 1$, hence

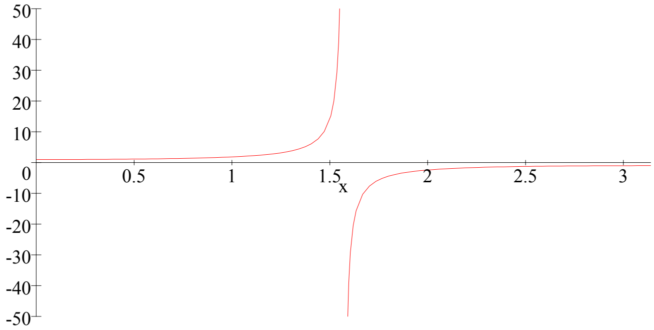
$$\begin{aligned} g'(x) &= \frac{dy}{dx} = \frac{1}{\sec^2 y} \\ &= \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2} \text{ for all } x \in \mathbb{R}. \end{aligned}$$

Example

$$\begin{aligned} \frac{d}{dx} \tan^{-1}(1-3x) &= \frac{1}{1+(1-3x)^2} \cdot (-3) \\ &= \frac{-3}{9x^2-6x+2}. \end{aligned}$$

Definition *Consider the* 1-1 *function*

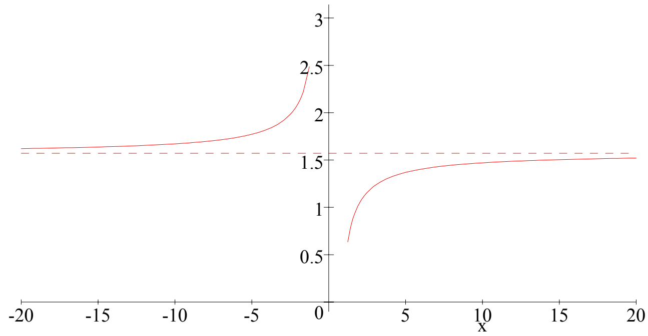
$$sec\ x\ :\ 0,\ \setminus\ \overline{\ \ }\ 2\qquad ,\ 1\qquad 1,\ \ .$$



$$f\ x\qquad sec\, x$$

Then $f\ x$ *has inverse the* ***inverse secant function***

$$g\ x\qquad sec^{\ 1}\ x\ :\qquad ,\ 1\qquad 1,\qquad 0,\ \setminus\ \overline{\ \ }\ 2\ .$$



$$g\ x\qquad sec^{\ 1}x,\ y\qquad \overline{\ \ }\ 2$$

Remarks

$f\ x\qquad \sec x$	$g\ x\qquad \sec^{\ 1} x$
$\lim_{x\ \ \overline{2}}\ \sec x$	$\lim_{x}\ \sec^{\ 1} x\qquad \overline{2}$
$\lim_{x\ \ \overline{2}}\ \sec x$	$\lim_{x}\ \sec^{\ 1} x\qquad \overline{2}$
vertical asymptote $x\qquad \overline{2}$	horizontal asymptote $y\qquad \overline{2}$
$\sec\ 0\qquad 1\ \&\ \sec\qquad 1$	$\sec^{\ 1}\ 1\qquad 0\ \&\ \sec^{\ 1}\qquad 1$
increasing on $\ 0,\ \overline{2}$	increasing on $\qquad ,\ 1$
increasing on $\ \overline{2},$	increasing on $\ 1,$

Theorem *Let $g\ x \quad sec^{\ 1}x$. Then*

$$g\ x \qquad \frac{1}{|x|\sqrt{x^2 \quad 1}},\ |x| \quad 1.$$

Proof Let $y \quad g\ x \quad \sec^{\ 1}x \quad \cos^{\ 1}\ \frac{1}{x}$ for $|x| \quad 1$.

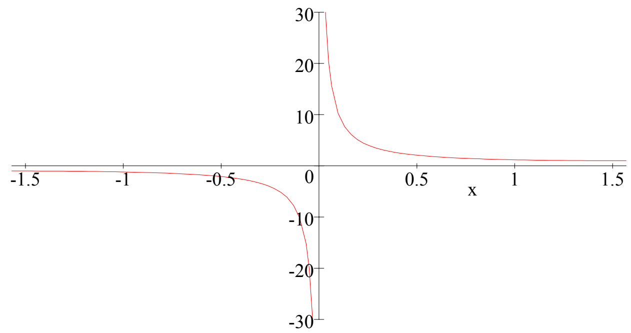
$$\begin{aligned} g\ x \quad & \frac{dy}{dx} \\ & \frac{1}{\sqrt{1 \quad \frac{1}{x}\ ^2}} \quad \frac{1}{x^2} \\ & \frac{\sqrt{x^2}}{\sqrt{x^2 \quad 1}} \quad \frac{1}{x^2} \\ & \frac{|x|}{\sqrt{x^2 \quad 1}} \quad \frac{1}{|x|\ ^2} \\ & \frac{1}{|x|\sqrt{x^2 \quad 1}} \text{ for } |x| \quad 1. \end{aligned}$$

Example

$$\frac{d}{dx} sec^{\ 1}\ e^x \qquad \frac{e^x}{|e^x|\sqrt{e^{2x} \quad 1}} \qquad \frac{1}{\sqrt{e^{2x} \quad 1}}.$$

Definition *Consider the* 1-1 *function*

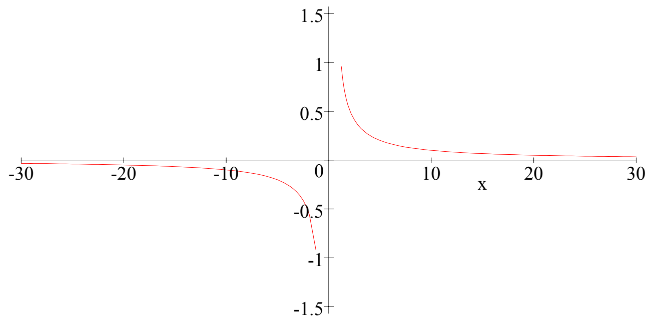
$$f(x) = csc(x) : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\} \to (-\infty, -1] \cup [1, \infty).$$



$$f(x) = csc(x)$$

Then $f(x)$ *has inverse the* ***inverse cosecant function***:

$$csc^{-1}(x) : (-\infty, -1] \cup [1, \infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$$



$$g(x) = csc^{-1}x$$

Remarks Consider the functions $f(x) = \csc x$ and its inverse $g(x) = \csc^{-1}x$.

$f(x) = \csc x$	$g(x) = \csc^{-1}x$
$\lim_{x \to 0^+} \csc x = \infty$	$\lim_{x \to \infty} \csc^{-1}x = 0$
$\lim_{x \to 0^-} \csc x = -\infty$	$\lim_{x \to -\infty} \csc^{-1}x = 0$
vertical asymptote $x = 0$	horizontal asymptote $y = 0$
$\csc\left(-\frac{\pi}{2}\right) = -1$ & $\csc\left(\frac{\pi}{2}\right) = 1$	$\csc^{-1}(-1) = -\frac{\pi}{2}$ & $\csc^{-1}(1) = \frac{\pi}{2}$
decresing on $\left[-\frac{\pi}{2}, 0\right)$	decresing on $(-\infty, -1]$
decresing on $\left(0, \frac{\pi}{2}\right]$	decresing on $[1, \infty)$

Theorem *Let $g(x) = \csc^{-1}x$. Then*

$$g'(x) = \frac{-1}{|x|\sqrt{x^2-1}},\ |x| > 1.$$

Proof Let $y = g(x) = \csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$ where $|x| > 1$. Then

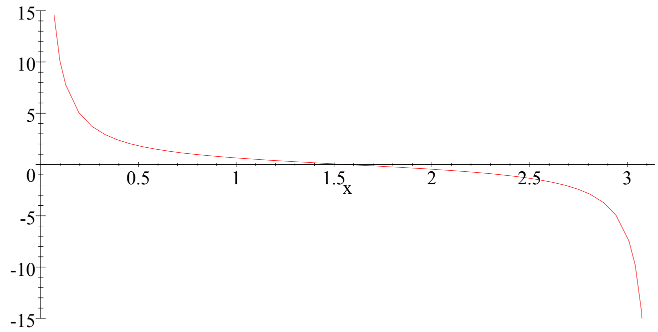
$$
\begin{aligned}
g'(x) &= \frac{dy}{dx} \\
&= \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\left(-\frac{1}{x^2}\right) \\
&= \frac{\sqrt{x^2}}{\sqrt{x^2-1}}\left(-\frac{1}{x^2}\right) \\
&= \frac{|x|}{\sqrt{x^2-1}}\left(-\frac{1}{(|x|)^2}\right) \\
&= \frac{-1}{|x|\sqrt{x^2-1}} \text{ for } |x| > 1.
\end{aligned}
$$

Example

$$
\begin{aligned}
\frac{d}{dx}\csc^{-1}(\ln x) &= \frac{-1}{|\ln x|\sqrt{\ln^2 x-1}}\left(\frac{1}{x}\right) \\
&= \frac{-1}{x|\ln x|\sqrt{\ln^2 x-1}}.
\end{aligned}
$$

Definition *Consider the* 1-1 *function*

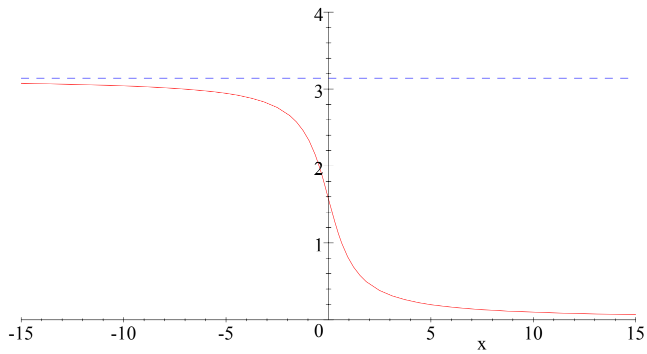
$$f\ x \quad cot x :\ 0, \qquad ,\ .$$



$$f\ x \quad cot x$$

Then $f\ x$ *has inverse the* ***inverse cotan function***

$$g\ x \quad cot\ ^{1}x :\quad ,\qquad 0,\ .$$



$$g\ x \quad cot\ ^{1}x,\ y$$

Remarks

$f\ x \quad \cot x$	$g\ x \quad \cot\ ^{1}x$
$\lim_{x\ 0} \cot$	$\lim_{x} \cot\ ^{1}x \quad 0$
$\lim_{x} \cot x$	$\lim_{x} \cot\ ^{1}x$
vertical asymptotes $x \quad 0$ & x	horizontal asymptotes $y \quad 0$ & y
$\cot\ \overline{\ 2\ } \quad 0$	$\cot\ ^{1}\ 0 \quad \overline{\ 2\ }$
decreasing	decreasing
continuous	continuous

Theorem *Let $g(x) = \cot^{-1}x$. Then*

$$g'(x) = -\frac{1}{1+x^2} \text{ for all } x \in \mathbb{R}.$$

Proof Let $y = g(x) = \cot^{-1}x$. Then $\cot y = x$ and we get with implicit differentiation $-\csc^2 y \,\frac{dy}{dx} = 1$, hence

$$g'(x) = \frac{dy}{dx} = -\frac{1}{\csc^2 y} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2} \text{ for all } x \in \mathbb{R}.$$

Example

$$\frac{d}{dx}\cot^{-1}(\sin x) = -\frac{1}{1+\sin^2 x}\cdot\cos x = -\frac{\cos x}{1+\sin^2 x}.$$

Summary

$f\ x$	***Domain***	***Range***	***Derivative***	
$sin\ ^{1}x$	$1,1$	$\frac{}{2},\frac{}{2}$	$\frac{1}{\sqrt{1\ \ x^2}}$	$x\quad 1,1$
$cos\ ^{1}x$	$1,1$	$0,$	$\frac{1}{\sqrt{1\ \ x^2}}$	$x\quad 1,1$
$tan\ ^{1}x$	$,$	$\frac{}{2},\frac{}{2}$	$\frac{1}{1\ \ x^2}$	$x\quad R$
$sec\ ^{1}x$	$,\ 1\quad 1,$	$0,\ \ \backslash\ \frac{}{2}$	$\frac{1}{\lvert x\rvert\sqrt{x^2\ \ 1}}$	$x\quad ,\ 1\quad 1,$
$csc\ ^{1}x$	$,\ 1\quad 1,$	$\frac{}{2},\frac{}{2}\ \backslash\ 0$	$\frac{1}{\lvert x\rvert\sqrt{x^2\ \ 1}}$	$x\quad ,\ 1\quad 1,$
$cot\ ^{1}x$	$,$	$0,$	$\frac{1}{1\ \ x^2}$	$x\quad R$

Summary

$sec\ ^{1}x$		$cos\ ^{1}\ \frac{1}{x}\ ,$	$\lvert x\rvert\quad 1$	
$sin\ ^{1}x\quad cos\ ^{1}x$		$\frac{}{2},$	$x\quad 1,1$	
$cos\ sin\ ^{1}x$		$\sqrt{1\quad x^2},$	$x\quad 1,1$	
$tan\ sin\ ^{1}x$		$\frac{x}{\sqrt{1\ \ x^2}},$	$x\quad 1,1$	
$sin\ cos\ ^{1}x$		$\sqrt{1\quad x^2},$	$x\quad 1,1$	
$sec\ tan\ ^{1}x$		$\sqrt{1\quad x^2},$	$x\quad R.$	
$sin\ sec\ ^{1}x$		$\frac{\sqrt{x^2\ \ 1}}{\lvert x\rvert},$	$\lvert x\rvert\quad 1.$	

4.5 L'Hopital's Rule

Theorem *Suppose that f & g are functions, such that:*

1. $\lim_{x \quad a} f\,x \qquad 0 \qquad \lim_{x \quad a} g\,x$;
2. $f\,x$ & $g\,x$ *are differentiable on an open interval containing "a" (except possibly at "a" itself);*
3. $\lim_{x \quad a} \frac{f\,x}{g\,x} \qquad L, \quad or \qquad$ (*where* $L \qquad R$).

Then

$$\lim_{x \quad a} \frac{f\,x}{g\,x} \qquad \lim_{x \quad a} \frac{f\;\,x}{g\;\,x}.$$

Remark *The statement of the previous theorem remains true in the case* $x \qquad a\;,\; x \qquad a\;,\; x \qquad$ *or* $x \qquad$ (*with the second condition modified as appropriate*).

Example *Let* $b \qquad 0, \quad \backslash\; 1$. *Then*

$$\lim_{x \quad 0} \frac{b^x \quad 1}{x} \qquad \lim_{x \quad 0} \frac{b^x \ln\; b}{1} \qquad \ln\; b\;.$$

Example

$$\lim_{x \quad 0} \frac{\ln\;\cos x}{x} \qquad \lim_{x \quad 0} \frac{1/\cos x \qquad \sin x}{1} \qquad 0.$$

Example

$$\lim_{x \quad 0} \;\frac{1}{x} \qquad \frac{1}{e^x \quad 1} \qquad \lim_{x \quad 0} \frac{e^x \quad 1 \quad x}{x\;e^x \quad 1}$$

$$\lim_{x \quad 0} \frac{e^x \quad 1}{1 \quad e^x \quad 1 \quad x \quad e^x}$$

$$\lim_{x \quad 0} \frac{e^x \quad 1}{e^x\;1 \quad x \quad 1}$$

$$\lim_{x \quad 0} \frac{e^x}{e^x \quad 1\; x \quad e^x \quad 1}$$

$$\lim_{x \quad 0} \frac{e^x}{e^x\;2 \quad x}$$

$$\lim_{x \quad 0} \frac{1}{2 \quad x} \qquad \frac{1}{2}.$$

Theorem *Suppose that f & g are functions, such that:*

1. $\lim_{x \quad a} f\ x \qquad \lim_{x \quad a} g\ x$;
2. $f\ x$ & $g\ x$ *are differentiable on an open interval containing "a" (except possibly at "a" itself);*
3. $\lim_{x \quad a} \frac{f\ x}{g\ x} \quad L, \quad or \quad (where\ L \quad R).$

Then

$$\lim_{x \quad a} \frac{f\ x}{g\ x} \qquad \lim_{x \quad a} \frac{f\ \ x}{g\ \ x}.$$

Remark *The statement of the previous theorem remains true in the case* $x \quad a$, $x \quad a$, $x \quad$ *or* $x \quad$ *(with the second condition modified as appropriate).*

Example

$$\lim_{x \ 0} \frac{ln\ sin x}{ln\ x} \qquad \lim_{x \ 0} \frac{1/sin x \quad cos x}{1/x}$$

$$\lim_{x \ 0} \frac{x\, cos x}{sin x}$$

$$\lim_{x \ 0} \frac{x}{sin x} \qquad \lim_{x \ 0} cos x$$

$$1 \quad 1 \quad 1.$$

Example

$$\lim_{x} \quad e^x \quad x^2 \qquad \lim_{x} \quad x^2 \quad \frac{e^x}{x^2} \quad 1 \quad .$$

Now

$$\lim_{x} \frac{e^x}{x^2} \qquad \lim_{x} \frac{e^x}{2x} \qquad \lim_{x} \frac{e^x}{2} \qquad .$$

Hence

$$\lim_{x} \quad e^x \quad x^2 \qquad \lim_{x} x^2 \quad \lim_{x} \frac{e^x}{x^2} \quad 1 \qquad .$$